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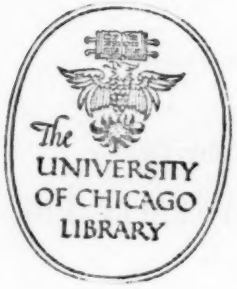

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796

INDEX

PAGE

ADAMS, C. R. On the linear ordinary q-difference equation 195

ADAMS, C. R. Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial. . . . 626

ALBERT, A. A. On the structure of normal division algebras 322

ALBERT, A. A. Normal division algebras in $4p^2$ units, p an odd prime . 583

ALBERT, A. A. The structure of any algebra which is a direct product of rational generalized quaternion division algebras 621

ALEXANDROFF, P. Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension 101

ALTSHILLER-COURT, N. On five mutually orthogonal spheres 613

ARCHIBALD, R. G. The impossibility of a separation of types of linear odd divisors of binary quadratic forms 12

ARWIN, A. On cubic fields 1

BELL, E. T. An interpretation of certain decomposable algebraic forms as functions of divisors 429

BROWNE, E. T. On the signature of a quadratic form 517

BURINGTON, R. S., and H. K. HOLT. Canonical forms of plane cubic curves under euclidean transformations 52

CRAIG, C. C. The frequency function of y/x 471

EISENHART, L. P. Contact transformations 211

EISENHART, L. P. Dynamical trajectories and geodesics 591

GARVER, R. Quartic equations with certain groups 47

GARVER, R. On the Brioschi normal quintic 607

HICKEY, D. M. The equilibrium point of Green's function for an annular region 373

HILLE, E. Note on the preceding paper by Mr. Peek 270

HOLLCROFT, T. R. Multiple lines with fixed coincident tangent planes . 451

HOLT, H. K. and R. S. BURINGTON. Canonical forms of plane cubic curves under euclidean transformations 52

JEFFERY, R. L. The sequence of functions which define a definite integral containing a parameter. 311

KNEBELMAN, M. S. Tensors with invariant components 339

LANE, E. P. On the fundamental transformation of surface 457

LATIMER, C. G. On forms which repeat under multiplication 61

LEHMER, D. H. On the multiple solutions of the Pell equation . . . 66

MCFARLAN, L. H. A parametric problem of the calculus of variations and its treatment as a problem of Lagrange 23

MACKIE, E. L. The Jacobi condition for a problem of Mayer with variable end-points 393

MITRA, S. C. On certain hitherto unsolved cases of the complex multiplication of elliptic functions 73

PAGE

MOORE, T. W. Extended results in elimination 92
MOORE, W. L. On the geometry of the Weddle surface 492
PEEK, R. L. Solution to a problem in diffusion employing a non-orthogonal sine series 265
PIEDVACHE, R. Sur la forme quadratique fondamentale d'un espace de Riemann en coordonnées normales 272
PIEDVACHE, R. Sur le développement de Clebsch-Gordan 281
PIERPONT, J. On the complex roots of a transcendental equation occurring in the electron theory 81
PORITSKY, H. On certain oscillation theorems 441
SHEFFER, I. M. Systems of linear differential equations of infinite order, with constant coefficients 250
SHEFFER, I. M. Linear differential equations of infinite order, with polynomial coefficients of degree one 345
SLOTNICK, M. M. Semi-parallel transformations of lines of curvature . . 188
SRIVASTAVA, P. L. On a class of Taylor's series 39
SRIVASTAVA, P. L. On a class of integral functions 384
THOMAS, J. M. Riquier's existence theorems 285
TRJITZINSKY, W. J. On quasi-analytic functions 526
VANDIVER, H. S. On a theorem of Kummer's concerning power characters of units in a cyclotomic field 487
VANDIVER, H. S. On the first case of Fermat's last theorem 552
VANDIVER, H. S. An algorithm for transforming Kummer criteria in connection with Fermat's last theorem 559
WARD, M. Certain expansions involving doubly infinite series 578
WEYL, H. Der Zusammenhang zwischen der symmetrischen und der linearen Gruppe 499
WHYBURN, W. M. On the fundamental existence theorems for differential systems 31
WILLIAMS, K. P. A generalization of the Cauchy-Riemann equations . . 206
WILSON, N. R. On finding ideals 411
WONG, B. C. A certain point-to-line transformation in space of four dimensions 547

ERRATA

VOLUME 29, PAGE 76. Omit the Corollary.

ON CUBIC FIELDS*.

By A. Arwin.

I have in this short sketch set myself the task of investigating two problems. The first of these concerns the "außerwesentlichen" primefactors for any given cubic and also higher irrationality. They are, as is known†, characterized by the fact that with a given equation $F(x) = 0$

$$F(x) \equiv \prod(x) + p \cdot M(x) \tag{1}$$

where $\prod(x)$ is the product of all irreducible factors of $F(x)$ (mod. p) and for some rational b

$$\prod(b) \equiv 0, \qquad \prod'(b) \equiv 0, \qquad M(b) \equiv 0 \pmod{p}.$$

The general method for determining the ideal primefactors fails for such primes. We shall see how, by modification, it can be used in this case also. The other task is to find any systematic way to decide when two ideals are equivalent, and to construct units. This is done by the construction of chains. It will be seen that each equivalence from the chain gives rise to an equivalence of ideals. The converse theorem is not necessarily true. It is indeed connected with some difficulty to give a method of forming periodic developments with higher irrationalities than quadratic, and therefore it will be necessary to leave the domain $K(1)$, where only the quadratic irrationnalities are periodic. It is however possible to form a continued set of inequalities of approximation in three dimensions and construct chains with coefficients from the cubic field $K(\omega)$ itself, which really are periodic or have, as we shall say, periodic convergence.

As to the former problem we start from the fundamental equation

$$F(x) = x^3 + A_1 x^2 + A_2 x + A_3 = 0, \tag{2}$$

and have to treat the cases when $F(x) \equiv 0$ (mod. p) has a double or a triple root. In the first case

$$F(x) \equiv (x-a)(x-b)^2 + p^i \cdot M(x),$$

* Received March 19, 1928.

† Bachmann, P.: Allgemeine Arithmetik der Zahlenkörper, p. 277.

where for example $M(b) \equiv 0\,(p^m)$. By substituting $x-b=y$ we have the equation

$$y^3+B_1\,y^2+p^i\,B_2\,y+p^{i+m}\,B_3 = 0$$

where $B_i \not\equiv 0\,(p)$. Multiplying by p^{2m-i}/y^3 we have

$$p^{2m-i}+p^{m+i}\,B_1\,(p^m/y)+B_2\,(p^m/y)^2+B_3\,(p^m/y)^3 = 0.$$

Hence, if δ is a root of (2), $p^m/\delta-b$ is an algebraic integer relative to p and, since

$$F(x)/(x-b) = (x-a)\,(x-b)+p^i\,(e_0\,x+e_1)+p^i\,(p^m/(x-b))\,(d_0\,x+d_1),$$

we infer that $(\delta-a)\,(\delta-b)/p^i$ is an algebraic integer. Assuming $m=i$, we transform

$$p^m\,z = y^2+B_1\,y+p^m\,B_2$$

and have, since

$$z\,y = -p^m\,B_3,$$

the equation

$$z^3-B_2\,z^2+B_1\,B_3\,z-p^m\,B_3^2 = 0. \tag{3}$$

Let, for example, $1, \eta, \varrho = (\eta^2+B_1\,\eta+B_2\,p^m)/p^m$ be a base of the field defined by (2); then we compute the discriminant $\Delta(K)$ of the field

$$\Delta(K) = \begin{vmatrix} 1, & \eta, & \varrho \\ 1, & \eta', & \varrho' \\ 1, & \eta'', & \varrho'' \end{vmatrix}^2 = \begin{vmatrix} 1, & p^m\,B_3/\varrho, & \varrho \\ 1, & p^m\,B_3/\varrho', & \varrho' \\ 1, & p^m\,B_3/\varrho'', & \varrho'' \end{vmatrix} = \left(\frac{p^m\,B_3}{\varrho\,\varrho'\,\varrho''}\right)^2 \begin{vmatrix} 1, & \varrho, & \varrho^2 \\ 1, & \varrho', & \varrho'^2 \\ 1, & \varrho'', & \varrho''^2 \end{vmatrix}^2$$

and

$$\Delta(\varrho) = B_3^2\,\Delta(K),$$

whence index of $\Delta(\varrho)$, $I(\varrho) = B_3^2$, is free from p, and (3) is regular. Without any restriction in the method we presume $i=1$ and have $m>1$. For $\varrho = (\delta-a)\,(\delta-b)/p$ it is

$$\Delta(\varrho) = \begin{vmatrix} 1, & \varrho, & \varrho^2 \\ 1, & \varrho', & \varrho'^2 \\ 1, & \varrho'', & \varrho''^2 \end{vmatrix}^2 = \frac{1}{p^b} \begin{vmatrix} 1, & (\delta-a)\,(\delta-b), & (\delta-a)^2\,(\delta-b)^2 \end{vmatrix}^2$$

and, putting $\Psi(x) = x^3+2\,A_1\,x^2+(A_1^2+A_2)\,x+A_1\,A_2-A_3$, we compute

$$I(\varrho)\cdot\Delta(K) = \Delta(\varrho) = \left[\frac{\psi(a+b)}{p^3}\right]^2\Delta(\delta) = \left[\frac{\psi(a+b)}{p^3}\right]^2 I(\delta)\,\Delta(K),$$

from which we see that here also the exponent can be reduced in index as in the following example.

$$x^3-x^2+129 = 0, \qquad -\Delta(\delta) = 7^2\cdot 3\cdot 43\cdot 71,$$
$$x^3-x^2+129 = (x-2)(x-3)^2+7\cdot[(x-3)x+7\cdot 3],$$

whence $I(\delta) = 7^2$. Transforming

$$7z = x^2-5x+6,$$

having $I = (\delta-2)(\delta-3)/7$ we compute the equation

$$z^3-2z^2-33z-57 = 0,$$
$$z^3-2z^2-33z-57 = (z-1)(z+2)(z-3)-7\cdot(4z+9),$$

which relatively 7 is regular and, since $\psi(5) = -49$, the prime 7 shall not be in index $I(\varrho)$. Next, let us assume that also the transformed equation in ϱ is irregular and therefore has p as factor in index. Transforming $\delta-b = y$, $\delta-a = z$ we find

$$\begin{aligned} y^3+B_1y^2+B_2py+B_3p^{m+1} &= 0,\\ z^3+C_1z^2+C_2z \quad +C_3p^{m_1} &= 0, \end{aligned} \tag{4}$$

where $m_1 \geqq 1$. Consider the ideals $[p;\ \delta-a]$, $[p;\ \delta-b]$. They are this time not both primeideals, and this is just the disadvantage in this case. The product

$$[p;\ \delta-a]\,[p;\ \delta-b] = [p]\left[p;\ \delta-a;\ \delta-b;\ \frac{(\delta-a)(\delta-b)}{p}\right] = [p] \tag{5}$$

shows that the product $[p;\ \delta-a]\,[p;\ \delta-b]$ contains all primeideals of p, and forming the norm

$$N[p;\ \delta-a]$$
$$\leqq [p^3;\ p^2\textstyle\sum(\delta-a);\ p(\delta-a)(\delta'-a);\ (\delta-a)(\delta'-a)(\delta''-a)] = [p],$$

since in (4) $C_i \not\equiv 0\,(p)$, and hence $[p;\ \delta-a]$ is a primeideal. As for $[p;\ \delta-b]$ its norm is p^2, whence $[p;\ \delta-b]$ is divided in two equal or different ideals. If the equation $\chi(u) = 0$ having ϱ as a root, is not regular, then we have in

$$N(\varrho) = \prod(\delta-a)(\delta-b)/p = B_3C_3p^{m+1+m_1-3}$$

$m+m_1>3$. Further, $\chi(u) = 0$ has in $u \equiv 0\,(p)$ one, two or three equal roots. In the first case $[p;\ \varrho]$ is also a primeideal, since $N[p;\ \varrho] = p$; the two ideals $[p;\ \varrho]$ and $[p;\ \delta-a]$ are not identical, since by means of

$$[p]\,[\delta-b;\ \varrho] = [\delta-b]\,[p;\ \delta-a]$$

the norm

$$N[\delta - b; \varrho] = -B_3 p^{m+1+1-3}$$

must for $m > 1$ contain a factor p. In the same way we infer from

$$[p][\delta - a; \varrho] = [\delta - a][p; \delta - b]$$

and

$$N[\delta - a; \varrho] = -C_3 p^{m_1+2-3}$$

that this first case is characterized by $m_1 = 1$. The other root of $\chi(u) \equiv 0$ is double $u \equiv k(p)$ and $N[p; \varrho - k] = p^2$ as before $N[p; \delta - b] = p^2$. As these two ideals are not identical, the "cutideal" $[p; \delta - b; \varrho - k]$ gives the third primeidealfactor of p, if the two already found, written in three elements, are $[p; \delta - a; \varrho - k]$ and $[p; \varrho; \delta - b]$. In the second case $u \equiv 0(p)$ a double rout we see that $N[p; \varrho] = p^2$, $N[p; \varrho - k] = p$ and, since therefore $[p; \delta - a]$ and $[p; \varrho]$ are not relatively prime, $m_1 > 1$. Hence the three idealfactors of p, determined as "cutideals" are this time $[p; \delta - a; \varrho]$, $[p; \delta - b; \varrho]$, $[p; \varrho - k; \delta - b]$. There remains then the last case with three equal roots, which is impossible. We have namely ϱ^2/p an integer and from

$$[p; \varrho][p; \varrho] = [p][p; \varrho; \varrho^2/p]$$

we conclude that $N[p; \varrho] = p^2$, $N[p; \varrho; \varrho^2/p] = p$ and hence $[p; \varrho; \varrho^2/p] = P_1$ is a primeideal. Further we obtain $p = P_1 P_2^2$, $[p; \varrho] = P_1 P_2$, but as $[p; \delta - a]$ and $[p; \varrho; \varrho^2/p]$ are not identical, as before, $[p; \delta - a] = P_2$ and therefore $[p; \delta - b] = P_1 P_2 = [p; \varrho]$, leading to a contradiction. Hence the case of two equal roots of (2) is disposed of.

Assuming three equal roots of (2) in, for example, $x \equiv 0(p)$, and assuming that the transformed equation $\chi(u) = 0$ ($\varrho = \delta^2/p$ since $a = b = 0$) has two equal roots, p must contain two primeideals and hence from

$$[p; \delta][p; \delta] = [p][p; \delta; \delta^2/p]$$

we once more conclude $p = P_1 P_2^2$, $[p; \delta] = P_1 P_2$, $[p; \delta; \varrho] = P_1$. Hence $[p; \varrho]$ is primeideal and $N[p; \varrho - k] = p^2$, $[p; \varrho - k] = P_2^2$ that is $[p; \delta; \varrho - k] \equiv P_2$. There remains only the last case with three equal roots in $\chi(u) = 0$. As now $[p; \varrho]$ is a primeideal, $\chi(u) = 0$ is regular relative to p and $(p) = [p; \varrho]^3$.

Example. $F(x) = x^3 - x^2 - 2x - 8 = 0$, $-\Delta(\delta) = 2^2 \cdot 503$.

$$x^3 - x^2 - 2x - 8 = x^2(x - 1) - 2(x + 4),$$

hence $\varrho = \delta(\delta-1)/2$ and the equation in u

$$u^3-2u^2+3u-10 \equiv u(u-1)^2+2(u-5)$$

and therefore the three primeideals in (2): $[2;\ \delta;\ \varrho]$, $[2;\ \delta;\ \varrho-1]$, $[2;\ \delta-1;\ \varrho-1]$.

Example. $F(x) = x^3+6x+8$, $-\Delta(\delta) = 2^5\cdot 3^4$, where $\varrho = \delta^2/2$ and the equation in u

$$u^3+6u^2+9u-8 \equiv u(u+3)^2-8\cdot 1$$

whence $(2) = [2;\ \delta;\ \varrho]\ [2;\ \delta;\ \varrho+1]^2$.

Example. $F(x) = x^3+4x-4$, $\Delta(\delta) = 4^2\cdot 43$, with $\varrho = \delta^2/2$ and

$$u^3+4u^2+4u-2 \equiv u^3+2(2u^2+2u-1),$$

which is regular, whence $(2) = [2;\ \varrho]^3 = (\varrho)^3$.

Higher irrationalities than cubic are treated in the same way and with advantage in the concrete examples, but it will be necessary to use cuts of more than two ideals.* This method can also be used, if it is desired to determine a base of the algebraic field.

We shall now consider the problem of deciding when two ideals are equivalent. In quadratic domains this problem is fully solved by means of periodic chains. In order to obtain a closed formation of chains in a cubic field it is necessary, as already indicated, to use the coefficients in the cubic field itself. We shall first prove that each equivalence of fractions must give an equivalence of ideals. Let therefore $[R;\ \psi(\omega)]$, where $\psi(\omega) = \alpha\omega^2+\beta\omega+\gamma$ with α, β and γ in $K(1)$, denote any ideal, ϱ_1 any algebraic integer, and let the chain be

$$\frac{\alpha\omega^2+\beta\omega+\gamma}{R} = \varrho_1 \pm \frac{1}{\dfrac{\alpha_1\omega^2+\beta_1\omega+\gamma_1}{R_1}} = \varrho_1 \pm \frac{1}{\dfrac{\psi_1(\omega)}{R_1}}.$$

We form

$$\begin{aligned}\psi_1(\omega)[R;\ \psi(\omega)] &= [R\psi_1(\omega);\ \psi(\omega)\psi_1(\omega)]\\ &= R[\psi_1(\omega);\ \varrho_1\psi_1(\omega)\pm R_1] = R[R_1;\ \psi_1(\omega)],\end{aligned}$$

from which we see the equivalence of the ideals $[R;\ \psi(\omega)]$ and $[R_1;\ \psi_1(\omega)]$. Let once more

$$F(x) = x^3+A_1x^2+A_2x+A_3 = 0 \tag{2}$$

determine a cubic field with two conjugated real roots. Let

* See, Berwick, W. E. H.: Integral Basis, 1927.

$$\beta_n = r_n^{(1)}\omega^2 + r_n^{(2)}\omega + r_n^{(3)}, \qquad \alpha_n = m_n^{(1)}\omega^2 + m_n^{(2)}\omega + m_n^{(3)},$$

$$D = \begin{vmatrix} 1, & \omega, & \omega^2 \\ 1, & \omega', & \omega'^2 \\ 1, & \omega'', & \omega''_2 \end{vmatrix}^2$$

and call the conjugate quantities ω', ω'', β_n', β_n'' etc. Set generally $\omega_1 = (a^{(1)}\omega^2 + b^{(1)}\omega + c^{(1)})/R_1$; $Q_{n+1} = Q_{n-1} - \varrho_n Q_n = \pm(\alpha_n\omega_1 - \beta_n)$, $Q_0 = \omega_1$, $Q_1 = 1$, $\beta_{n+1} = \varrho_n\beta_n + \beta_{n-1}$ and quite formally

$$\frac{a^{(n)}\omega^2 + b^{(n)}\omega + c^{(n)}}{R_n} = \omega_n = \frac{\alpha_{n-1}\omega_1 - \beta_{n-1}}{-\alpha_n\omega_1 + \beta_n};$$

then

$$\omega_1 = \frac{\beta_n\omega_n + \beta_{n-1}}{\alpha_n\omega_n + \alpha_{n-1}}, \qquad \beta_n\alpha_{n-1} - \beta_{n-1}\alpha_n = (-1)^{n-1}.$$

We start from a known theorem concerning approximation in integers,* which specialized to three variables says: The inequalities

$$\begin{aligned} |\alpha_0 x + \alpha_1 y + \alpha_2 z| &\leqq M_1 \\ |\beta_0 x + \beta_1 y + \beta_2 z| &\leqq M_2 \\ |\gamma_0 x + \gamma_1 y + \gamma_2 z| &\leqq M_3 \end{aligned} \tag{6}$$

has always integer solutions x, y, z when $M_1 M_2 M_3 \geqq |d|$, where d is the determinant of the three forms. The theorem is true for β_i and δ_i conjugated complex. This says that within ore on the sides of the parallelepiped, determined by the inequalities (6), there exist always lattice-points if $M_1 M_2 M_3 \geqq |d|$.

Let us then form the following system of inequalities

$$\begin{aligned} |Q_{n-1} - Q_n(x\omega^2 + y\omega + z)| &\leqq C_n^{(1)}|Q_n| \\ |Q'_{n-1} - Q'_n(x\omega'^2 + y\omega' + z)| &\leqq C_n^{(2)}|Q'_n| \\ |Q''_{n-1} - Q''_n(x\omega''^2 + y\omega'' + z)| &\leqq C_n^{(3)}|Q''_n| \end{aligned} \tag{6'}$$

with the determinant $\Delta = |Q_n Q'_n Q''_n|\,|\sqrt{D}|$. The parallelepiped F, as we briefly say, being on both sides divided by Q_n, Q'_n, Q''_n respectively, does not here belong to the origin but is transformed out in the space. The theorem is yet true if the new middlepoint is an integer-point; if not, there must lie at least one lattice-point within a sphere of radius 1, and it is always possible to dilate F by means of the constants $C_n^{(i)}$ so as to obtain further lattice-points on and within F. Let us take the first lattice-point x_n, y_n, z_n on F and set $\varrho_n = x_n\omega^2 + y_n\omega + z_n$, whence

* Hecke, E.: Theorie der algebr. Zahlen, p. 116.

$$(6'') \qquad \begin{aligned} |Q_{n+1}| &= |Q_{n-1} - \varrho_n Q_n| \leqq C_n^{(1)} |Q_n| \\ |Q'_{n+1}| &= |Q'_{n-1} - \varrho'_n Q'_n| \leqq C_n^{(2)} |Q'_n| \\ |Q''_{n+1}| &= |Q''_{n-1} - \varrho''_n Q''_n| \leqq C_n^{(3)} |Q''_n| . \end{aligned}$$

The equations in x, y, z $Q_{n+1}(xyz) = 0$, $Q'_{n+1}(xyz) = 0$, $Q''_{n+1}(xyz) = 0$ represent the equations of the three diametral planes in F and the three quantities in (6'')

$$|Q_{n+1}/Q_n| \cdot 1/s, \qquad |Q'_{n+1}/Q'_n| \cdot 1/s, \qquad |Q''_{n+1}/Q''_n| \cdot 1/s,$$

where $s = \sqrt{\omega^4 + \omega^2 + 1}$, are the expressions of the three distances to the diametral planes above. If this procedure (6') with, for instance, the least $C_n^{(i)}$ gives us expressions in Q_n, Q'_n, Q''_n which are limited, the periodically convergent formation is deduced, as we shall see. Let us therefore assume, that at least one of them might increase for $n \to \infty$; then and only then we are constructing sets of Q_n which will be done as follows. Suppose therefore that for a given n

$$|Q_n| > |Q'_n| > |Q''_n| .$$

Take in (6') $C_n^{(1)} = C_n^{(2)} = 1$, $C_n^{(3)}$ arbitrarily large in order to bring suitable lattice-points, within F. Referring to (6'') we have

$$(7) \qquad |Q_{n+1}| \leqq |Q_n|, \qquad |Q'_{n+1}| \leqq |Q'_n|, \qquad |Q''_{n+1}| \leqq |Q''_n| C_n^{(3)}$$

with the least possible $C_n^{(3)}$. If the order remains $|Q_{n+1}| > |Q'_{n+1}| > |Q''_{n+1}|$, we repeat the process. If Q_n, Q'_n, Q''_n remain limited, $\eta_n^{(1)}$ in

$$\begin{aligned} R_1 Q_n &= a_n \omega^2 + b_n \omega + c_n = \eta_n^{(1)}, \\ R_1 Q'_n &= a_n \omega'^2 + b_n \omega' + c_n = \eta_n^{(2)}, \\ R_1 Q''_n &= a_n \omega''^2 + b_n \omega'' + c_n = \eta_n^{(3)}, \end{aligned}$$

where a_n, b_n, c_n are rational integers, are limited for all n. Subtracting these relations two by two we have, since $\omega + \omega' + \omega'' = -A_1$,

$$-\omega - A_1 + \frac{b_n}{a_n} = \frac{\eta_n^{(2)} - \eta_n^{(3)}}{a_n(\omega' - \omega'')}$$

and likewise for ω' and ω''. This is impossible when $|a_n| \to \infty$, so that a_n, b_n, c_n are limited, the formation is periodic and at least one $\eta_n^{(1)} \not\to 0$. Thus it is also necessary that a change in the order in (7) shall follow, and we now form the next order of approximation with, say, $C_n^{(2)} = C_n^{(3)} = 1$ and $C_n^{(1)}$ free etc. By this proceeding it is excluded that

any one of the elements $|Q_n|$, $|Q'_n|$, $|Q''_n|$ shall remain the greatest after a certain index n, whence the two conjugates $\to\infty$ with $|Q_n|$. Moreover it is possible to find by dilation lattice-points such that Q_n, as entering in the expression of the distance to the diametral planes, will take a previously given sign + or —. Supposing $|\alpha_n\omega_1-\beta_n|=|Q_n|\to\infty$ we are able for $n>N$ to continue with the sign + for all Q_n. If now Q_n is limited, the chain is periodic, as already proved. If $Q_n\to\infty$ we infer from $Q_n=\pm(\alpha_n\omega_1-\beta_n)$ that $|\alpha_n|$ or $|\beta_n|$ or both $\to\infty$. From the proved existence of an infinitely repeated change of order in the size of Q_n we infer the existence of an infinite set of inequalities $Q_{n_1}<Q_{n_1-1}$, which we say belong to an n_1-set. Then we have

$$0<Q_{n_1+1}=Q_{n_1-1}-\varrho_{n_1}Q_{n_1}<C^{(1)}_{n_1}Q_{n_1},\qquad 0<\omega_{n_1}-\varrho_{n_1}<C^{(1)}_{n_1}$$

and

$$\omega_{n_1}=\varrho_{n_1}+\frac{1}{\omega_{n_1+1}},\qquad \omega_{n_1}=Q_{n_1-1}/Q_{n_1}>\frac{1}{C^{(1)}_{n_1-1}}.$$

If in the general formula $\alpha_{n_1+1}=\varrho_{n_1}\alpha_{n_1}+\alpha_{n_1-1}$ ($n-i$, $i=1, 0, -1$, three successive integers) the α_{n_1} and α_{n_1+1} are of the same sign, this ϱ_{n_1} will be accepted. If not, let us for instance have $\alpha_{n_1}>0$, $\varrho_{n_1}>0$ but $\alpha_{n_1+1}<0$, and therefore $-|\alpha_{n_1+1}|=\varrho_{n_1}\alpha_{n_1}-|\alpha_{n_1-1}|$ and make the approximation $\omega_{n_1-1}-\varrho_{n_1-1}<C^{(1)}_{n_1-1}$ ($=1$) with decreasing $C^{(1)}_{n_1-1}$. Thus, ω_{n_1} is arbitrarily increased and, since the following approximation $\omega_{n_1}-\varrho_{n_1}<C^{(1)}_{n_1}$ is made by our general rule, by which $C^{(1)}_n=1$ or a constant dependent only on the discriminant D of the field, ϱ_{n_1} will be increased as ω_{n_1} and positive. We remark incidentally that by decreasing $C^{(1)}_{n_1-1}$ the Q_{n_1} is diminished. Further, ϱ_{n_1-1} is, as we see, not affected in its order of size, for it is varied with a quantity η_{n_1-1}, $|\eta_{n_1-1}|<1$ and hence $\alpha_{n_1}=\varrho_{n_1-1}\alpha_{n_1-1}+\alpha_{n_1-2}$ with $\eta_{n_1-1}\alpha_{n_1-1}$. If now α_{n_1} is changing in sign, we have already $\alpha_{n_1}/\alpha_{n_1+1}>0$; if not, it must result from continually decreasing $C^{(1)}_{n_1-1}$ that $\varrho_{n_1}\alpha_{n_1}>|\alpha_{n_1-1}|$, and α_{n_1+1} is compelled to take the same sign as α_{n_1}. Reasoning in the same manner, if $\alpha_{n_1}<0$ and $\alpha_{n_1+1}=-\varrho_{n_1}|\alpha_{n_1}|+\alpha_{n_1-1}$, we are permitted to presume that $\alpha_{n_1}/\alpha_{n_1+1}>0$ in the new n_1-set. The existence of infinite sets of lattice-points for our purpose is certain, since there already exists an infinite set of lattice-points on or in the neighborhood of each vector in the space.* After the above construction it follows that either Q_n is limited or $Q_n\to\infty$. In the latter case we repeat the same construction and shall finally have either Q_n limited or an infinite n_1-set. Let us then assume this latter possibility. From

* Perron, O.: Irrationalzahlen, p. 153.

$$\omega_1 = \frac{\beta_{n_1+1}\,\omega_{n_1+1} + \beta_{n_1}}{\alpha_{n_1+1}\,\omega_{n_1+1} + \alpha_{n_1}}, \qquad \beta_{n_1+1}\,\alpha_{n_1} - \beta_{n_1}\,\alpha_{n_1+1} = (-1)^{n_1}$$

and

$$\pm Q_{n_1+1} = \alpha_{n_1+1}\,\omega_1 - \beta_{n_1+1}$$
$$= \frac{(-1)^{n_1-1}}{\alpha_{n_1+1}\,(\omega_{n_1+1} + \alpha_{n_1}/\alpha_{n_1+1})} = \frac{(-1)^{n_1-1}\,\omega_1}{\beta_{n_1+1}\,(\omega_{n_1+1} + \beta_{n_1}/\beta_{n_1+1})}$$

we deduce, since $\omega_{n_1+1} > 0$, $\alpha_{n_1}/\alpha_{n_1+1} > 0$, $|\alpha_{n_1+1}| \to \infty$, the contradiction $Q_n \to 0$. Consequently it is possible by means of the approximation (6′) to form in three lines the periodically convergent chains. In the case of a real root ω, and ω', ω'' conjugate imaginary we replace the two last equations in (6′) by $1/2\,[Q'_n(xyz) + Q''_n(xyz)]$, $1/2\,i\,[Q'_n(xyz) - Q''_n(xyz)]$ and approximate as before.

It is remarkable that epressions for units are deduced from the periodic chains (6′), almost as in the quadratic field. From the equivalence

$$\omega_\tau(\omega) = \frac{l_\tau(\omega)\,\omega_\tau(\omega) \pm l_{\tau-1}(\omega)}{s_\tau(\omega)\,\omega_\tau(\omega) \pm s_{\tau-1}(\omega)}, \qquad l_\tau(\omega)\,s_{\tau-1}(\omega) - l_{\tau-1}(\omega)\,s_\tau(\omega) = (-1)^{\tau-1}$$

we compute

$$s_\tau\,\omega_\tau^2 - (l_\tau \mp s_{\tau-1})\,\omega_\tau \mp l_{\tau-1} = 0,$$

$$\omega_\tau = \frac{l_\tau \mp s_{\tau-1}}{2\,s_\tau} \pm \frac{\sqrt{(l_\tau \mp s_{\tau-1})^2 \pm 4\,s_\tau\,l_{\tau-1}}}{2\,s_\tau} \qquad \left(= \frac{\sqrt{D_\tau(\omega)}}{2\,s_\tau}\right);$$

hence

$$\pm\sqrt{D_\tau(\omega)} = 2\,s_\tau\,\omega_\tau - (l_\tau \mp s_{\tau-1}).$$

For $X_\tau = l_\tau \pm s_{\tau-1}$ we find

$$X_\tau^2 - D_\tau = (l_\tau \pm s_{\tau-1})^2 - (l_\tau \mp s_{\tau-1})^2 \mp 4\,s_\tau\,l_{\tau-1}$$
$$= \pm 4\,(l_\tau\,s_{\tau-1} - l_{\tau-1}\,s_\tau) = \pm 4;$$

hence

$$(X_\tau - \sqrt{D_\tau})\,(X_\tau + \sqrt{D_\tau}) = \pm 4, \tag{8}$$

and therefore always a unit, when the ideals in 2 are not principal or (2) itself is primeideal; otherwise it is not sure but possible that 2 can be divided off, and a unit must perhaps be computed with help of more than one expression (8).

We shall illustrate with an example.

Example. $x^3 - 7x + 3 = 0$, $\Delta = 1129$, $\delta \sim 0{,}44$, $\delta' \sim 2{,}40$, $\delta'' \sim -2{,}85$, $3^{1}./3^{3}.\sqrt{1129} - 7{,}5$, $[2] = [2]$, $[3] = [3; \delta]\,[3; \delta - 1]\,[3; \delta + 1]\,[5] = [5]$, $[7] = [7]$.

Proceeding from the initial values $Q_0 = \delta_1 = \delta$, $Q'_0 = \delta_2 = \delta'$, $Q''_0 = \delta_3 = \delta''$, $Q_1 = Q'_1 = Q''_1 = 1$ we have to solve the system

$$|Q_0 - Q_1(x\delta^2 + y\delta + z)| = |\delta - (x\delta^2 + y\delta + z)| < C|Q_1|,$$
$$|Q_0' - Q_1'(x\delta'^2 + y\delta' + z)| = |\delta' - (x\delta'^2 + y\delta' + z)| < C|Q_1'|,$$
$$|Q_0'' - Q_1''(x\delta''^2 + y\delta'' + z)| = |\delta'' - (x\delta''^2 + y\delta'' + z)| < C|Q_1''|,$$

where as a first trial we take $C^3 = \sqrt{1129}$, $C = 3{,}226$ in virtue of the general theorem. The middlepoint of F is determined by solving the equations in x, y, z, and it leaves as result $x = 0$, $y = 1$, $z = 0$. Starting from this point we find as a suitable lattice-point $x = 0$, $y = 0$, $z = 1$, i. e., $\varrho_1 = 1$, and hence

$$Q_2 = \delta - 1 = -0{,}56, \quad Q_2' = \delta' - 1 = 1{,}40, \quad Q_2'' = \delta'' - 1 = -3{,}85.$$

Next we approximate in

$$|Q_1 - Q_2(x\delta^2 + y\delta + z)| < C|Q_2|, \quad \text{etc.},$$

compute the new middlepoint $x = 1/3$, $y = 1/3$, $z = -2$ in F, use the lattice-point $x = 0$, $y = 0$, $z = -2$, $\varrho_2 = -2$, and form $Q_3 = Q_1 - \varrho_2 Q_2 = 1 + 2(\delta - 1) = 2\delta - 1 = -0{,}12$, $Q_3' = 2\delta' - 1 = 3{,}80$, $Q_3'' = 2\delta'' - 1 = -6{,}70$. From the next approximation

$$|Q_2 - Q_3(x\delta^2 + y\delta + z)| < C|Q_3|, \quad \text{etc.},$$

we have the middlepoint $x = -2/3$, $y = -1/3$, $z = 5$ and an integer point $x = 0$, $y = 1$, $z = 1$, $\varrho_3 = \delta - 1$, since $1/Q_3\{Q_2 - Q_3(\delta + 1)\} \sim 3{,}29$ and the conjugates $-3{,}04$, $2{,}38$ respectively; hence $Q_4 = -2\delta^2$, etc. In the same way we compute in order $\varrho_4 = 1$, $Q_5 = 2\delta^2 + 2\delta - 1$; $\varrho_5 = -1$, $Q_6 = 2\delta - 1$; $\varrho_6 = \delta + 2$, $Q_7 = -\delta + 1$; $\varrho_7 = -\delta$, $Q_8 = -\delta^2 + 3\delta - 1$ and, since $N(Q_8) = -1$, the chain is in this case closed. The successive links in the chain are

$$\delta = 1 + \cfrac{1}{\cfrac{1}{\delta - 1}} = 1 + \cfrac{1}{\cfrac{\delta^2 + \delta - 6}{3}},$$

$$\frac{1}{\delta - 1} = -2 + \cfrac{1}{\cfrac{\delta - 1}{2\delta - 1}} = -2 + \cfrac{1}{\cfrac{-2\delta^2 - \delta + 15}{3}},$$

$$\frac{\delta - 1}{2\delta - 1} = \delta + 1 + \cfrac{1}{\cfrac{2\delta - 1}{-2\delta^2}} = \delta + 1 + \cfrac{1}{\cfrac{\delta^2 + 3\delta - 7}{-18}},$$

$$\frac{2\delta - 1}{-2\delta^2} = 1 + \cfrac{1}{\cfrac{-2\delta^2}{2\delta^2 + 2\delta - 1}} = 1 + \cfrac{1}{\cfrac{2\delta^2 + 8\delta - 10}{39}},$$

$$\frac{-2\delta^2}{2\delta^2+2\delta-1} = -1+\frac{1}{\frac{2\delta^2+2\delta-1}{2\delta-1}} = -1+\frac{1}{\frac{2\delta^2+4\delta-9}{3}},$$

$$\frac{2\delta^2+2\delta-1}{2\delta-1} = \delta+2+\frac{1}{\frac{2\delta-1}{-\delta+1}} = \delta+2+\frac{1}{\frac{\delta^2+\delta}{-3}},$$

$$\frac{2\delta-1}{-\delta+1} = -\delta+\frac{1}{\frac{-\delta+1}{-\delta^2+3\delta-1}} = -\delta+\frac{1}{\frac{-\delta^2-\delta+5}{1}}.$$

Hence in virtue of

$$\delta^2-4+\frac{1}{|-2}+\frac{1}{|\delta+1}+\frac{1}{|1}+\frac{1}{|-1}+\frac{1}{|\delta+2} = \frac{\delta^2-4}{1}$$

and

$$\delta^2-4+\frac{1}{|-2}+\frac{1}{|\delta+1}+\frac{1}{|1}+\frac{1}{|-1}+\frac{1}{|\delta+2}+\frac{1}{|-\delta} = \frac{2\delta^2-3\delta-6}{2-\delta},$$

there follows for $\omega = \delta^2+\delta-5$ the equivalence

$$\omega = \frac{(2\delta^2-3\delta-6)\,\omega-(\delta^2-4)}{(2-\delta)\,\omega-1}$$

and therefore

$$\sqrt{D_\tau(\delta)} = 2(2-\delta)\,\omega-(2\delta^2-3\delta-5) = 3\delta-9,$$
$$X_\tau(\delta) = t_\tau(\delta)+s_{\tau-1}(\delta) = 2\delta^2-3\delta-7.$$

Hence

$$X_\tau+\sqrt{D_\tau} = 2(\delta^2-8),$$
$$X_\tau-\sqrt{D_\tau} = 2(\delta^2-3\delta+1),$$
$$(\delta^2-8)(\delta^2-3\delta+1) = +1.$$

We have already found this unit. In order distinguish ideal equivalences we have by our theorem above

$$[3;\,\delta^2+\delta] = [3;\,\delta]\,[3;\,\delta+1]\sim 1,$$

whence for the remaining ideal $[3;\,\delta-1]\sim 1$, and likewise from $[3;\,2\delta^2+\delta]$ $= [3;\,\delta]\,[3;\,\delta-1]\sim 1$ there follows $[3;\,\delta]\sim 1$, and finally $[3;\,\delta+1]\sim 1$. Further from the second row of the chain we deduce $1^3-7\cdot 1\cdot 2^2+3\cdot 2^3 = -3$, and from the easily found solution $2^3-7\cdot 2\cdot 1^2+3\cdot 1^3 = -3$ belonging to the same congruence root we deduce according to a general rule the unit $-\frac{1}{3}(1-2\delta)\,(2-\delta')\,(2-\delta'') = \delta^2+2\delta-1$ and as the conjugate $\frac{1}{3}(2-\delta)(1-2\delta')(1-2\delta'') = 2\delta^2+\delta-14$ with $(\delta^2+2\delta-1)(2\delta^2+\delta-14) = -1$, and the two units $\varepsilon_1 = \delta^2-3\delta+1$, $\varepsilon_2 = \delta^2+2\delta-1$ leave two independent units in $K(\delta)$.

MALMÖ, SWEDEN.

THE IMPOSSIBILITY OF A SEPARATION OF TYPES OF LINEAR ODD DIVISORS OF BINARY QUADRATIC FORMS.*

By Ralph G. Archibald.

1. **Introduction.** It is the object of this paper to develop theorems on quadratic congruences in two unknowns and to utilize these results in proving the impossibility of a separation of types of linear odd divisors of binary quadratic forms.

In A. M. Legendre's *Théorie des nombres*† and in his *Zahlentheorie* (a German translation, 1893, by H. Maser of the third French edition) tables‡ are given of the quadratic divisors and of the linear odd divisors of a form t^2+au^2 for certain integral values of a. Moreover, in many cases the forms of the linear odd divisors are paired off in sets with a single quadratic divisor of t^2+au^2. The present paper is concerned with the remaining cases in Tables IV–VII in which the forms of the linear odd divisors of t^2+au^2 are paired off in sets with more than one quadratic divisor. In the construction of the tables of odd divisors of t^2+au^2, only those odd divisors relatively prime to a are considered.

Consider a to be a positive integer. Let $4ax+\alpha$ be one of the forms of the linear odd divisors of a given quadratic form t^2+au^2. Let y and z as well as t and u be relatively prime. Then, when $pr-q^2=a$, every odd divisor of t^2+au^2 contained in the arithmetic progression $4ax+\alpha$ is represented by a quadratic divisor $py^2+2qyz+rz^2$ of t^2+au^2. Suppose the linear odd divisor $4ax+\alpha$ corresponds to two or more quadratic divisors of t^2+au^2. Since, for any positive integer n as modulus, every integer x is congruent to one of $0, 1, 2, \cdots, (n-1)$, the question is raised whether it is possible to find a positive integer n such that all the divisors of t^2+au^2 contained in each of

* Received February 7, 1928.

† First edition, 1798; second edition, 1808; third edition, 1830; fourth edition, 1900. For errata in these tables see Bull. Amer. Math. Soc., vol. 8, pp. 401–2.

‡ Merely the linear forms of the odd divisors of the quadratic forms t^2+au^2 are given by P. L. Tschebyscheff in his *Theorie der Congruenzen* (Elemente der Zahlentheorie), 1889. These tables of Tschebyscheff are reproduced in E. Cahen's *Éléments de la théorie des nombres*, 1900.

$$
\begin{aligned}
&4any+\alpha,\\
&4any+(4a+\alpha),\\
&4any+(8a+\alpha),\\
&\qquad\cdot\qquad\cdot\\
&\qquad\cdot\qquad\cdot\\
&4any+[(n-2)4a+\alpha],\\
&4any+[(n-1)4a+\alpha],
\end{aligned}
\tag{1}
$$

are represented, respectively, by one and only one of the quadratic divisors of t^2+au^2. Let d be an integer, positive or zero, such that $d4a+\alpha$ is represented by a quadratic divisor of t^2+au^2. Now, if, for every positive integer n, $4any+d4a+\alpha$ contains a number represented by another (distinct) quadratic divisor (to which $4ax+\alpha$ corresponds), then such a separation as indicated in (1) is impossible. By the general method employed in §§ 3, 4, the above query is answered in the negative for Legendre's Tables IV–VII.

2. Theorems on quadratic congruences in two unknowns.

THEOREM 1. *If p is an odd prime dividing neither a nor b and if k is any integer, $ax^2+by^2\equiv k \pmod p$ has integral solutions.**

COROLLARY. *If no one of a, b, k is divisible by the odd prime p, $ax^2+by^2\equiv k \pmod p$ has integral solutions with x and y not both divisible by p.*

THEOREM 2. *Let p be an odd prime, k any integer, a and b integers both relatively prime to p, and c an integer satisfying the congruence $ac\equiv 1 \pmod p$. If $-bc$ is a quadratic residue of p when k is divisible by p, then $ax^2+by^2\equiv k \pmod p$ has integral solutions with x and y not both divisible by p.*

Since a is relatively prime to p, c can always be found. If k is relatively prime to p, the theorem follows by the above Corollary. If $k\equiv 0 \pmod p$, we obtain $x^2\equiv -bcy^2 \pmod p$ on multiplying the given quadratic congruence by c. If $\zeta^2\equiv -bc \pmod p$, ζ is relatively prime to p; hence we get $x^2\equiv(\zeta y)^2 \pmod p$. Taking for y any arbitrary integer relatively prime to p, we have solutions with neither x nor y divisible by p.

THEOREM 3. *Let n be an arbitrary positive integer. Then, under the hypothesis of Theorem 2, $ax^2+by^2\equiv k \pmod{p^n}$ has integral solutions with x and y not both divisible by p.*

By Theorem 2 there are solutions when $n=1$ such that x and y are not both divisible by p. To prove the theorem by induction on n, let

* This follows from a theorem given by Lebesgue, Journ. Math., 2 (1837), pp. 269–270.

$ax^2+by^2 \equiv k \pmod{p^m}$ have a solution ξ, η such that ξ and η are not both divisible by p. Where q is an integer, write

$$a\xi^2+b\eta^2 = k+p^m q, \qquad x = \xi+p^m X, \qquad y = \eta+p^m Y.$$

Then,* for $m \geqq 1$

$$\begin{aligned} ax^2+by^2 &= k+p^m q+2p^m(a\xi X+b\eta Y)+p^{2m}(aX^2+bY^2) \\ &\equiv k+p^m q+2p^m L \pmod{p^{m+1}}, \end{aligned}$$

where $L = a\xi X+b\eta Y$. X and Y can be chosen so that $q+2L \equiv 0 \pmod{p}$. For example, if ξ is not divisible by p, take $Y = 0$. Then $ax^2+by^2 \equiv k \pmod{p^{m+1}}$ has a solution with x and y not both divisible by p. The induction is therefore complete.

COROLLARY. *If p is an odd prime not dividing b, and if $-b$ is a quadratic residue of p when k is divisible by p, then for arbitrary k and n, $x^2+by^2 \equiv k \pmod{p^n}$ has integral solutions with x and y not both divisible by p.*

THEOREM 4. *Let p be an odd prime, a any integer not divisible by p, and b any integer different from zero and divisible by p; moreover, let k be any integer which is a multiple of b in case k is divisible by p, and let c be an integer satisfying the congruence $ac \equiv 1 \pmod{p}$. If ck is a quadratic residue of p when k is not divisible by p, and if $\frac{k}{b}$ is a quadratic residue of p^n when k is divisible by b, then*

$$ax^2+by^2 \equiv k \pmod{p^n} \tag{2}$$

has integral solutions when n is any positive integer.

If k is divisible by p, $k = br$ and the theorem follows at once. On the other hand, let k be relatively prime to p. For $n = 1$ the quadratic congruence on multiplication by c becomes $x^2+bcy^2 \equiv ck \pmod{p}$. Since ck is a quadratic residue of p and b is divisible by p, $x^2+bcy^2 \equiv ck \pmod{p}$ has, when y is an *arbitrary* integer, an integral solution with x not divisible by p. Hence $ax^2+by^2 \equiv k \pmod{p^n}$ has solutions when $n = 1$ such that x and y are both relatively prime to p. To prove the theorem by induction on n, let $ax^2+by^2 \equiv k \pmod{p^m}$ have a solution ξ, η such that ξ and η are both relatively prime to p. Where q is an integer, write

$$a\xi^2+b\eta^2 = k+p^m q, \qquad x = \xi+p^m X, \qquad y = \eta+p^m Y.$$

* The method of proof used here is that employed by Professor L. E. Dickson in proving a similar theorem in his paper "Ternary Quadratic Forms and Congruences", Annals of Math., 28 (1927), pp. 333-341.

Then, for $m \geq 1$

$$ax^2 + by^2 \equiv k + p^m q + 2p^m a\xi X \pmod{p^{m+1}},$$

since $b \equiv 0 \pmod{p}$. Choose Y arbitrarily. Since ξ and a are both relatively prime to p, choose X so that $2a\xi X + q \equiv 0 \pmod{p}$. Then $ax^2 + by^2 \equiv k \pmod{p^{m+1}}$ has solutions with x and y both relatively prime to p. Therefore the induction is complete.

COROLLARY. *Let p be an odd prime, t any integer, and u any integer relatively prime to p. If k is a quadratic residue of p when it is not divisible by p, and if $k = tpu^2$ in case k is divisible by p, then $x^2 + tpy^2 \equiv k \pmod{p^n}$ has integral solutions for any positive integer n.*

THEOREM 5. *Let a and b be integers and k an odd integer. If $ax^2 + by^2 \equiv k \pmod 8$ is solvable in integers, then $ax^2 + by^2 \equiv k \pmod{2^n}$ has integral solutions when n is an arbitrary positive integer.**

COROLLARY. *Let b be any integer and k an odd integer. If $x^2 + by^2 \equiv k \pmod 8$ is solvable, then $x^2 + by^2 \equiv k \pmod{2^n}$ has integral solutions when n is an arbitrary positive integer.*

THEOREM 6. *Let q be any prime number, k an odd integer, a an integer, relatively prime to q if q is odd, b any integer different from zero; moreover, let $ax^2 + by^2 \equiv k \pmod 8$ have integral solutions if $q = 2$. In case q is odd, let c be an integer satisfying the congruence $ac \equiv 1 \pmod q$ and let condition (3) or condition (4) be satisfied according as q does not or does divide b.*

(3) *$-bc$ is a quadratic residue of q when $k \equiv 0 \pmod q$.*

(4) *If $k \equiv 0 \pmod q$, $k \equiv 0 \pmod b$ and $\frac{k}{b}$ is a quadratic residue of q^n; if $k \not\equiv 0 \pmod q$, ck is a quadratic residue of q.*

Under the restrictions stated, the congruence $ax^2 + by^2 \equiv k \pmod{q^n}$ has integral solutions when n is any positive integer.

If $q = 2$, the theorem follows from Theorem 5. If q is odd, Theorem 3 or Theorem 4 can be applied according as b is not or is divisible by q.

The following theorem can now be readily proved.

THEOREM 7. *Let N be an arbitrary integer expressed as $N = q^{n_1} r^{n_2} \cdots w^{n_\nu}$, where $q, r, \cdots, w$ are distinct primes and $n_1, n_2, \cdots, n_\nu$ are positive integers. If the definitions and restrictions of Theorem 6 hold true for each of the primes $q, r, \cdots, w$, the congruence $ax^2 + by^2 \equiv k \pmod N$ has integral solutions.*

The following theorem follows as a corollary of Theorems 6 and 7.

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* This theorem follows at once from a known theorem on the number of roots of a congruence. Cf. Bachmann, *Die Arithmetik der Quadratischen Formen*, I, p. 483. A simple proof can also be given by induction.

THEOREM 8. *Let n be the index of the highest power of an arbitrarily chosen prime factor q contained in the arbitrary integer N (different from zero), k an odd integer, and b any integer different from zero; moreover, let $x^2+by^2\equiv k \pmod 8$ be solvable in integers when N has a factor 4. In case q is odd, let condition (5) or condition (6) be satisfied according as q does not or does divide b.*

(5) *$-b$ is a quadratic residue of q when $k\equiv 0 \pmod q$.*

(6) *If $k\equiv 0 \pmod q$, $k\equiv 0 \pmod b$ and $\frac{k}{b}$ is a quadratic residue of q^n; if $k\not\equiv 0 \pmod q$, k is a quadratic residue of q.*

Under these restrictions for each prime factor q of N, the congruence $x^2+by^2\equiv k \pmod N$ has integral solutions.

THEOREM 9. *Let a, b, k, N be any four integers such that k and N are relatively prime. Then, if $ax^2+by^2\equiv k \pmod N$ has any integral solution, it has one with x and y relatively prime.*

Let $x=\xi_1$, $y=\eta_1$ be any solution. Let the greatest common divisor of ξ_1, η_1 be α_1, and let the greatest integer dividing ξ_1 and relatively prime to α_1 be ξ. Then $x=\xi_1=\xi\alpha\alpha_1$, $y=\eta_1=\eta\alpha_1$. Hence ξ is relatively prime to each of the integers α_1, α, η, and every prime factor of α is contained in α_1. Consequently, k and N being relatively prime, every divisor of α_1 (and α) is relatively prime to N, and $x=\xi_1=\xi\alpha\alpha_1$, $y=\eta_1+N\xi=\eta\alpha_1+N\xi$ is a solution of the congruence with x and y relatively prime.

The next two theorems are fundamental to §§ 3, 4.

Consider the following definitions and conditions:

(7) *b, k, N are integers such that N is greater than one, k odd, b different from zero and relatively prime to k.*

(8) *$x^2+by^2\equiv k \pmod 8$ is solvable in integers when N has a factor 4.*

(9) *For every odd prime factor q of N, condition (5) or condition (6) of Theorem 8 is satisfied according as the arbitrary odd prime factor q of N does not or does divide b.*

THEOREM 10. *Under the foregoing restrictions (7)–(9),*

$$x^2+by^2\equiv k \pmod N \tag{10}$$

has an integral solution with x and y relatively prime.

It is to be noted that N is an arbitrary integer greater than unity. By Theorem 8, the congruence (10) has integral solutions.

Let each of $\lambda_1, \lambda_2, \cdots, \lambda_r, \nu_1, \nu_2, \cdots, \nu_r$ be a positive integer or zero; moreover, let the greatest common divisor of k and N be $d_1^{\lambda_1} d_2^{\lambda_2} \cdots d_r^{\lambda_r}$, where $d_1, d_2, \cdots, d_r$ are distinct primes. If $\nu_1, \nu_2, \cdots, \nu_r$ are the exponents of the highest powers of $d_1, d_2, \cdots, d_r$, respectively, contained in N,

write $N = d_1^{\nu_1} d_2^{\nu_2} \cdots d_r^{\nu_r} N_1$. If all $\lambda_i = 0$ $(i = 1, 2, \cdots, r)$, the present theorem follows immediately from Theorem 9. Henceforth it is assumed than at least one $\lambda_i > 0$. We now define $d = d_1^{\nu_1} d_2^{\nu_2} \cdots d_{r-1}^{\nu_{r-1}} d_r^{\nu_r}$, $\delta_i = \dfrac{d}{d_i^{\nu_i}}$ $(i = 1, 2, \cdots, r)$. Since δ_i is relatively prime to d_i, ϱ_i's can be found to satisfy the r congruences $\delta_i \varrho_i \equiv 1 \pmod{d_i^{\nu_i}}$, where $i = 1, 2, \cdots, r$. By Theorem 3, the r congruences

$$x^2 + by^2 \equiv k \pmod{d_i^{\nu_i}} \tag{11}$$

have integral solutions $x = \alpha_i$, $y = \beta_i$ $(i = 1, 2, \cdots, r)$, respectively, with α_i and β_i both relatively prime to d_i $(i = 1, 2, \cdots, r)$. Then

$$\begin{aligned} X_0 &= \delta_1 \varrho_1 \alpha_1 + \delta_2 \varrho_2 \alpha_2 + \cdots + \delta_r \varrho_r \alpha_r, \\ Y_0 &= \delta_1 \varrho_1 \beta_1 + \delta_2 \varrho_2 \beta_2 + \cdots + \delta_r \varrho_r \beta_r \end{aligned} \tag{12}$$

satisfy the r congruences (11), and hence satisfy the congruence

$$x^2 + by^2 \equiv k \pmod{d}. \tag{13}$$

Moreover, X_0 and Y_0 are both relatively prime to $d = d_1^{\nu_1} d_2^{\nu_2} \cdots d_r^{\nu_r}$.

Next, consider

$$x^2 + by^2 \equiv k \pmod{N_1}. \tag{14}$$

Congruence (14) has solutions since (10) has. Since k and N_1 are relatively prime, it is seen from Theorem 9 that congruence (14) has an integral solution $x = \xi$, $y = \eta$, with x and y relatively prime. Since N_1 and d are relatively prime, let σ and τ be integers such that

$$N_1 \sigma \equiv 1 \pmod{d}, \qquad d\tau \equiv 1 \pmod{N_1}.$$

Then $X = N_1 \sigma X_0 + d\tau\xi$, $Y = N_1 \sigma Y_0 + d\tau\eta$ is a solution of both (13) and (14) and hence of (10). X is relatively prime to d since neither $N_1\sigma$ nor X_0 is divisible by d. Similarly Y is relatively prime to d. A common divisor of X and N_1 must divide ξ, and a common divisor of Y and N_1 must divide η. Hence a common divisor of X and $N = dN_1$ must divide ξ, and a common divisor of Y and $N = dN_1$ must divide η. But, since ξ and η are relatively prime, a common divisor (greater than one) of X and Y is not a divisor of N.

Let $Y = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$, where $p_1, p_2, \cdots, p_s$ are distinct primes and $e_1, e_2, \cdots, e_s$ are positive integers. If X and Y are relatively prime, the theorem is proved. On the other hand, if X and Y are not relatively

prime, let p_1, say, divide X. Then $X_1 = X + N$, $Y_1 = Y$ is a solution of the congruence (10) and X_1 is relatively prime to p_1. If p_2, say, divides X_1, then $X_2 = X_1 + p_1 N$, $Y_2 = Y$ is a solution of (10), and X_2 is relatively prime to p_1 and p_2: for, if p_2 is a divisor of X_2, it is a divisor of N, and hence of X — a contradiction. Again, if p_3 divides X_2, then $X_3 = X_2 + p_1 p_2 N$, $Y_3 = Y$ is a solution of the congruence (10) and X_3 is relatively prime to p_1, p_2, p_3. Continuing in this way, we eventually obtain a solution $x = X_j$, $y = Y_j$ (where $1 \leqq j \leqq s$) of (10) with x and y relatively prime.

THEOREM 11. *Let N be an arbitrary integer and p, q, r be integers such that $d = pr - q^2$ contains no odd square factor greater than one; moreover, let p be 2, a power of an odd prime, or double a power of an odd prime, and, in case $q \neq 0$, let the greatest odd divisor of p be equal to the greatest odd divisor of q when p and q have a common odd divisor greater than one.*

If ξ and η are relatively prime integers such that $pN = \xi^2 + d\eta^2$, then

$$px^2 + 2qxy + ry^2 \tag{15}$$

*represents N with x and y relatively prime.**

By hypothesis, $\xi^2 + d\eta^2 \equiv 0 \pmod{p}$, whence $\xi^2 - q^2\eta^2 = (\xi - q\eta)(\xi + q\eta) \equiv 0 \pmod{p}$. If p is even, $\xi - q\eta$ and $\xi + q\eta$ are both even. If $p = 2$, N is represented by (15) for $x = \dfrac{\xi - q\eta}{2}$, $y = \eta$. If $p > 2$, let $p_1 = p$ in case p is odd and let $p_1 = \dfrac{p}{2}$ in case p is even. Now p is relatively prime to η since ξ and η are relatively prime. When p_1 is relatively prime to q, one of $\xi - q\eta$, $\xi + q\eta$ is divisible by p_1 and the other one is not. When p_1 and q have a common divisor greater than one, both q and ξ are divisible by p_1. Hence, in view of the identity

$$px^2 + 2qxy + ry^2 = \frac{1}{p}[(px + qy)^2 + dy^2],$$

place $px + qy = \xi$, $y = \eta$ or $-\eta$ according as $\xi - q\eta \equiv 0 \pmod{p}$ or $\xi + q\eta \equiv 0 \pmod{p}$. Consequently, N is represented by (15) for these (relatively prime) values of x and y.

3. The impossibility of a separation of cases in Legendre's Table IV. In Legendre's Table IV (*Zahlentheorie*, Maser's translation, I, pages 402–406; see also I, nos. 216–218, pages 282–285), at least one set of the linear odd divisors of $t^2 + au^2$, where $a = 17, 29, 41, 53, 61, 65, 69, 73, 77, 89, 97, 101$, corresponds to two or more quadratic divisors

* The theorem is obviously true in the trivial case $p = 1$.

of the particular quadratic form t^2+au^2. It will be found, either by direct numerical computation or by the application of Theorems 10 and 11 of § 2, that a separation (in the sense of § 1) of types of linear odd divisors of these forms t^2+au^2 is not possible. We shall consider throughout representations in which y and z are relatively prime.*

Form t^2+29u^2.

	Quadratic divisors.	*Linear odd divisors.*
(16)	$y^2+2yz+30z^2=(y+z)^2+29z^2$	$116x+1$, 5, 9, 13, 25; 33, 45, 49, 53, 57; 65, 81, 93, 109.
(17)	$5y^2+2yz+6z^2=\frac{1}{5}[(5y+z)^2+29z^2]$	
(18)	$2y^2+2yz+15z^2=\frac{1}{2}[(2y+z)^2+29z^2]$	$116x+3$, 11, 15, 19, 27; 31, 39, 43, 47, 55; 75, 79, 95, 99.
(19)	$10y^2+2yz+3z^2=\frac{1}{3}[(3z+y)^2+29y^2]$	

Treating first the forms (16) und (17), we find the least numbers given by the forms $116x+1$, 5, $\cdots$, 109, respectively, which are represented *properly* (that is, with y and z relatively prime) by either (16) or (17). If it should happen that the least number of any one of these linear forms represented by one of the forms (16), (17), is represented by both forms (16), (17), then no further theory is necessary to show that a separation is impossible for that linear form.

$116\cdot 0+1=1$ is represented by (16) for $y=1$, $z=0$, but not by (17). By Theorem 10 with $k=5$, $N=2^2\cdot 5\cdot 29n$ (where n is a positive integer definite but arbitrary), the congruence $X^2+29Y^2\equiv 5 \pmod{580n}$ has a solution $X=\xi$, $Y=\eta$, where ξ and η are relatively prime. Let $\xi^2+29\eta^2=5\,(116nM+1)$. Then, by Theorem 11 with $p=5$, $q=1$, $r=6$, $N=116nM+1$, it is seen that $5y^2+2yz+6z^2$ represents $116nM+1$ with y and z relatively prime. Hence, for this form $116x+1$ a separation is impossible. In this case, as in some of the others, a separation can be seen by easy computation to be impossible since $116\cdot 1+1=117$ is represented by forms (16) and (17) for the values $y=-1$, $z=2$; $y=3$, $z=-4$, respectively.

Again, $116\cdot 0+57=57$ is represented by (17) for $y=3$, $z=1$. By Theorem 10 with $k=57$, $N=2^2\cdot 29n$, the congruence $X^2+29Y^2\equiv 57 \pmod{116n}$ has a solution $X=\xi$, $Y=\eta$ in which ξ and η are relatively prime. Letting $\xi^2+29\eta^2=116nM+57$, we see by Theorem 11 with

* It is to be noted that although these tables were obtained on the supposition that y and z and also t and u are relatively prime, the same forms of linear odd divisors are obtained if y and z have a common odd divisor greater than one (see Legendre's *Zahlentheorie*, I, nos. 207–208, pp. 270–273; I, nos. 210–211, pp. 275–278).

$p = 1$, $q = 1$, $r = 30$, $N = 116nM + 57$, that $y^2 + 2yz + 30z^2$ represents $116nM + 57$ with y and z relatively prime. For this form $116x + 57$ a separation is impossible. By trial it is found that the least number of the form $116x + 57$ represented (properly) by both (16) and (17) is $116 \cdot 3 + 57 = 405$, which is represented by (16) and (17) for $y = 15$, $z = 2$; $y = 1$, $z = 8$, respectively.

Consider, next, forms (18) and (19). In order to apply the foregoing theory directly to the forms (18) and (19), it is necessary to obtain for each of the corresponding linear odd divisors a number represented by (18), since the k of Theorem 10 is to be *odd*. We can, however, transform (18) into an equivalent quadratic form

$$(20) \qquad 2y_1^2 + 6y_1z_1 + 19z_1^2 = \frac{1}{19}[(19z_1 + 3y_1)^2 + 29y_1^2]$$

by the transformation $y_1 = y - z$, $z_1 = z$. We can now apply the theory directly to the forms (20) and (19) by obtaining from the particular form of the linear odd divisor the least number representable (properly) by *either* (20) *or* (19). It is seen that for each linear odd divisor a separation is impossible. Theorems 10 and 11 are applicable in each case where a congruence in X, Y is tabulated.

Linear odd divisor	*Values for (16)*	*Values for (17)*	*Congruence solvable with X, Y relatively prime*
$116 \cdot 0 + 1 = 1$	$y = 1 \quad z = 0$		$X^2 + 29Y^2 \equiv 5 \pmod{580n}$
$116 \cdot 0 + 5 = 5$		$y = 1 \quad z = 0$	$X^2 + 29Y^2 \equiv 5 \pmod{116n}$
$116 \cdot 0 + 9 = 9$		$y = 1 \quad z = -1$	$X^2 + 29Y^2 \equiv 9 \pmod{116n}$
$116 \cdot 0 + 13 = 13$		$y = 1 \quad z = 1$	$X^2 + 29Y^2 \equiv 13 \pmod{116n}$
$116 \cdot 0 + 25 = 25$		$y = 1 \quad z = -2$	$X^2 + 29Y^2 \equiv 25 \pmod{116n}$
$116 \cdot 0 + 33 = 33$	$y = 1 \quad z = 1$	$y = 1 \quad z = 2$	
$116 \cdot 0 + 45 = 45$	$y = 3 \quad z = 1$	$y = 3 \quad z = -1$	
$116 \cdot 1 + 49 = 165$	$y = 5 \quad z = 2$	$y = 3 \quad z = 4$	
$116 \cdot 0 + 53 = 53$		$y = 1 \quad z = -3$	$X^2 + 29Y^2 \equiv 53 \pmod{116n}$
$116 \cdot 0 + 57 = 57$		$y = 3 \quad z = 1$	$X^2 + 29Y^2 \equiv 57 \pmod{116n}$
$116 \cdot 0 + 65 = 65$	$y = 5 \quad z = 1$	$y = 1 \quad z = 3$	
$116 \cdot 0 + 81 = 81$		$y = 3 \quad z = 2$	$X^2 + 29Y^2 \equiv 81 \pmod{116n}$
$116 \cdot 0 + 93 = 93$	$y = 7 \quad z = 1$	$y = 1 \quad z = -4$	
$116 \cdot 0 + 109 = 109$		$y = 1 \quad z = 4$	$X^2 + 29Y^2 \equiv 109 \pmod{116n}$

Linear odd divisor	*Values for (20)*	*Values for (19)*	*Congruence solvable with X, Y relatively prime*
$116 \cdot 0 + 3 = 3$		$y = 0 \quad z = 1$	$X^2 + 29Y^2 \equiv 57 \pmod{2204n}$
$116 \cdot 0 + 11 = 11$		$y = 1 \quad z = -1$	$X^2 + 29Y^2 \equiv 19 \cdot 11 \pmod{2204n}$
$116 \cdot 0 + 15 = 15$	$y_1 = -1 \quad z_1 = 1$	$y = 1 \quad z = 1$	
$116 \cdot 0 + 19 = 19$	$y_1 = 0 \quad z_1 = 1$		$X^2 + 29Y^2 \equiv 57 \pmod{348n}$
$116 \cdot 0 + 27 = 27$	$y_1 = 1 \quad z_1 = 1$		$X^2 + 29Y^2 \equiv 81 \pmod{348n}$
$116 \cdot 0 + 31 = 31$		$y = -1 \quad z = 3$	$X^2 + 29Y^2 \equiv 589 \pmod{2204n}$
$116 \cdot 0 + 39 = 39$	$y_1 = 2 \quad z_1 = 1$	$y = -2 \quad z = 1$	

Linear odd divisor	*Values for (20)*		*Values for (19)*		*Congruence solvable with X, Y relatively prime*
$116\cdot 0+43=43$			$y=1$	$z=3$	$X^2+29Y^2\equiv 43\cdot 19 \pmod{2204n}$
$116\cdot 0+47=47$			$y=2$	$z=1$	$X^2+29Y^2\equiv 47\cdot 19 \pmod{2204n}$
$116\cdot 0+55=55$	$y_1=3$	$z_1=1$	$y=-2$	$z=3$	
$116\cdot 0+75=75$	$y_1=4$	$z_1=1$	$y=-1$	$z=5$	
$116\cdot 0+79=79$			$y=2$	$z=3$	$X^2+29Y^2\equiv 79\cdot 19 \pmod{2204n}$
$116\cdot 0+95=95$			$y=1$	$z=5$	$X^2+29Y^2\equiv 95\cdot 19 \pmod{2204n}$
$116\cdot 0+99=99$	$y_1=5$	$z_1=1$	$y=3$	$z=1$	

As an illustration of the utility of the method here employed as compared with the method of direct computation, we note the following results:

The least number of the form $116x+3$ represented (properly) by both (20) and (19) is found by trial to be $116\cdot 2+3=235$, which is represented by (20) and (19) for $y_1=9, z_1=1$; $y=1, z=9$, respectively; that of the form $116x+11$ is $116\cdot 8+11=939$, which is represented by (20) and (19) for $y_1=20, z_1=1$; $y=-8, z=13$, respectively; that of the form $116x+31$ is $116\cdot 4+31=495$, which is represented by (20) and (19) for $y_1=14, z_1=1$; $y=-2, z=13$, respectively; that of the form $116x+95$ is $116\cdot 2+95=327$, which is represented by (20) and (19) for $y_1=11, z_1=1$; $y=-5, z=7$, respectively.

4. **The method extended to Legendre's Tables V, VI, VII.** The methods of § 3 will now be extended to Legendre's Table V. As a typical case consider the form t^2+39u^2. Two of the quadratic divisors are

$$y^2+39z^2, \tag{21}$$

$$3y^2+13z^2=\frac{1}{3}[(3y)^2+39z^2]=\frac{1}{13}[(13z)^2+39y^2]; \tag{22}$$

and the corresponding linear odd divisors are $78x+1, 25, 43, 49, 55, 61$. Since 3 and 13 are divisors of 39, the form (22) is to be replaced* by an equivalent form. Under the transformation $y_1=y-z, z_1=-y+2z$ the form (22) becomes

$$25y_1^2+38y_1z_1+16z_1^2=\frac{1}{25}[(25y_1+19z_1)^2+39z_1^2]. \tag{23}$$

* If e, f, g are integers having no common divisor greater than one and if not both e and g are even, the quadratic form $ex^2+2fxy+gy^2$ contains infinitely many primes for integral values of the variables (see paper by G. Lejeune Dirichlet, Journal für Mathematik, vol. 21 (1840), pp. 98–100). If the form represents the prime p, let $p=e\alpha^2+2f\alpha\beta+g\beta^2$. Since α and β are relatively prime integers, there exist integers γ, δ such that $\alpha\gamma-\beta\delta=1$. By means of the transformation $x=\alpha X+\delta Y, y=\beta X+\gamma Y$ the quadratic form becomes $pX^2+2(e\delta\alpha+f\alpha\gamma+f\beta\delta+g\beta\gamma)XY+(e\delta^2+2f\gamma\delta+g\gamma^2)Y^2$. Now, since the coefficients 3, 0, 13 of the form (22) have no common divisor greater than one, (22) contains an infinite number of primes of which 61 is one. We therefore know that there is a transformation of determinant one which transforms (22) into an equivalent form in which 61 is the coefficient of a square term.

The theory of § 3 can now be applied to the two quadratic forms (21) and (23). For forms t^2+au^2 in which a is of the form $8m+3$, the theory of § 3 is applicable only after modification of Table* V.

As a typical case of Table VI consider the form t^2+34u^2. The first two quadratic divisors are

$$y^2+34z^2, \tag{24}$$

$$2y^2+17z^2 = \frac{1}{17}[(17z)^2+34y^2]. \tag{25}$$

The corresponding linear odd divisors are $136x+1$, 9, 19, 25, 33; 35, 43, 49, 59, 67; 81, 83, 89, 115, 121; 123. Since 17 is a divisor of 34, we shall transform (25) into an equivalent form (26). Under the transformation $y_1 = y-z$, $z_1 = z$ the form (25) becomes

$$2y_1^2+4y_1z_1+19z_1^2 = \frac{1}{19}[(19z_1+2y_1)^2+34y_1^2]. \tag{26}$$

The theory of § 3 can now be applied to the quadratic forms (24) and (26). Legendre's Table VII can be treated in a similar manner.

The writer has considered in detail the forms t^2+17u^2, t^2+29u^2, t^2+41u^2, t^2+65u^2† of Table IV; t^2+19u^2, t^2+39u^2 of Table V; t^2+34u^2 of Table VI; and t^2+38u^2 of Table VII.

* See Legendre's *Zahlentheorie* (Maser), I, nos. 219–224, pp. 285–290.

† In this case, for the second pair of quadratic divisors, $y_1 = y-z$, $z_1 = z$; $y_1 = y-2z$, $z_1 = z$, respectively, will be found suitable transformations.

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A PARAMETRIC PROBLEM OF THE CALCULUS OF VARIATIONS AND ITS TREATMENT AS A PROBLEM OF LAGRANGE.*

By Lee H. McFarlan.

Introduction. The simplest problem of the Calculus of Variations stated in parametric form with arc length, s as parameter is the following: Of all curves of a certain class connecting two points $P_1\,P_2$, we are to determine that one, $x = x(s)$, $y = y(s)$ along which the integral

$$I = \int_{s_1}^{s_2} f(x, y, x', y')\, ds,$$

between the two points is a minimum. The minimizing arc joining $P_1\,P_2$ will be called E_{12}. It is well known that the functions $x(s)$, $y(s)$ must satisfy the Euler equations

$$(1) \qquad f_x - \frac{d f_{x'}}{d s} = 0 \qquad f_y - \frac{d f_{y'}}{d s} = 0.$$

The above problem may be restated as a special case of the Lagrange Problem in the following manner.

Given the integral

$$I_1 = \int_{x_1}^{x_2} f(x, y, z, y', z')\, dx,$$

a fixed point (x_1, y_1, z_1) and a fixed curve $x = X(t)$, $y = Y(t)$, $z = Z(t)$, let us determine among all sets of functions $y(x)$, $z(x)$, satisfying conditions of the form

$$\varphi(x, y, z, y', z') = 0, \qquad y(x_1) = y_1, \qquad z(x_1) = z_1,$$
$$x_2 = X(t), \qquad y[X(t)] = Y(t), \qquad z[X(t)] = Z(t),$$

one which gives I_1 a minimum value. This is a Problem of Lagrange with one variable end-point. It may be reduced to the parametric problem in the plane, described above, as follows. Let $f(x, y, z, y', z')$ be independent of x and suppose that the arc $X(t)$, $Y(t)$, $Z(t)$ has the form

$$(2) \qquad X(t) = t = x, \qquad Y(t) = y_2, \qquad Z(t) = z_2.$$

* Received December 21, 1927.

Let the equation $\varphi(x, y, z, y', z') = 0$ be replaced by $y'^2 + z'^2 - 1 = 0$. Furthermore suppose that the function $f(y, z, y', z')$ satisfies the "homogeneity" condition with respect to y', z' customarily imposed in the theory of the parametric problem in the plane,

$$f(y, z, \varkappa y', \varkappa z') = \varkappa \cdot f(y, z, y', z'). \tag{3}$$

The problem of minimizing I_1 in xyz-space under these conditions is now equivalent to the problem of minimizing I in sxy-space as described above. The general curve L, defined by the functions $X(t)$, $Y(t)$, $Z(t)$ for the integral I_1 is replaced by a straight line passing through the point $(0, x_2, y_2)$ and parallel to the s-axis in the problem of minimizing the integral I.

The object of this paper is to identify some of the known necessary conditions, arising in the special case of the Problem of Lagrange as stated, with those which are found in the plane problem in the parametric form. One of the most interesting results is the reduction of the determinant defining the focal point of the curve L on E_{12}, to the determinant whose zero give the parameter value of the conjugate point of P_2 on E_{12} for the plane parametric problem.

1. **Necessary conditions on the arc E_{12}.** The work in this paper will be based upon the following assumptions:

(A) The functions $y(x)$, $z(x)$, defining the minimizing arc E_{12}, are continuous on the interval $x_1 x_2$ and have continuous first derivatives on this interval. It will also be assumed that y'' and z'' exist everywhere on this interval.

(B) For values of (x, y, z, y', z') in the neighborhood of those defining E_{12}, f and φ have continuous derivatives of at least the first four orders.

(C) The derivatives $\varphi_{y'}$ and $\varphi_{z'}$ shall not vanish simultaneously along E_{12}.

It is well known that for such a minimizing arc E_{12} there exists a function $F = \lambda_0 f + \lambda(x) \cdot \varphi$ such that the functions $y(x)$, $z(x)$ and $\lambda(x)$ are solutions of the equations

$$\text{I.} \qquad F_y - \frac{dF_{y'}}{dx} = 0, \quad F_z - \frac{dF_{z'}}{dx} = 0, \quad \varphi(x, y, z, y', z') = 0.$$

Let us assume that the extremal arc E_{12} is normal, according to the definition in a previous paper by the author* in which case the constant λ_0 may be taken equal to unity and the function $\lambda(x)$ is then uniquely determined for this arc. We shall assume that E_{12} is normal on every subinterval of $x_1 x_2$. The further assumption will be made that

* The focal point for the problem of Lagrange with one variable end point, Annals of Math., vol. 28 (1927), p. 184.

$$(4) \qquad R(x, y, z, y', z', \lambda) = \begin{vmatrix} F_{y'y'} & F_{y'z'} & y' \\ F_{z'y'} & F_{zz'} & z' \\ y' & z' & 0 \end{vmatrix}$$

does not vanish along E_{12}.

The arc E_{12} must satisfy further necessary conditions, the Analogue of the Weierstrass condition and the Clebsch condition. These conditions are given in section 3, where use is made of them.

Equations I have a four parameter family of solutions which may be written in the form

$$(5) \quad y = y(x, a_1, a_2, a_3, a_4),\ z = z(x, a_1, a_2, a_3, a_4),\ \lambda = \lambda(x, a_1, a_2, a_3, a_4)$$

and which contain E_{12} for particular parameter values $a_{10}, a_{20}, a_{30}, a_{40}$.*

The conditions mentioned are for the case of fixed end-points and must still be satisfied for the case of variable end-points. For the end-point 2 restricted to the curve L, the transversality condition must hold, namely, *at the point 2 the condition*

$$\text{II.} \qquad (F - y'F_{y'} - z'F_{z'})X' + Y'F_{y'} + Z'F_{z'}\Big|^2 = 0$$

must be satisfied.†

The determinant whose zero gives the focal point on E_{12} of the curve L for $x = x_2$ may be written

$$(6) \qquad H(x, x_2) = \begin{vmatrix} A & A_1 & A_2 & A_3 & A_4 \\ -\varrho_1 & y_{a_1}(x_2) & y_{a_2}(x_2) & y_{a_3}(x_2) & y_{a_4}(x_2) \\ -\varrho_2 & z_{a_1}(x_2) & z_{a_2}(x_2) & z_{a_3}(x_2) & z_{a_4}(x_2) \\ 0 & y_{a_1}(x) & y_{a_2}(x) & y_{a_3}(x) & y_{a_4}(x) \\ 0 & z_{a_1}(x) & z_{a_2}(x) & z_{a_3}(x) & z_{a_4}(x) \end{vmatrix}$$

where

$$\varrho_1 = Y' - y'X'\Big|^2, \qquad \varrho_2 = Z' - z'X'\Big|^2,$$

$$(7) \quad \begin{aligned} A = {} & (F - y'F_{y'} - z'F_{z'})X'' + Y''F_{y'} + Z''F_{z'} + (F_x - y'F_y - z'F_z)X'^2 \\ & + 2F_yX'Y' + 2F_zX'Z' + \varrho_1^2 F_{y'y} + \varrho_1\varrho_2(F_{y'z} + F_{yz'}) + \varrho_2^2 F_{z'z}\Big|^2, \\ A_i = {} & F_{y'y'}\varrho_1 y'_{a_i} + F_{y'z'}(\varrho_2 y'_{a_i} + \varrho_1 z'_{a_i}) + F_{z'z'}\varrho_2 z'_{a_i} \\ & + \varphi_{y'}\varrho_1\lambda_{a_i} + \varphi_{z'}\varrho_2\lambda_{a_i}\Big|^2. \end{aligned}$$

We may state the new necessary condition on E_{12} as follows

* Bolza, Vorlesungen über Variationsrechnung, pp. 592–594.

† McFarlan, loc. cit., p. 187.

III. *In order that E_{12} shall render I_1 a minimum value, it is necessary that the focal point 3 on E_{12} of the curve L, whose abscissa is the zero of $H(x, x_2)$, shall not be on the interval $x_1 x_2$.**

The proof of the above statement depends upon the fact that at least one of the quantities ϱ_1, ϱ_2 is different from zero.

2. First necessary conditions for the problem in the yz-plane where x is arc length. Equations I for our problem are

$$(8)\quad F_y - \frac{dF_{y'}}{dx} = 0, \qquad F_z - \frac{dF_{z'}}{dx} = 0, \qquad \frac{1}{2}(y'^2 + z'^2 - 1) = 0$$

where

$$F = f + \frac{\lambda}{2}(y'^2 + z'^2 - 1).$$

The numerical factor $\frac{1}{2}$ has been introduced to simplify our work. The assumption of § 1, that the determinant $R \neq 0$ along E_{12}, has been shown by Bolza† to be sufficient for the existence of y'', z'' and λ'. By use of this fact and the assumptions B and C of the first section, one may write the first two of equations I in the differentiated form. To these have been added the result of differentiating $\frac{1}{2}(y'^2 + z'^2 - 1) = 0$ with respect to x. We thus obtain

$$(9)\quad \begin{aligned} y'' F_{y'y'} + z'' F_{y'z'} + \lambda' y' &= L, \\ y'' F_{z'y'} + z'' F_{z'z'} + \lambda' z' &= M, \\ y'' y' + z'' z' &= 0 \end{aligned}$$

where

$$(10)\quad \begin{aligned} L &= f_y - y' f_{y'y} - z' f_{y'z}, \\ M &= f_z - y' f_{z'y} - z' f_{z'z}. \end{aligned}$$

Equations (9) may be solved for y'', z'' and λ' since the determinant of the coefficients in these equations in R, which is different from zero. Thus we have

$$(11)\quad \begin{aligned} y'' &= \frac{z'(My' - Lz')}{R}, \qquad z'' = \frac{y'(Lz' - My')}{R}, \\ \lambda' &= \frac{L(z' F_{y'z'} - y' F_{z'z'}) + M(y' F_{y'z'} - z' F_{y'y'})}{R} = \frac{S}{R} \end{aligned}$$

where S is merely a notation for the numerator on the right.

The projection K_{12} of E_{12} in the yz-plane has the equations $y = y(x)$, $z = z(x)$. Since x is the arc length along the projected curve, we have

* McFarlan, loc. cit., p. 192.

† loc. cit., p. 589.

$$y' = \cos\psi, \qquad z' = \sin\psi, \qquad \frac{d\psi}{dx} = \frac{y'z'' - z'y''}{(y'^2+z'^2)^{3/2}} = \frac{Lz' - My'}{R}$$

where ψ is the angle which the tangent to K_{12} makes with the positive y axis. The last equations (11) therefore take on the form

$$\begin{aligned} &\frac{dy}{dx} = \cos\psi, \qquad \frac{dz}{dx} = \sin\psi, \\ &\frac{d\psi}{dx} = \frac{Lz' - My'}{R}, \qquad \frac{d\lambda}{dx} = \frac{S}{R}. \end{aligned} \tag{12}$$

It is easily shown that these equations are equivalent to the original system (8). They may be given, however, a more simple form if one makes use of the following properties of f, due to its homogeneity property expressed by means of equation (3),

$$f = y' f_{y'} + z' f_{z'}, \tag{13}$$

$$f_y = y' f_{y'y} + z' f_{z'y}, \qquad f_z = y' f_{y'z} + z' f_{z'z}, \tag{14}$$

$$y' f_{y'y'} + z' f_{y'z'} = 0, \qquad y' f_{y'z'} + z' f_{z'z'} = 0. \tag{15}$$

By virtue of the last two identities, there exists a function f_1 defined by the expressions

$$f_1 = f_{z'z'}/y'^2 = -f_{y'z'}/y'z' = f_{y'y'}/z'^2. \tag{16}$$

The use of the above identities gives the two relations

$$L = z'(f_{yz'} - f_{y'z}), \qquad M = y'(f_{y'z} - f_{yz'}),$$

and

$$y'L + z'M = 0.$$

Also one obtains for S the expression

$$\begin{aligned} S &= y'(-y'z'M - y'^2 L) f_1 - z'(z'^2 M + y'z'L) f_1 - \lambda(y'L + z'M) \\ &= -[f_1 + \lambda](y'L + z'M) \equiv 0. \end{aligned}$$

Hence λ is a constant along E_{12}. The value of λ may be obtained from the Transversality condition II. If that expression is solved for λ, one has

$$\lambda = \frac{(f - y'f_{y'} - z'f_{z'})X' + Y'f_{y'} + Z'f_{z'}}{X' - y'Y' - z'Z'}.$$

The denominator is seen to have the value 1, by referring to equations (2). Equation (13) shows that the coefficient of X' in the numerator vanishes.

Since Y' and Z' vanish also it follows that the numerator vanishes and λ has the value zero for $x = x_2$. Hence λ has the value zero along E_{12}. The determinant $R(x, y, z, y', z', \lambda)$ is easily shown to have the value $-(f_1 + \lambda)$ of $-f_1$ since λ is zero. The non-vanishing of R along E_{12} amounts to the assumption that the problem is a "regular" one.* Equations (12) are readily reduced to the system†

$$\frac{dy}{dx} = \cos\,\psi, \qquad \frac{dz}{dx} = \sin\,\psi, \qquad \frac{d}{dx} = \frac{f_{zy'} - f_{z'y}}{f_1}$$

since λ and $d\lambda/dx$ are now zero.

3. **The Weierstrass and Clebsch conditions.** Let us designate by the term admissible set (x, y, z, p, q), a set of values which satisfy the equation $\varphi(x, y, z, p, q) = 0$ and lies in the region for which the continuity properties (B) on f and φ hold.

The analogue of the Weierstrass Condition is then as follows.

In order that E_{12} shall render I_1 a strong minimum with the adjoined condition $\varphi = 0$, it is necessary that the inequality

$$E(x, y, z, y', z', p, q, \lambda) = F(x, y, z, p, q, \lambda) - F(x, y, z, y', z', \lambda)$$
$$-(p - y')\,F_{y'}(x, y, z, y', z', \lambda) - (q - z')\,F_{z'}(x, y, z, y', z', \lambda) \geqq 0.$$

holds for every element (x, y, z, y', z') of the arc E_{12} and for all admissible sets $(x, y, z, p, q) = (x, y, z, y', z')$.

The Clebsch necessary condition on the minimizing arc E_{12} may be stated as follows:

For every point of the minimizing arc E_{12} the inequality

$$F_{y'y'}\,\pi_1^2 + 2F_{y'z'}\,\pi_1\pi_2 + F_{z'z'}\,\pi_2^2 \geqq 0 \tag{16a}$$

must be satisfied for all pairs of functions (π_1, π_2) not both zero which are solutions of the equations,

$$\varphi_{y'}\,\pi_1 + \varphi_{z'}\,\pi_2 = 0. \tag{17}$$

The determination of $\lambda = 0$ enables one to write the analogue of the Weierstrass condition in the form

$$f(y, z, p, q) - f(y, z, y', z') - p f_{y'}(y, z, y', z') + y' f_{y'}(y, z, y', z')$$
$$- q f_{z'}(y, z, y', z') + z' f_{z'}(y, z, y', z') \geqq 0.$$

* Bolza, loc. cit. p. 123.

† Bliss, G. A., A generalization of the notation of angle. Trans. Amer. Math. Soc., vol. 7 (1906), p. 188.

This readily reduces to

$$p[f_{y'}(y,z,p',q')-f_{y'}(y,z,y',z')]+q[f_{z'}(y,z,p,q)-f_{z'}(y,z,y',z')]\geqq 0,$$

which is the well known form of the Weierstrass condition for our parametric problem in the plane.

In considering the Clebsch condition, we note that the equation (17) becomes

$$y'\pi_1+z'\pi_2=0.$$

Hence one has $\pi_1=\sigma z'$, $\pi_2=-\sigma y'$. The inequality (16a) becomes, by use of expressions (16) for f_1,

$$z'^2 f_1\pi_1^2-2y'z'f_1\pi_1\pi_2+y'^2 f_1\pi_2^2\geqq 0$$

which reduces to $f_1\sigma^2\geqq C$. Hence the Clebsch Condition reduces to $f_1\geqq C$,* which is the well known Legendre condition for our parametric problem in the plane.

4. **The focal point condition.** In order that we may interpret the focal point determinant for the special case where L is represented by the equations

$$X(t)=t, \qquad Y(t)=y_2, \qquad Z(t)=z_2,$$

we note that $X'=1$, $Y'=0$, $Z'=0$, $\varrho_1=-y'(x_2)$, $\varrho_2=-z'(x_2)$. *The solutions of the Euler Equations may be written in the form*†

$$y=y(x,\alpha,\beta,\lambda), \qquad z=z(x,\alpha,\beta,\lambda), \qquad \lambda=\lambda,$$

or since f and φ are free of x, these may be written

$$(18)\quad y=y(x-x_2,\alpha,\beta,\lambda), \qquad z=z(x-x_2,\alpha,\beta,\lambda), \qquad \lambda=\lambda.$$

The four arbitrary parameters are x_2, α, β and λ. In the expression (7) for A_i in $H(x,x_2)$, one can easily show by use of the known properties (14) and (15) of f that A_1, A_2, A_3 are all zero. The only remaining term A_4, contains the non-vanishing term

$$\frac{\partial\lambda}{\partial\lambda}[\varrho_1 y'+\varrho_2 z']|^2=-1$$

since ϱ_1, ϱ_2 have the values $-y'(x_2)$ and $-z'(x_2)$ respectively and $y'^2+z'^2=1$ for all x. The expression for A vanishes identically by virtue of (14) and (15) and $f_x=0$. In differentiating with respect to the

* The factor σ is not zero since otherwise π_1 and π_2 would both vanish.

† Bolza, loc. cit., pp. 464 and 490.

parameter x_2, we note that the derivatives of the functions (18) with respect to x and x_2 are numerically equal but opposite in sign.

The determinant of the focal point of the line $x = t$, $y = y_2$, $z = z_2$ becomes

$$\begin{vmatrix} 0 & 0 & 0 & 0 & -1 \\ y'(x_2) & -y'(x_2) & y_\alpha(x_2) & y_\beta(x_2) & y_\lambda(x_2) \\ z'(x_2) & -z'(x_2) & z_\alpha(x_2) & z_\beta(x_2) & z_\lambda(x_2) \\ 0 & -y'(x) & y_\alpha(x) & y_\beta(x) & y_\lambda(x) \\ 0 & -z'(x) & z_\alpha(x) & z_\beta(x) & z_\lambda(x) \end{vmatrix}$$

which reduces readily to

$$\theta_1(x) \cdot \theta_2(x_2) - \theta_1(x_2) \cdot \theta_2(x)$$

where

$$\theta_1(x) = z'(x) \cdot y_\alpha(x) - y'(x) \cdot z_\alpha(x),$$
$$\theta_2(x) = z'(x) \cdot y_\beta(x) - y'(x) \cdot z_\beta(x).$$

Hence the above determinant for the focal point is identified with the well known expression $\theta(x, x_2)$* whose zeros determine the *conjugate point* of x_2 on E_{12}.

* Bolza, loc. cit., p. 233.

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ON THE FUNDAMENTAL EXISTENCE THEOREMS FOR DIFFERENTIAL SYSTEMS.*

By William M. Whyburn.†

It is the object of this paper to treat existence theorems for systems of first order differential equations where both the equations and the initial conditions differ from those that occur in the ordinary theory. The equations are different in that the quantities that appear on the right-hand side of the equations do not, as a whole, satisfy a Lipschitz condition or a condition of uniform boundedness by a summable function.‡ The boundary conditions differ in that they apply to more than one point of the interval.

Throughout the paper we work entirely in the real domain and treat the system of ordinary, first order differential equations

$$(1) \qquad \frac{dy_i}{dx} = f_i(x, y_1, \cdots, y_n) + \sum_{j=1}^{j=n} A_{ij}(x, y_1, \cdots, y_n) y_j \quad (i = 1, \cdots, n)$$

where the following conditions are imposed:§

1. For each fixed x on X: $a \leqq x \leqq b$, the functions f_i and A_{ij} are continuous in $(y_1, \cdots, y_n)$ for all real values of $y_1, \cdots, y_n$.

2. For each fixed $(y_1, \cdots, y_n)$, the functions f_i and A_{ij} are summable in x on X.

3. There exists a summable function $M(x)$ on X such that for all x on X and for all $(y_1, \cdots, y_n)$: $|A_{ij}|, |f_i| \leqq M(x), \quad (i, j = 1, \cdots, n)$.

4. There exists a function $L(x)$ that is summable on X such that for each x on X and for every $(y_1, \cdots, y_n)$ and $(z_1, \cdots, z_n)$

$$\left\{\begin{matrix} |A_{ij}(x, y_1, \cdots, y_n) - A_{ij}(x, z_1, \cdots, z_n)| \\ |f_i(x, y_1, \cdots, y_n) - f_i(x, z_1, \cdots, z_n)| \end{matrix}\right\} \leqq L(x) \sum_{r=1}^{r=n} |y_r - z_r|, \quad (i, j = 1, \cdots, n).$$

By a solution of system (1) we mean a set of absolutely continuous functions $y_1(x), \cdots, y_n(x)$ that satisfies (1) *almost everywhere* on X. When

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‡ Summable is used in the sense of Lebesgue, i. e., $f(x)$ is summable on $a \leqq x \leqq b$ if the Lebesgue integral $\int_a^b f(x)\,dx$ exists.

§ It is to be noted that conditions 3. and 4. are not met by $A_{ij} y_j$.

no confusion arises from so doing, we use the notation $F(x, y^{(k)})$ to denote $F(x, y_1^{(k)}[x], \cdots, y_n^{(k)}[x])$. We state that a property holds on S_m to signify that that property holds for $i = 1, 2, \cdots, n$, each x on X, and for $k = 0, 1, 2, \cdots, m$. We use S to denote the presence of this property for $k = 0, 1, 2, \cdots$, $i = 1, \cdots, n$, and for every x on X.

1. An existence theorem.

THEOREM I. *If $x = c$ is any point of X and $\alpha_1, \cdots, \alpha_n$ are arbitrarily assigned constants, there exists a unique solution of* (1) *satisfying the conditions*

$$y_i(c) = \alpha_i \qquad (i = 1, \cdots, n). \tag{2}$$

Proof. A theorem of Carathéodory's* shows that under our hypotheses the functions A_{ij} and f_i $(i, j = 1, \cdots, n)$, are summable functions of x on X when $y_1, \cdots, y_n$ are replaced by any continuous functions of x on X. It follows from the summability of the product of a continuous function by a summable function that the functions $A_{ij}\, y_j$ are summable on X when $y_1, \cdots, y_n$ are continuous in x on X. If system (1), (2) has a solution it will therefore satisfy the integral system

$$y_i(x) = \alpha_i + \int_c^x \left[f_i(t, y) + \sum_{j=1}^{j=n} A_{ij}(t, y) y_j(t) \right] dt \qquad (i = 1, \cdots, n), \tag{3}$$

and conversely, as a consequence of the properties of indefinite integrals, it is true that any solution of (3) is also a solution of system (1), (2). Hence we may establish theorem I by proving a corresponding theorem for system (3).

Let the absolutely continuous functions $y_i^{(k)}$, $(i = 1, \cdots, n)$, $(k = 0, 1, \cdots)$, be defined as follows:

$$y_i^{(0)}(x) = \alpha_i \qquad (i = 1, \cdots, n)$$

$$y_i^{(k)}(x) = \alpha_i + \int_c^x \left[f_i(t, y^{(k-1)}) + \sum_{j=1}^{j=n} A_{ij}(t, y^{(k-1)})\, y_j^{(k-1)}(t) \right] dt,$$

$$(i = 1, \cdots, n), \quad (k = 1, 2, 3, \cdots).$$

LEMMA. *There exists a constant G such that $|y_i^{(k)}(x)| < G$ on S.*

Proof of Lemma. Let a constant K be chosen greater than unity and so that $K > \sum_{i=1}^{i=n} |\alpha_i| + \int_a^b M(t)\, dt$ and then consider the inequality

$$|y_i^{(k)}(x)| < K \sum_{r=0}^{r=k} \left[n \left| \int_c^x M(t)\, dt \right| \right]^r / r!. \tag{4}$$

* Cf.: *Vorlesungen über Reelle Funktionen*, Leipzig, 1927, p. 665.

This inequality is valid on S_0, where if $x = c$ we replace the righthand side of (4) by K. Let m be the smallest positive integer such that (4) fails on S_m. For the sake of definiteness, suppose this occurs for an x such that $x > c$, since obviously no essential difference exists between the argument for this case and the case $x < c$.

$$|y_i^{(m)}(x)| \leqq |\alpha_i| + \int_c^x |f_i(t, y^{(m-1)})|\,dt + \sum_{j=1}^{j=n} \int_c^x |A_{ij}(t, y^{(m-1)})|\,|y_j^{(m-1)}(t)|\,dt$$

$$< |\alpha_i| + \int_c^x M(t)\,dt + K\int_c^x \left\{ n M(t) \sum_{r=0}^{r=m-1} \left[n\int_c^t M(s)\,ds \right]^r / r! \right\} dt.$$

Integrating the last term

$$|y_i^{(m)}(x)| < K\left[1 + \sum_{r=0}^{m-1} \left\{ n\int_c^x M(t)\,dt \right\}^{r+1} / (r+1)!\right] = K \sum_{r=0}^{r=m} \left[n\int_c^x M(t)\,dt \right]^r / r!$$

and hence (4) is valid on S_m. This contradiction establishes the validity of (4) on S. The inequality (4) is strengthened if we replace the lower and upper limits of integration respectively by a and b. Also, since the series of positive constants thus obtained converges — as k becomes infinite — to a limit $G = K e^{n\int_a^b M(t)\,dt}$, we have $|y_i^{(k)}(x)| < G$ on S.

Proof of Theorem I. Let $z_i^{(0)}(x) = y_i^{(0)}(x)$, $z_i^{(k)}(x) = y_i^{(k)}(x) - y_i^{(k-1)}(x)$, $(i = 1, \cdots, n)$, $(k = 1, 2, \cdots)$. Let $N(x)$ be the summable function that is equal to the greater of $M(x)$ and $L(x)$ for each x on X. The inequality

$$|z_i^{(k)}(x)| < K\left[(n^2 G + 2n)\left|\int_c^x N(t)\,dt\right|\right]^k / k! \tag{5}$$

holds on S_0. Its truth on S_1 follows from

$$|z_i^{(1)}(x)| \leqq \left|\int_c^x \left[|f_i(t, y^{(0)})| + \sum_{j=1}^{j=n} |A_{ij}(t, y^{(0)})|\,|y_j^{(0)}(t)| \right] dt \right|$$

$$< \left|\int_c^x [N(t) + n N(t) G]\,dt\right| < K\left|\int_c^x N(t)\,dt\right| [n^2 G + 2n],$$

while the inequalities

$$|z_i^{(k)}(x)| = \left|\int_c^x \left\{ \sum_{j=1}^{j=n} A_{ij}(t, y^{(k-1)})\, z_j^{(k-1)} + y_j^{(k-2)} [A_{ij}(t, y^{(k-1)}) - A_{ij}(t, y^{(k-2)})] \right.\right.$$

$$\left.\left. + [f_i(t, y^{(k-1)}) - f_i(t, y^{(k-2)})] \right\} dt \right|$$

$$< \left| K\int_c^x \left\{ (2n + Gn^2) N(t) \left[(2n + Gn^2) \left| \int_c^t N(s)\,ds \right| \right]^{k-1} / (k-1)! \right\} dt \right|$$

$$< K\left[(n^2 G + 2n)\left|\int_c^x N(t)\,dt\right|\right]^k / k!$$

show that its validity on S_{k-1} implies its validity on S_k, $(k = 2, 3, \cdots)$. It follows by an application of inductive reasoning that (5) holds on S.

The inequality (5) is strengthened if c and x as limits of integration are replaced by a and b respectively. Furthermore, since the series of constants thus obtained converges as k becomes infinite, the series $\sum_{k=0}^{k=\infty} z_i^{(k)}(x)$, $(i = 1, \cdots, n)$, converge absolutely and uniformly on X to continuous functions $y_i(x)$. Also, since $y_i^{(k)}(x) = \sum_{r=0}^{r=k} z_i^{(r)}(x)$, we have that $\lim_{k\to\infty} y_i^{(k)}(x) = y_i(x)$. From the continuity of the functions A_{ij} and f_i, $(i, j = 1, 2, \cdots, n)$, we have $\lim_{k\to\infty} A_{ij}(x, y^{(k)}) y_j^{(k)} = A_{ij}(x, y) y_j$ and $\lim_{k\to\infty} f_i(x, y^{(k)}) = f_i(x, y)$, for $i = 1, \cdots, n$, and for every x on X. Since these limit functions are summable on X and the sequences are uniformly bounded by the summable function $GM(x)$, we may apply Lebesgue's theorem* concerning the limit of the integrals of a sequence of summable functions to get that the set $y_1(x), \cdots, y_n(x)$ is a solution of (3) and consequently of the system (1), (2).

The uniqueness of this solution is established by assuming the existence of another solution $u_1, \cdots, u_n$ and showing that this assumption leads to a contradiction. Since $y_1, \cdots, y_n, u_1, \cdots, u_n$ are continuous on X these functions are all bounded in absolute value by a constant H which we take greater than 1. The functions $A_{ij} y_j$ and f_i satisfy the Lipschitz condition 4) where L is replaced by $HN(x)$. The argument given by Carathéodory† may now be applied to obtain a contradiction.

2. **A more general system.** Let r be any constant such that $0 < r < 1$. Since $\int_c^x M(t)\,dt$ is absolutely continuous when x and c both lie on X, we can determine a positive constant d_1 such that the absolute value of this integral is less than or equal to r/n when $|x - c| \leqq d_1$. Also, from the absolute continuity of $I = \int_c^x [2 + (nB + r)/(1 - r)]\, N(t)\,dt$, where the constant B is greater than 1, $|a_i|$, $(i = 1, \cdots, n)$, and $N(x)$ is the summable function that is equal to the greater of $M(x)$ and $L(x)$ at each point of X, it follows that if R is any constant such that $0 < R < 1$, a positive quantity d_2 can be determined so that $|I| \leqq R/n$ whenever $|x - c| \leqq d_2$ and x and c both lie on X. Let r, R, d_1, d_2 be so chosen and let δ be the smaller of d_1 and d_2.

THEOREM II. *If $a_1, \cdots, a_n$ are points (not necessarily all distinct) all of which lie on a single sub-interval D: $c \leqq x \leqq d$ of X of length less than*

* Lebesgue: *Leçons sur l'intégration etc.*, Paris, 1904, p. 114.

† Loc. cit., p. 673.

or equal to δ, there exists a unique solution of system (1) *satisfying the conditions*

$$y_i(a_i) = \alpha_i \qquad (i = 1, \cdots, n). \tag{6}$$

Proof. We first prove that this theorem is true on D and then extend it to X by means of theorem I. Let the variable x be confined to D. The argument that was used in proving theorem I establishes the equivalence of system (1), (6) and the integral system

$$y_i(x) = \alpha_i + \int_{a_i}^{x} \left[f_i(t, y) + \sum_{j=1}^{j=n} A_{ij}(t, y)\, y_j(t) \right] dt \qquad (i = 1, \cdots, n). \tag{7}$$

Proceeding as before, we define the absolutely continuous functions $y_i^{(k)}(x)$, $(i = 1, \cdots, n)$, $(k = 0, 1, 2, \cdots)$, by the equations

$$y_i^{(0)}(x) = \alpha_i,$$

$$y_i^{(k)}(x) = \alpha_i + \int_{a_i}^{x} \left[f_i(t, y^{(k-1)}) + \sum_{j=1}^{j=n} A_{ij}(t, y^{(k-1)})\, y_j^{(k-1)}(t) \right] dt$$

$$(k = 1, 2, 3, \cdots), \quad (i = 1, \cdots, n).$$

LEMMA. *The functions $|y_i^{(k)}(x)|$ are bounded on S by a constant G.*

Proof of Lemma. Reasoning similar to that used in establishing the lemma under theorem I establishes the inequality

$$|y_i^{(k)}(x)| < (B + r/n) \sum_{j=0}^{j=k} r^j.$$

Our lemma follows when we note that the positive increasing sequence on the right-hand side of this inequality converges to $G = (B + r/n)/(1 - r)$ when k becomes infinite.

Proof of Theorem II. Let $z_i^{(0)} = y_i^{(0)}$, $z_i^{(k)} = y_i^{(k)} - y_i^{(k-1)}$, $(i = 1, \cdots, n)$, $(k = 1, 2, 3, \cdots)$, and note that the inequality $|z_i^{(k)}(x)| < BR^k$ is valid on S_0. That this relation also holds on S_1 follows from

$$|z_i^{(1)}(x)| = \left| \int_{a_i}^{x} \left[\sum_{j=1}^{j=n} A_{ij}(t, y^{(0)})\, \alpha_j + f_i(t, y^{(0)}) \right] dt \right|$$

$$< \left| \int_{a_i}^{x} [nBN(t) + N(t)]\, dt \right| < BR$$

while the inequalities

$$|z_i^{(k)}(x)| \leqq \left| \int_{a_i}^{x} \left[\sum_{j=1}^{j=n} |A_{ij}(t, y^{(k-1)})|\, |z_j^{(k-1)}| + |y_j^{(k-2)}|\, |A_{ij}(t, y^{(k-1)}) \right.\right.$$

$$\left.\left. - A_{ij}(t, y^{(k-2)})| + |f_i(t, y^{(k-1)}) - f_i(t, y^{(k-2)})| \right] dt \right|$$

$$< nBR^{k-1} \int_{c}^{d} [2 + (nB + r)/(1 - r)]\, N(t)\, dt < BR^k$$

show its validity on S_k, $(k = 2, 3, \cdots)$, as a consequence of its truth on S_{k-1}. Since the absolute value of each term of the series $\sum_{k=0}^{k=\infty} z_i^{(k)}(x)$ is less than the corresponding term of the convergent power series $B\sum_{k=0}^{k=\infty} R^k$, it follows that these series converge absolutely and uniformly on D to continuous functions $y_i(x)$. An application of Lebesgue's theorem as in the proof of theorem I yields the result that $y_1(x), \cdots, y_n(x)$ is a solution of system (1), (6) on the interval D.

This solution is unique. For suppose $u_1, \cdots, u_n$ is another solution of (7). For at least one value of x on D the continuous function $v(x) = \sum_{i=1}^{i=n} |y_i - u_i|$ is greater than zero and hence its maximum on D is greater than zero. Let $x = h$ be a point of D where $v(x)$ assumes this maximum. As a consequence of the above lemma and the convergence of the sequence $y_i^{(k)}(x)$, $(k = 0, 1, 2, \cdots)$, to $y_i(x)$, we have $|y_i(x)| \leqq (B + r/n)/(1 - r)$ on D.

$$|y_i(h) - u_i(h)| = \left| \int_{a_i}^{h} \left\{ [f_i(t, y) - f_i(t, u)] + \sum_{j=1}^{j=n} [A_{ij}(t, y) - A_{ij}(t, u)]\, y_j(t) + A_{ij}(t, u)\,[y_j - u_j] \right\} dt \right|$$

$$\leqq \int_c^d [M(t) + (Bn + r)\, L(t)/(1 - r) + L(t)]\, v(h)\, dt \leqq Rv(h)/n.$$

Summing for $i = 1, \cdots, n$ yields $v(h) \leqq Rv(h)$ or $1 \leqq R$. This however contradicts $R < 1$ and yields the uniqueness theorem.*

It remains to show that this solution can be extended over X. Let $y_i(c) = \beta_i$. By theorem I there exists on X (or any subinterval of X) a unique solution $u_1, \cdots, u_n$ of system (1) such that $u_i(c) = \beta_i$. This solution must therefore coincide with $y_1, \cdots, y_n$ on D and hence must satisfy the conditions (6). $u_1(x), \cdots, u_n(x)$ is the solution whose existence and uniqueness was to be established.

COROLLARY I. *If* $A_{ij}(x, 0, 0, \cdots, 0) \equiv 0$, $f_i(x, 0, 0, \cdots, 0) \equiv 0$, $\alpha_i \equiv 0$, $(i, j \equiv 1, \cdots, n)$, *then the unique solution of system* (1), (6) *is* $y_1(x) \equiv 0$, $\cdots$, $y_n(x) \equiv 0$.

COROLLARY II.† *There do not exist two non-identically equal solutions,* $(y_1, \cdots, y_n)$ *and* $(u_1, \cdots, u_n)$, *of* (1) *such that* $y_1 - u_1, \cdots, y_n - u_n$ *all have roots on a sub-interval of* X *of length* δ.

* This proof is similar to one given by Carathéodory (loc. cit., p. 673).

† See a theorem of Fite, Trans. Amer. Math. Soc., vol. 19 (1918), p. 351.

COROLLARY III.* *If* $A_{ij} \equiv 0$, $(i, j = 1, \cdots, n)$, *theorem* II *is valid if* δ *is chosen so that* $\left|\int_c^x N(t)\,dt\right| \leqq R/n$ *when* $|x - c| \leqq \delta$.

3. The solution as a function of the initial values. Consider $y_1, \cdots, y_n$ as functions of $(x, a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_n)$ in the $(2n+1)$-dimensional domain T: $c \leqq a_i \leqq d$, $c \leqq x \leqq d$, $-E \leqq \alpha_i \leqq +E$, $0 < E < B$ $(i = 1, \cdots, n)$.

THEOREM III. *The functions* $y_i(x, a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_n)$ *are continuous in all of their arguments throughout* T.

Proof. Examine the functions $y_i^{(k)}$, $(k = 0, 1, 2, \cdots)$, used in proving theorem II. Certainly $y_i^{(0)}$ has the continuity asserted in our theorem. We show by induction that $y_i^{(k)}$ must also be continuous throughout T if $y_i^{(k-1)}$ has this continuity.

$$y_i^{(k)} = \alpha_i + \int_{a_i}^{b}\left[\sum_{j=1}^{j=n} A_{ij}(t, y^{(k-1)})\, y_j^{(k-1)} + f_i(t, y^{(k-1)})\right] dt + \int_{b}^{x}\left[\sum_{j=1}^{j=n} A_{ij}(t, y^{(k-1)})\, y_j^{(k-1)} + f_i(t, y^{(k-1)})\right] dt,$$

where $x = b$ is any fixed point of D. Since $A_{ij}(t, y^{(k-1)})\, y_j^{(k-1)}$ and $f_i(t, y^{(k-1)})$ are continuous in $(y_1^{(k-1)}, \cdots, y_n^{(k-1)})$ for t fixed and since $(y_1^{(k-1)}, \cdots, y_n^{(k-1)})$ are continuous throughout T, it follows from well known theorems in function theory that the functions $A_{ij}(t, y^{(k-1)})\, y_j^{(k-1)}$ and $f_i(t, y^{(k-1)})$ are continuous in $(a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_n)$ for each fixed t on D and summable in t for each fixed $(a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_n)$ in the domain under consideration. From condition 3) on A_{ij} and f_i and the continuity of $y_j^{(k-1)}$ follows the boundedness of $A_{ij} y_j^{(k-1)}$ and f_i throughout T by a summable function of t. We may therefore apply a theorem due to Carathéodory that has recently been isolated by Ettlinger† to each of the integral terms ($2n+2$ in number) in the above expression for $y_i^{(k)}$ to get that $y_i^{(k)}$ is continuous throughout T.

Since the sequences of continuous functions $y_i^{(k)}$, $(i = 1, \cdots, n)$, $(k = 0, 1, 2, \cdots)$ converge‡ absolutely und uniformly on T to $y_i(x, a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_n)$, $(i = 1, \cdots, n)$, it follows that these limit functions are continuous throughout T.

4. Remarks. It is worthy of note, perhaps, that the length of the interval over which the points in theorem II are distributed may be increased in the neighborhood of any particular point of X. The nature

* This follows when we observe that in this case theorem II can be established without the aid of the lemma.

† Bull. Amer. Math. Soc., vol. 33 (1927), pp. 37–38.

‡ This is established implicitly in the proof of theorem II.

of the limitation on the length of this interval is well exhibited by the system $y_1' = ky_2$, $y_2' = -ky_1$, where k is a constant different from zero. Theorem II is valid for this system if y_1 and y_2 are assigned at any two points of the real axis whose distance apart is less than $\pi/2k$. If these points are at a distance apart equal to $\pi/2k$ the uniqueness fails and it is easily verified by solving the system that there is at least a one parameter family of solutions.

No difficulties arise in treating systems of the type (1) where A_{ij} and f_i depend upon one or more parameters. The methods of this paper can be used to establish Carathéodory's theorem* where A_{ij} and f_i are assumed to satisfy the same hypotheses that his functions f_k satisfy. In theorem II of the present paper δ will in general depend upon the parameters but this can be avoided in case the constant B can be chosen greater than $|\alpha_1|, \cdots, |\alpha_n|$ for all values of the parameters under consideration.

In connection with this work it is of interest to read an article of Niccoletti's† in which he treats a single equation of the nth order, and by making explicit use of the fact that his equation is of the nth order, he is able to establish an existence theorem where the solution and some of its derivatives are assigned at more than one point of the interval (it being necessary, however, that $y, y', \cdots, y^{(i-1)}$ be assigned at a point where $y^{(i)}$ is assigned). Niccoletti does not get uniqueness. There seems to be no overlapping of his theorems and those of the present paper.

A system of the type (1), (6) would arise in a consideration of dynamical systems where all of the coördinates are not assigned initial values simultaneously but this assignment is distributed over a time interval of length less than or equal to δ. Theorems II and III shed some light on the effects of such variation from the systems ordinarily considered.

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* Loc. cit., page 678, Satz 5.

† Atti della R. Accademie Delle Scienze Di Torino, vol. 33 (1897), p. 746 ff.

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ON A CLASS OF TAYLOR'S SERIES.

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1. In this paper I propose to study the analytic continuation of the function represented by the series $\sum_0^\infty \varphi(n) z^n$, where $\varphi(z)$ is an integral function of z, of order e^{kz}, where $k<\pi$. The paper contains several interesting results, the chief of which is embodied in Theorem II.

My thanks are due to Prof. G. H. Hardy, under whose guidance these results were obtained.

2. First of all, I shall establish the following theorem:

THEOREM I. *If* $a(z) = \sum_0^\infty \frac{c_\nu z^\nu}{\nu!}$ *is an integral function of* z, *and satisfies the relation*†

$$a(z) = O(e^{k|z|}),$$

where $k<\pi$, *everywhere in the plane, then,*‡ *if* $s=\sigma+it$, *and* $|t|\leqq\pi$,

$$F(s) = \sum_0^\infty a(n) e^{-sn}$$

$$= \frac{1}{2}c_0 + \sum_{\nu=0}^\infty \frac{c_\nu}{s^{\nu+1}} + \sum_{n=1}^\infty \sum_{\nu=0}^\infty c_\nu \left(\frac{1}{(s+2n\pi i)^{\nu+1}} + \frac{1}{(s-2n\pi i)^{\nu+1}} \right)$$

where the single series converges absolutely for $|s|>k$, *and the double series is absolutely and uniformly convergent in the strip* $t\leqq\pi$.

The Dirichlet's series is absolutely and uniformly convergent in the region $\sigma\geqq k+\delta>k$, $|t|\leqq\pi$; and making use of a formula of Lindelöf,§ we have the equation

$$(2.1)\quad F(s) = \frac{1}{2}c_0 + \int_0^\infty a(x) e^{-sx}\,dx + i\int_0^\infty \frac{a(i\varrho)\,e^{-si\varrho} - a(-i\varrho)\,e^{si\varrho}}{e^{2\pi\varrho}-1}\,d\varrho.$$

* Received July 20, 1928.

† k must necessarily be positive, if $a(z)$ does not reduce to a constant.

‡ S. Wigert, *Sur une certaine classe de séries de puissances*, Arkiv för Mat., Ast. och Fys., vol. 12, no. 7, 1918. Wigert obtains this result in a different way.

§ *Le calcul des résidus*, ch. III. The formula referred to is the following: *If* (i) $f(z)$ *is an analytic function of* z *in the half-plane* $R(z)\geqq 0$; (ii) $\sum_1^\infty f(\nu)$ *is convergent*; (iii) $f(z)$ *is subject to the condition*

$$\lim_{\varrho\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varrho\, e^{-2\pi\varrho|\sin\psi|}\, |f(\varrho e^{i\psi})|\,d\psi \to 0;$$

then

$$\sum_1^\infty f(\nu) = -\frac{1}{2}f(0) + \int_0^\infty f(x)\,dx + i\int_0^\infty \frac{f(it)-f(-it)}{e^{2\pi t}-1}\,dt.$$

We now wish to find expressions for the functions represented by the integrals on the right side of (2.1) in the form of series.

First consider the integral $\int_0^\infty a(x)\,e^{-sx}\,dx$. We have, by a theorem of Cauchy,

$$c_\nu = O((k+\varepsilon)^\nu), \qquad \text{for every } \varepsilon > 0. \tag{2.2}$$

Therefore, supposing s real and $>k$, in the first instance,

$$\int_0^\infty \left(\sum_0^\infty \frac{c_\nu x^\nu}{\nu!}\right) e^{-sx}\,dx = \sum_0^\infty \frac{c_\nu}{\nu!}\int_0^\infty x^\nu e^{-sx}\,dx = \sum_0^\infty \frac{c_\nu}{s^{\nu+1}},$$

since

$$\int_0^\infty \sum_0^\infty \left|\frac{c_\nu x^\nu}{\nu!}\right| |e^{-sx}|\,dx$$

converges like $\int_0^\infty O(e^{-(s-k-\varepsilon)x})\,dx$.

The series $\sum_0^\infty c_\nu/s^{\nu+1}$ is absolutely convergent outside the circle $|s| = k$, and represents there a regular and uniform function of s.

Consider next the second integral, whose integrand is regular at the lower limit. It is easy to show that this integral represents an analytic function of $s = \sigma + it$ in the strip $|t| \leqq \pi$. Suppose then, in the first instance, that s is purely imaginary and is equal to it, where $|t| \leqq \pi$. We have

$$\begin{aligned} I &\equiv i\int_0^\infty \frac{a(i\varrho)e^{-si\varrho} - a(-i\varrho)e^{si\varrho}}{e^{2\pi\varrho}-1}\,d\varrho \\ &= ic_0\int_0^\infty \frac{e^{-si\varrho}-e^{si\varrho}}{e^{2\pi\varrho}-1}\,d\varrho + i\int_0^\infty e^{-si\varrho}\left(\sum_{\nu=1}^\infty \frac{(i\varrho)^\nu}{\nu!}\,c_\nu\right)\frac{d\varrho}{e^{2\pi\varrho}-1} \\ &\qquad - i\int_0^\infty e^{si\varrho}\left(\sum_{\nu=1}^\infty \frac{(-i\varrho)^\nu}{\nu!}\,c_\nu\right)\frac{d\varrho}{e^{2\pi\varrho}-1} \\ &= I_1 + I_2' + I_2'', \end{aligned}$$

say. Now

$$\begin{aligned} I_1 &= ic_0\int_0^\infty (e^{-si\varrho}-e^{si\varrho})\left(\sum_{n=1}^\infty e^{-2\nu\pi\varrho}\right)d\varrho \\ &= ic_0\sum_{n=1}^\infty \int_0^\infty \{e^{-(2n\pi+si)\varrho} - e^{-(2n\pi-si)\varrho}\}\,d\varrho \qquad (2.3) \\ &= c_0\sum_{n=1}^\infty \left\{\frac{1}{(s-2n\pi i)} + \frac{1}{(s+2n\pi i)}\right\} = \sum_{n=1}^\infty \frac{2c_0 s}{s^2+4n^2\pi^2}, \end{aligned}$$

the expansion of $1/(e^{2\pi\varrho}-1)$ in infinite series and the subsequent integration term-by-term being justified by the fact* that

* Bromwich, *Infinite Series*, 2nd ed., pp. 500–502.

$$\int_0^\infty |e^{-si\varrho} - e^{si\varrho}| \sum_{n=1}^\infty |e^{-2n\pi\varrho}|\, d\varrho = 2\int_0^\infty \frac{\sinh t\varrho}{e^{2\pi\varrho}-1}\, d\varrho$$

is convergent.

Consider next I_2'.

$$I_2' = i\int_0^\infty e^{-si\varrho}\left(\sum_{\nu=1}^\infty \frac{(i\varrho)^\nu}{\nu!} c_\nu\right)\frac{d\varrho}{e^{2\pi\varrho}-1} = i\sum_{\nu=1}^\infty\sum_{n=1}^\infty \frac{(i)^\nu c_\nu}{\nu!}\int_0^\infty \varrho^\nu e^{-(2n\pi+si)\varrho}\, d\varrho$$

$$= \sum_{\nu=1}^\infty\sum_{n=1}^\infty \frac{c_\nu}{(s-2n\pi i)^{\nu+1}},$$

the inversion of the order of summation and integration being justified by the fact that the double series

$$\sum_{\nu=1}^\infty\sum_{n=1}^\infty \frac{|c_\nu|}{\nu!}\int_0^\infty |\varrho^\nu e^{-(2n\pi+si)\varrho}|\, d\varrho$$

is convergent and converges like $\sum_{\nu=1}^\infty\sum_{n=1}^\infty |c_\nu|/(2n\pi - t)^{\nu+1}$, c_ν satisfying the equality $c_\nu = O((k+\varepsilon)^\nu)$.

Similarly

$$I_2'' = \sum_{\nu=1}^\infty\sum_{n=1}^\infty \frac{c_\nu}{(s+2n\pi i)^{\nu+1}}.$$

Therefore

$$(2.4)\qquad I_2' + I_2'' = \sum_{\nu=1}^\infty\sum_{n=1}^\infty \left\{\frac{c_\nu}{(s+2n\pi i)^{\nu+1}} + \frac{c_\nu}{(s-2n\pi i)^{\nu+1}}\right\}.$$

Now to show that the double series is uniformly convergent in the strip $|t| \leqq \pi$, and hence represents an analytic function of s there, we observe that

$$|s \pm 2n\pi i| \geqq (2n-1)\pi \quad \text{for } n \geqq 1, \qquad |t| \leqq \pi,$$

so that

$$\left|\sum_{\nu=1}^\infty\sum_{n=1}^\infty\right| < 2\sum_{n=1}^\infty\sum_{\nu=1}^\infty \frac{|c_\nu|}{((2n-1)\pi)^{\nu+1}},$$

which converges by virtue of (2.2). Also the right side of (2.3) represents an analytic function of s in the strip $|t| \leqq \pi$.

Therefore the equality

$$I = I_1 + I_2' + I_2'' = \sum_{n=1}^\infty\sum_{\nu=0}^\infty c_\nu\left\{\frac{1}{(s+2n\pi i)^{\nu+1}} + \frac{1}{(s-2n\pi i)^{\nu+1}}\right\}$$

is true for all values of s in the strip $|t| \leqq \pi$, and we have the desired result

$$F(s) = \frac{1}{2}c_0 + \sum_{\nu=0}^{\infty} \frac{c_\nu}{s^{\nu+1}}$$

$$+ \sum_{n=1}^{\infty} \sum_{\nu=0}^{\infty} c_\nu \left\{ \frac{1}{(s+2n\pi i)^{\nu+1}} + \frac{1}{(s-2n\pi i)^{\nu+1}} \right\}, \qquad (|t| \leqq \pi). \tag{2.5}$$

This completes the proof of the theorem.

The most obvious consequence of the relation (2.5) is that $F(s)$ has no singularities in the strip $|t| \leqq \pi$ other than those of the function represented by the Laurent's series $\sum_0^\infty c_\nu/s^{\nu+1}$.

Remarks. (α) A particular case of Carlson's theorem,* *viz., that if* $a(z)$ *is an integral function, which satisfies everywhere the relation* $a(z) = O(e^{k|z|})$, $k < \pi$, *and vanishes for positive integral values of* z, *then* $a(z) \equiv 0$, follows from (2.5).

For if $a(z) \not\equiv 0$, we have from (2.5)

$$c_0 = \frac{1}{2}c_0 + \sum_{\nu=0}^{\infty} \frac{c_\nu}{s^{\nu+1}} + \sum_{n=1}^{\infty} \sum_{\nu=0}^{\infty}$$

or

$$\frac{1}{2}c_0 = \sum_{\nu=0}^{\infty} \frac{c_\nu}{s^{\nu+1}} + \sum_{n=1}^{\infty} \sum_{\nu=0}^{\infty},$$

which is impossible, unless all the c's are zero, since the function represented by the first series must have a singularity either on or within the circle $|s| = k$, and the double series represents a regular function in the strip $|t| \leqq \pi$. Therefore, all the c's are zero, i. e., $a(z) \equiv 0$.

It also follows from this that *if* $a(z)$ *is an integral function, not identically zero, and satisfies the relation*

$$a(z) = O(e^{k|z|}), \qquad k < \pi,$$

then the Taylor's series $\sum_0^\infty a(n)x^n$ *can not represent an integral function of* x.

This result is no longer true if $a(z)$ be taken to be regular in the half-plane $R(z) \geqq 0$ only. Thus, for example, the Taylor's series $\sum_0^\infty x^n/\Gamma(1+n)$ represents an integral function of x. Here $\varphi(z) = 1/\Gamma(1+z)$ is regular in the half-plane $R(z) \geqq 0$, and $= O\left(e^{(\frac{\pi}{2}+\varepsilon)|z|}\right)$, throughout this region, provided $|z|$ exceeds a certain finite limit, ε being an arbitrary positive number.

* *Sur une classe de séries de Taylor*, Thése, Upsalla, 1914, p. 58. Also see Wigert, *Sur un théorème concernant les fonctions entières*, Arkiv för Mat., 1916.

(β) The equation (2.5) has been obtained on the assumption that $|t| \leqq \pi$. But the right side represents a regular and uniform function* of s all over the plane outside the circles of radius k round the singular points $s = \pm 2ni$, $n = 0, 1, 2, 3, \cdots$.

Hence (2.5) persists for all such values of s for which the right side exists.

3. THEOREM 2. *A necessary and sufficient condition that* $f(z) = \sum_0^\infty a_n z^n$ *should be regular and uniform in the region exterior to the curve*

$$\left(\log \frac{1}{r}\right)^2 = \gamma^2 - \theta^2, \qquad (0 < \gamma < \pi), \tag{3.1}$$

and including infinity, and have this curve as a barrier of singularities, is that there should be an integral function $a(z) = \sum_0^\infty c_\nu z^\nu/\nu!$ *which assumes the values* $a_1, a_2, a_3, \cdots$ *for* $z = 1, 2, 3, \cdots$, *and for which the associated Taylor's series* $\sum_0^\infty c_\nu z^\nu$ *has the circle of its convergence* $|z| = 1/\gamma$ *as a barrier of singularities.*

First I shall show that the condition is sufficient.

If the circle of convergence of the series $\sum c_\nu z^\nu$ is precisely $1/\gamma$, then the integral function $a(z) = \sum_0^\infty c_\nu z^\nu/\nu! = O(\exp.(\gamma + \varepsilon)|z|)$, for every $\varepsilon > 0$, and all sufficiently large values of $|z|$.

Now

$$f(z) = (a_0 - c_0) + \sum_{n=0}^\infty \left(\sum_{\nu=0}^\infty \frac{c_\nu}{\nu!} n^\nu \right) z^n. \tag{3.2}$$

Putting $z = e^{-s}$, $s = \sigma + it$, and supposing $|t| \leqq \pi$, we have from (2.5)

$$f(z) = \left(a_0 - \frac{1}{2}c_0\right) + \sum_{\nu=0}^\infty \frac{c_\nu}{s^{\nu+1}} + \sum_{n=1}^\infty \sum_{\nu=0}^\infty c_\nu \left(\frac{1}{(s + 2n\pi i)^{\nu+1}} + \frac{1}{(s - 2n\pi i)^{\nu+1}} \right), \tag{3.3}$$

where the first series on the right side represents a regular and uniform of s, outside the circle $|s| = \gamma$, and for which the circle is a barrier of singularities, since $\sum_0^\infty c^\nu z^\nu$ has the circle $z = 1/\gamma$ as its singular line; and the double series represents a regular function of s in the strip $|t| \leqq \pi$.

Transforming these results to the z-plane, we observe that the double series represents a regular and uniform function of z, when a cut has been made along the negative real axis, with the possible singularities at 0 and ∞. The first series also represents a regular and uniform function of z in the cut plane, with the possible singularities at 0 and for this function the curve,

* This is in accordance with Cramér's Théorème V, pp. 30–33, *Sur une classe de séries de Dirichlet*, Thése, Upsalla, 1917.

$$|s| = \gamma, \text{ or } \left(\log \frac{1}{r}\right)^2 = \gamma^2 - \theta^2, \qquad (z = re^{i\theta}),$$

forms a barrier of singularities.

We have thus proved that the curve (3.1) is a barrier of singularities for the function $f(z)$.

Now we proceed to show that $f(z)$ is regular and uniform in the region exterior to the curve (3.1), and including infinity.

We can write, following Wigert,*

$$(3.4) \qquad f(z) = (a_0 - c_0) + \frac{c_0}{1-z} + \sum_{\nu=1}^{\infty} (-1)^\nu \frac{c^\nu}{\nu!} P_{(\nu+1)}\left(\frac{1}{1-z}\right),\dagger$$

where

$$P_{(\nu+1)}(t) = (t-1) \sum_{m=1}^{\nu} (-1)^m m!\ C_{\nu-m}^{(-m)}\ t^m,$$

and C_μ^{-m} is the coefficient defined by the development

$$\frac{1}{(1-x)(1-2x)\cdots(1-mx)} = \sum_{\mu=0}^{\infty} C_\mu^{(-m)}\ x^\mu.$$

Now each term of the series on the right side of (3.4) is a uniform function of $1/(1-z)$, and the series converges uniformly in any finite region exterior to the curve (3.1); its sum is zero both at the origin and at infinity, hence it represents an analytic function of $1/(1-z)$ outside the curve (3.1). It follows, therefore, that $f(z)$ is a regular and uniform function of z outside the curve (3.1) and assumes the values $(a_0 - c_0)$ and a_0 for $z = \infty$ and $z = 0$, respectively.

The first half of the theorem is, therefore, proved.

Now we shall show that the condition is necessary. That is, we shall now suppose that $f(z)$ is uniform and regular throughout the region exterior to the curve (3.1) and including infinity; and further that the curve (3.1) forms a barrier of singularities for the function $f(z)$. In these circumstances, we have to show that there is an integral function $a(z) = \sum_0^\infty c_\nu z^\nu/\nu!$ which assumes the values, $a_1, a_2, \cdots$ for $z = 1, 2, \cdots$, and for which the associated series $\sum_0^\infty c_\nu z^\nu$ has the circle of its convergence $|z| = 1/\gamma$ as a barrier of singularities.

In what follows, I proceed on lines of argument used by Prof. Hardy.‡ We have

* Loc. cit., pp. 10–11.

† This result can also be obtained by using theorem V (p. 30) of Cramér's thesis, loc. cit.

‡ *On two theorems of F. Carlson and S. Wigert.* Acta Math., vol. 42, pp. 337–38. The results of this paper bear a close connection to those obtained here.

$$a_n = \frac{1}{2\pi i}\int \frac{f(z)}{z^{1+n}}\,dz, \tag{3.5}$$

the path of integration being a closed contour surrounding the origin but leaving the curve (3.1) entirely to its right. It is evident that if $n \geqq 1$,* we may deform this contour into that formed by (i) the right portion of the circle $r = e^{-\delta}$, where $\delta > \gamma$, and (ii) the parts of the radii $\theta = \pm\lambda$, where

$$\gamma < \lambda < \pi.$$

Let us now effect the transformation $z = e^{-s}$, $s = \log\frac{1}{z}$ where that value of the logarithm is chosen whose imaginary part lies between $-\pi$ and π.

The contour in the z-plane becomes a contour in the s-plane formed by three sides of an infinite rectangle whose vertices are $\delta \pm \lambda i$, $-\infty \pm \lambda i$, and we have

$$a_n = \frac{1}{2\pi i}\int e^{ns} f(e^{-s})\,ds, \tag{3.6}$$

the integration being effected along this contour, which now lies entirely outside the circles $|s| = \gamma$ and $|s \pm 2k\pi i| = \gamma$, $(k = 1, 2, 3, \cdots)$. Now $f(e^{-s})$ is regular in the region exterior to the circles just referred to. It is obvious that the contour may now be deformed into the circle $s = |l|$, where $l > \gamma$ but $< 2\pi - \gamma$. Also we can express $f(e^{-s})$ in the form of a Laurent's series

$$\sum_{\nu=0}^{\infty} \frac{d_{\nu+1}}{s^{\nu+1}} + \sum_{n=0}^{\infty} b_n s^n,$$

where the first series is convergent for $|s| > \gamma$ and has the circle $|s| = \gamma$ as a barrier of singularities, and the second series is convergent for $|s| < 2\pi - \gamma$.

We have, therefore,

$$a_n = \frac{1}{2\pi i}\int_{|s|=l} e^{ns}\left(\sum_0^{\infty}\frac{d_{\nu+1}}{s^{\nu+1}}\right)ds = \sum_{\nu=0}^{\infty} d_{\nu+1}\cdot\frac{1}{2\pi i}\int_{|s|=l}\frac{e^{ns}\,ds}{s^{\nu+1}} = \sum_{\nu=0}^{\infty}\frac{d_{\nu+1}n^{\nu}}{\nu!}.$$

That is, $a(z) = \sum_0^{\infty} z^{\nu} d_{\nu+1}/\nu!$, which is an integral function† of z, and is such that the associated Taylor's series $\sum_0^{\infty} d_{\nu+1} z^{\nu}$ has the circle of its

* The argument fails for $n = 0$, unless $f(\infty) = 0$.

† The function $a(z)$ is unique in virtue of the result established in remark (α), § 2.

convergence $|z| = 1/\gamma$ as a barrier of singularities, since the series $\sum_0^\infty d_{\nu+1}/s_{\nu+1}$ has the circle $|s| = \gamma$ as a barrier of singularities.

This completes the proof of the theorem.

As an example, consider the integral function $a(z) = \sum_0^\infty (kz)^{n^2}/n^2!$ where $0 < k < \pi$. Then the associated Taylor's series $\sum_0^\infty (kz)^{n^2}$ has the circle $|z| = 1/k$ as a singular line, and $f(z) = \sum_{n=0}^\infty \left(\sum_{\nu=0}^\infty (kn)^{\nu^2}/\nu^2! \right) z^n$ is a regular and uniform outside the curve

$$\left(\log \frac{1}{r}\right)^2 = k^2 - \theta^2,$$

which is the barrier of singularities for $f(z)$. $f(z) = 0$ at infinity, and $f(0) = 1$.

On the other hand, if $a(z) = \sum_0^\infty (kz)^n/n!$, the associated Taylor's series $\sum_0^\infty (kz)^n$ has the same circle of convergence, viz., $|z| = 1/k$, but the function $f(z) = 1/(1 - e^k z)$, and has a simple pole at $z = e^{-k}$.

4. Only trivial modifications in the proof of the last theorem are required to prove the following well-known theorem of Wigert.*

THEOREM 3. *A necessary and sufficient condition that* $f(z) = \sum_0^\infty a_n z^n$ *should be an integral function of* $1/(1-z)$ *is that there should be an integral function* $a(z)$ *of order* 1 *and type* 0 *which takes the values* $a_1, a_2, a_3 \cdots$ *for* $z = 1, 2, 3, \cdots$.

In my thesis† I have shown *that the alternative necessary and sufficient condition that* $f(z) = \sum_0^\infty a_n z^n$, *not regular for* $z = 1$, *be an integral function of* $1/(1-z)$ *is that the associated integral function* $a(z) = \sum_0^\infty a_n z^n/n!$ *should satisfy the asymptotic equality*

$$\limsup_{\varrho \to \infty} \frac{\log |a(\varrho e^{i\psi})|}{\varrho} = \cos \psi \qquad (|\psi| \leqq \pi).$$

As this is a particular case of a general theorem which I wish to consider in another paper, I content myself with only mentioning it here.

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* See, G. H. Hardy, loc. cit. References to the extensive literature connected with this theorem will be found in this paper.

† For the degree of Doctor of Philosophy, Oxford, 1927, (not yet published).

QUARTIC EQUATIONS WITH CERTAIN GROUPS.*

By Raymond Garver.

Seidelmann† has given expressions for the most general rational quartic equations having certain prescribed groups for the field of rational numbers. His article is simply a summary of his Erlangen Dissertation, and does not describe his methods in detail. However, they involve manipulations on rational functions of the roots of a quartic equation which belong to the desired group. His principal results are the following:

I. A general expression for the rational quartic (with second term removed) whose group for the field of rational numbers is the group $G_4 \equiv 1$, (12)(34), (13)(24), (14)(23) is

$$(1) \quad x^4 - 2x^2(e+f+efg^2) - 8efgx + (e-f-efg^2)^2 - 4ef^2g^2 = 0,$$

where e, f, g are rational parameters. For this equation to be irreducible, none of e, f, ef may be the square of a rational number.

II. A similar expression corresponding to a cyclic group C_4 is

$$(2) \quad \begin{aligned} &x^4 - 2x^2(1+e^2)(f^2+g) - 4fg(1+e^2)x \\ &\quad + (1+e^2)[(1+e^2)(f^2-g)^2 - g^2] = 0. \end{aligned}$$

In this equation g must be $\neq 0$ and $(1+e^2)$ must not be the square of a rational number.

III. A similar expression corresponding to a group G_8 of order 8 is

$$(3) \quad x^4 - 2x^2(e^2f+g) - 4efx + (e^2f-g)^2 - f = 0.$$

In order that this equation may be irreducible, and that its group may not reduce to G_4 or C_4, it is necessary that none of the quantities f, g^2-f, $f(g^2-f)$ be the square of a rational number.

The purpose of this paper is to obtain the above results in quite a different manner, by use of the Tschirnhaus transformation. This method seems to be worth presenting for two reasons: first, it acts as a unifying influence in the present work on the quartic equation, since a considerable part of the process can be carried through for all three cases simultaneously

* Received February 27, 1928.

† Math. Annalen, vol. 18 (1918), pp. 230–3.

and, second, it promises to be of some value in considering similar problems for equations of higher degree.

The method of the present paper may be summarized in a few words: to obtain a general expression for the equation of nth degree with rational coefficients having a given group for the field of rational numbers, proceed as follows.

1. Obtain a simple normal form to which all equations with the desired group can be reduced by means of a Tschirnhaus transformation with rational coefficients.

2. To this normal form, apply the general Tschirnhaus transformation of degree $(n-1)$ with rational coefficients.

By Lemmas 1 and 2 below the transformed equation thus obtained will be an expression of the type sought. A number of details will always have to be taken care of before the result is obtained in the most convenient form; these apparently will have to be worked out separately for different cases.

The required details for the quartic equation are given in the following six lemmas. The first two are general, yet easily proved, properties which really form the foundation of the method of this paper. The third and fourth are simply restatements of known theorems, the former furnishing the normal form which we shall make use of. The fifth and sixth are theorems which seem to be given here for the first time; it is necessary to use them in order to obtain the results in the simplest form.

LEMMA 1. *If an equation with rational coefficients (without a double root) is transformed into another such equation by a Tschirnhaus transformation with rational coefficients, the second can be transformed into the first by a similar transformation.*

The proof is very simple. Dickson* shows how to obtain a fractional transformation leading from the second equation back to the first, and in the present case its coefficients will be rational numbers. Further, the process of replacing a fractional transformation by the equivalent integral (Tschirnhaus) transformation introduces no irrationalities; hence the lemma is proved.

LEMMA 2. *If an equation with rational coefficients (without a double root) is transformed into another such equation by a Tschirnhaus transformation with rational coefficients, the group of the transformed equation for the field of rational numbers is the same as the group of the original equation.*

Let the group of the given equation be K. Every rational function with rational coefficients of the roots of the transformed equation which is

* Modern Algebraic Theories, Chicago, 1926, p. 211.

equal to a rational number is unaltered in value by every substitution of K, since it is also a rational function of the roots of the original equation which is equal to a rational number. Hence it follows* that K is a subgroup of the group of the transformed equation. But by Lemma 1 we can go from the transformed back to the original equation and reverse the reasoning. Thus the lemma is proved.

LEMMA 3. *Any quartic equation with rational coefficients whose group is G_4, C_4 or G_8 (as defined above) can be transformed by a Tschirnhaus transformation with rational coefficients into the form $x^4 + a_2 x^2 + a_4 = 0$, where a_2 and a_4 are rational.*

This theorem seems to be due to Wiman, who proved it in a more general form.† I have recently given a direct proof of this particular case.‡

LEMMA 4. *The equation obtained in Lemma 3 has the group G_4 in case a_4 is the square of a rational number r, provided also that none of the quantities $a_2^2 - 4r^2$, $2r - a_2$, $-2r - a_2$ is the square of a rational number. (These last restrictions insure the irreducibility of the equation.)*

The equation has the group C_4 if $a_4(a_2^2 - 4a_4)$ is the square of a rational number, while neither separate factor is such a square. And if none of the quantities a_4, $a_2^2 - 4a_4$, $a_4(a_2^2 - 4a_4)$ is a perfect square, the group is G_8.

The criteria which refer to irreducibility are easily enough obtained. The others are developed in a paper by Bucht,§ and are stated in a number of other places.

LEMMA 5. *There are two two-parameter families of Tschirnhaus transformations which transform $x^4 + a_2 x^2 + a_4 = 0$ into another quartic with second and fourth coefficients missing. These are*

$$z = b_0 x^3 + b_2 x \quad (b_0, b_2 \text{ arbitrary}), \tag{4}$$

$$z = b_0 x^3 + b_1 x^2 + b_0(a_2 \pm a_4^{1/2})\, x + \frac{b_1 a_2}{2} = 0 \quad (b_0, b_1 \text{ arbitrary}). \tag{5}$$

That transformation (4) satisfies the condition of the lemma is obvious. Transformation (5) will not be used in this paper; it is merely given to make the lemma complete. It is again easy to show that $\Sigma z = \Sigma z^3 = 0$, and that the condition of the lemma is satisfied.

LEMMA 6. *Any Tschirnhaus transformation which transforms $x^4 + a_2 x^2 + a_4 = 0$ into a quartic whose second coefficient is zero can be expressed as the product*

* Dickson, Modern Algebraic Theories, Theorem 5, p. 167.

† Ark. för Mat. Astr. Fys., vol. 3 (1906–7), No. 28.

‡ Bull. Amer. Math. Soc., vol. 34 (1928), pp. 73–4.

§ Ark. för Mat. Astr. Fys., vol. 6 (1910–11), No. 30.

of two transformations $z = b_0 x^3 + b_2 x$, $y = k_0 z^2 + z + k_0 k_2$, *where* k_2 *is chosen so that* $\Sigma y = k_0(\Sigma z^2 + 4k_2) = 0$.

The most general Tschirnhaus transformation which transforms $x^4 + a_2 x^2 + a_4 = 0$ into a quartic whose second coefficient is zero is

$$y = c_0 x^3 + c_1 x^2 + c_2 x + c_3. \tag{6}$$

where c_3 is chosen so that $\Sigma y = c_1 s_2 + 4c_3 = 0$. If we now form the product of the two transformations of lemma 6 by substituting for z its value in terms of x and reducing by means of the quartic, we have

$$y = b_0 x^3 + k_0 x^2 [b_0^2(a_2^2 - a_4) - 2b_0 b_2 a_2 + b_2^2] + b_2 x + k_0[b_0^2 a_2 a_4 - 2b_0 b_2 a_4 + k_2]. \tag{7}$$

To prove the lemma, we now choose $b_0 = c_0$, $b_2 = c_2$. The proof is completed if we can show that the two equations in k_0,

$$\begin{aligned} k_0[b_0^2(a_2^2 - a_4) - 2b_0 b_2 a_2 + b_2^2] &= c_1, \\ k_0[b_0^2 a_2 a_4 - 2b_0 b_2 a_4 + k_2] &= c_3, \end{aligned} \tag{8}$$

are consistent. However it is a matter of simple algebra to do this, requiring merely the substitution for k_2 and c_3 of their values as given above, the replacement of Σz^2 by its value $b_0^2 s_6 + 2b_0 b_2 s_4 + b_2^2 s_2$, and the use of Newton's formulas which give the s_i in terms of the a_i.

We are now ready to proceed to the main problem of the paper. Lemma 3 provides the rational normal form required by our method, and we apply to it the most general Tschirnhaus transformation with rational coefficients. However, to obtain convenient results, we use Lemma 6. That is, we may first apply $z = b_0 x^3 + b_2 x$ (b_0, b_2 arbitrary rational numbers), and then $y = k_0 z^2 + z + k_0 k_2$ ($\Sigma z^2 + 4k_2 = 0$). But by Lemma 5, the former of these does not change the form of the quartic in the least. Hence we may replace the symbol z by x, and still denote the first transformed equation by $x^4 + a_2 x^2 + a_4 = 0$. To this equation we now apply $y = k_0 x^2 + x + k_0 k_2$, where k_0 is an arbitrary rational number, and $k_2 = a_2/2$.

The transformed equation is of the form

$$y^4 + A_2 y^2 + A_3 y + A_4 = 0, \tag{9}$$

where A_2, A_3, A_4 can be computed by any one of several known methods. Their values are

$$\begin{aligned} A_2 &= a_2 - (a_2^2 - 4a_4) k_0^2/2, \\ A_3 &= -(a_2^2 - 4a_4) k_0, \\ A_4 &= (a_2^2 - 4a_4)^2 k_0^4/16 + a_2(a_2^2 - 4a_4) k_0^2/4 + a_4. \end{aligned} \tag{10}$$

Equation (9) is an expression for the most general quartic whose group is G_4, C_4 or G_8.

In conclusion it is desirable to distinguish between the three cases and show how Seidelmann's results can be obtained. Lemma 4 forms the basis of this part of the work.

If the group of the normal form of Lemma 3 is G_4 it is easy to see that it is of the form $x^4-2(e+f)x^2+(e-f)^2=0$. The irreducibility conditions now are simply that none of e, f, ef be the square of a rational number. Also $a_2^2-4a_4=16ef$, and if we put $k_0=g/2$ equation (9) becomes exactly Seidelmann's equation (1), except that y appears in place of x.

If the group of our normal form is C_4 it can be written in the form $x^4-2(1+e^2)gx^2+(1+e^2)g^2e^2=0$. For when the group is C_4, $a_4(a_2^2-4a_4)=r^2$ (r rational), and we have only to take $e=2a_4/r$, $g=-a_2/2(1+e^2)$. The restrictions that neither a_4 nor $(a_2^2-4a_4)$ be the square of a rational number are now both satisfied in case $(1+e_2)$ is not such a square, and $g\neq 0$. In fact, $a_2^2-4a_4=4g^2(1+e^2)$. Substituting these values in (10) and putting $k_0=f/g$, we find that equation (9) reduces to Seidelmann's equation (2).

When the group is G_8 the normal form can be written as $x^4-2gx^2+g^2-f=0$. This representation is clearly justifiable. To satisfy the three restrictions of lemma 4, none of f, g^2-f, $f(g^2-f)$ may be the square of a rational number. We have $a_2^2-4a_4=4f$, and putting $k_0=e$ Seidelmann's equation (3) is at once obtained.

CANONICAL FORMS OF PLANE CUBIC CURVES UNDER EUCLIDEAN TRANSFORMATIONS.*

By R. S. Burington and H. K. Holt.

1. **Introduction.** The first systematic classification of plane cubic curves was given by Newton in an appendix to his *Optiks*, first published in 1704.† By a difficult analysis, Newton showed that every cubic is included in one of the four "canonical forms":

$$\begin{aligned} xy^2 + ey &= ax^3 + bx^2 + cx + d, \\ xy &= ax^3 + bx^2 + cx + d, \\ y^2 &= ax^3 + bx^2 + cx + d, \\ y &= ax^3 + bx^2 + cx + d, \end{aligned}$$

the axes being oblique.

Plücker advanced an alternative classification using homogeneous coördinates.‡ An exhaustive analysis by Cayley§ is based upon these same "canonical forms" of Newton. The authors have been unable to find any canonical forms for the cubic essentially different from those given by Newton.

An essential objection to the "canonical forms" of Newton is the fact that they are not canonical forms in the modern sense of the word. That is, in not all cases can the coefffcients be obtained from invariants involving the original coefficients, or, what is the same thing, the transformations which leave a "canonical form" unaltered in form sometimes are more than finite in number.

In the present paper the general non-homogeneous binary cubic polynomial is reduced to eight canonical forms, and the group of transformations which leave each of these forms unaltered is determined. Newton's method does not lead to an easy classification by means of invariants, so an entirely different method was used. This may be described roughly as follows: the cubic has a quadratic covariant C_2, and when $C_2 = 0$ is a proper

* Received March 13, 1928. Presented to the American Mathematical Society, December 28, 1927.

† See an article by W. W. R. Ball, Proc. Lond. Math. Soc., vol. 22 (1890–1991), pp. 104–143.

‡ *System der analytischen Geometrie und der Theorie der Curven dritter Ordnung*, Berlin, 1835. This paper has not been consulted by the authors, but is summarized in the paper by Cayley.

§ Trans. Cambr. Phil. Soc., vol. 11, Pa. I, (1866), pp. 81–128. Collected Works, vol. 5, pp. 354–400.

conic in the familiar canonical form, the cubic is defined to be in canonical form. Degenerate cases are treated by special methods.

The authors wish to acknowledge their indebtedness to Professor C. C. MacDuffee of the Ohio State University under whose direction this investigation was carried out.

2. Invariants and covariants. We consider the binary cubic

$$\varphi = ax^3+3bx^2y+3cxy^2+dy^3+3ex^2+6fxy+3gy^2+3hx+3iy+j,$$

with real coefficients, and the real euclidean transformations

$$(1)\qquad x = x'\cos\omega - y'\sin\omega + \alpha, \qquad y = x'\sin\omega + y'\cos\omega + \beta.$$

Transformations (1) induce upon the coefficients of φ the transformations

$$\begin{aligned}
a' &= a\cos^3\omega + 3b\cos^2\omega\sin\omega + 3c\cos\omega\sin^2\omega + d\sin^3\omega,\\
b' &= b\cos^3\omega + (2c-a)\cos^2\omega\sin\omega + (d-2b)\cos\omega\sin^2\omega - c\sin^3\omega,\\
c' &= b\sin^3\omega + (a-2c)\sin^2\omega\cos\omega + (d-2b)\sin\omega\cos^2\omega + c\cos^3\omega,\\
d' &= -a\sin^3\omega + 3b\sin^2\omega\cos\omega - 3c\sin\omega\cos^2\omega + d\cos^3\omega,\\
e' &= (a\alpha+b\beta+e)\cos^2\omega + (b\alpha+c\beta+f)\sin 2\omega + (c\alpha+d\beta+g)\sin^2\omega,\\
f' &= (b\alpha+c\beta+f)\cos 2\omega + (c\alpha+d\beta+g-a\alpha-b\beta-e)\sin\omega\cos\omega,\\
(2)\quad g' &= (a\alpha+b\beta+e)\sin^2\omega - (b\alpha+c\beta+f)\sin 2\omega + (c\alpha+d\beta+g)\cos^2\omega,\\
h' &= (a\alpha^2+2b\alpha\beta+c\beta^2+2f\beta+2e\alpha+h)\cos\omega\\
&\qquad + (b\alpha^2+2c\alpha\beta+d\beta^2+2f\alpha+2g\beta+i)\sin\omega,\\
i' &= (b\alpha^2+2c\alpha\beta+d\beta^2+2f\alpha+2g\beta+i)\cos\omega\\
&\qquad - (a\alpha^2+2b\alpha\beta+c\beta^2+2e\alpha+2f\beta+h)\sin\omega,\\
j' &= a\alpha^3+3b\alpha^2\beta+3c\alpha\beta^2+d\beta^3\\
&\qquad +3e\alpha^2+6f\alpha\beta+3g\beta^2+3h\alpha+3i\beta+j.
\end{aligned}$$

The following functions are invariants and covariants of φ under (1):

$$I_1 = (a+c)^2+(b+d)^2, \qquad I_2 = \xi+\eta, \qquad I_3 = \xi\eta-\lambda^2,$$

$$I_4 = \begin{vmatrix} \xi & \lambda & \mu \\ \lambda & \eta & \nu \\ \mu & \nu & \zeta \end{vmatrix},$$

$$C_2 = \xi x^2+2\lambda xy+\eta y^2+2\mu x+2\nu y+\zeta,$$
$$C_3 = (a+c)x+(b+d)y+(e+g),$$

where

$$\xi = ac-b^2, \qquad \eta = bd-c^2, \qquad \zeta = eg-f^2, \qquad 2\lambda = ad-bc,$$
$$2\mu = ce+ag-2bf, \qquad 2\nu = de+bg-2cf.$$

We obtain the further invariants and covariants I_5, I_6, I_7, I_8, C_1, C_4, and C_5 by substituting $A, B, \cdots, J$ for $a, b, \cdots, j$ in I_1, I_2, I_3, I_4, φ, C_2, C_3, respectively, where

$$\begin{aligned}
A &= \xi h - af^2 + 2bef - ce^2, \\
3B &= 2\lambda h + \xi i + 2beg - 2afg + bf^2 - de^2, \\
3C &= 2\lambda i + \eta h + 2ceg - 2def + cf^2 - ag^2, \\
D &= \eta i - df^2 + 2cfg - bg^2, \\
3E &= (ag - cd)h + 2(be - af)i + \xi j - \zeta e, \\
6F &= (3bg - 2cf - de)h + (3ce - 2bf - ag)i + 2\lambda j - 2\zeta f, \\
3G &= 2(cg - df)h + (de - bg)i + \eta j - \zeta g, \\
3H &= 2\mu j - \zeta h - ai^2 + 2bhi - ch^2, \\
3I &= 2\nu j - \zeta i - bi^2 + 2chi - dh^2, \\
J &= \zeta j - ei^2 + 2fhi - gh^2.
\end{aligned}$$

We may explain briefly how these functions were obtained.* The covariant C_1 is the hessian of φ when φ is written in homogeneous form. Since C_1 is likewise cubic, the substitution of its coefficients $A, B, \cdots, J$ for those of φ in any invariant or covariant of φ produces another invariant or covariant of φ. The hessian of φ is C_2, and this quadratic has the well-known invariants, I_2, I_3 and I_4. The method of Lie† was now employed, and the partial differential equations defining the group (2) were actually solved to obtain C_3. The invariant I_1 follows from C_3.

3. **The generic canonical form.** We now consider the problem of reducing $\varphi = 0$ to canonical forms. We shall consider only the cases where φ is actually of the third degree, i. e., where the invariant

$$I_1 - 2I_2 = a^2 + 3b^2 + 3c^2 + d^2 \tag{4}$$

is not zero.

The covariant C_2 is of the second degree when and only when the invariant

$$I_2^2 - 2I_3 = \xi^2 + \eta^2 + 2\lambda^2 \tag{5}$$

is not zero. In this event, $C_2 = 0$ can be reduced to the form

$$\xi x^2 + \eta y^2 + \zeta = 0, \qquad \xi\eta \neq 0, \tag{6}$$

if and only if $I_3 \neq 0$.‡

* The authors have noticed a paper "Euclidean invariants of the plane cubic" by P. J. Federico and P. R. Neff read before the Maryland section of the Association, May 7, 1927, which, however, has not yet been published.

† C. C. MacDuffee, *Euclidean invariants of second degree curves*, Amer. Math. Monthly, vol. 33 (1926), pp. 243–252.

‡ MacDuffee, loc. cit., p. 248, l.8.

A transformation which reduced $C_2 = 0$ to (6) reduces $\varphi = 0$ to a form in which

$$\lambda = \mu = \nu = 0, \qquad \xi\eta \neq 0.$$

From these equations it follows directly that $f = 0$ and that

$$(7) \qquad a : c = b : d = e : -g = -\xi : \eta = k,$$

where k is finite and not zero.

With $\lambda = 0$, the invariant $I_2^2 - 4I_3$ reduces to $(\xi - \eta)^2$ and hence vanishes when and only when $\xi = \eta$. For our generic case, then, we assume that

$$2I_3 \neq I_2^2, \qquad 4I_3 \neq I_2^2, \qquad I_3 \neq 0,$$

and have the canonical form

$$(8) \quad \varphi = kcx^3 + 3kdx^2y + 3cxy^2 + dy^3 - 3kgx^2 + 3gy^2 + 3hx + 3iy + j,$$

where the parameters are all independent except that k is neither 0 nor -1 and $c^2 - kd^2 \neq 0$.

Transformations (1) induce upon the coefficients of C_2 the following transformations:

$$(9) \quad \begin{aligned}
\xi' &= \xi\cos^2\omega + 2\lambda\sin\omega\cdot\cos\omega + \eta\sin^2\omega, \\
\lambda' &= -\xi\sin\omega\cdot\cos\omega + \lambda(\cos^2\omega - \sin^2\omega) + \eta\sin\omega\cdot\cos\omega, \\
\eta' &= \xi\sin^2\omega - 2\lambda\sin\omega\cdot\cos\omega + \eta\cos^2\omega, \\
\mu' &= (\mu + \xi\alpha + \lambda\beta)\cos\omega + (\nu + \lambda\alpha + \eta\beta)\cdot\sin\omega, \\
\nu' &= -(\xi\alpha + \lambda\beta + \mu)\cdot\sin\omega + (\nu + \lambda\alpha + \eta\beta)\cdot\cos\omega, \\
\zeta' &= \xi\alpha^2 + 2\lambda\alpha\beta + \eta\beta^2 + 2\mu\alpha + 2\nu\beta + \zeta.
\end{aligned}$$

Since

$$\lambda = \mu = \nu = 0, \qquad \xi\eta \neq 0,$$

we have

$$\lambda' = -\xi\sin\omega\cdot\cos\omega + \eta\cos\omega\cdot\sin\omega.$$

To keep the property $\lambda' = 0$ we must have

$$(\xi - \eta)\cdot\sin\omega\cdot\cos\omega = 0,$$

and since $\xi - \eta \neq 0$, we must have $\omega = 0$, $\pi/2$, π, or $3\pi/2$. From the relations

$$\begin{aligned}
\mu' &= +\xi\alpha\cos\omega + \eta\beta\sin\omega = 0, \\
\nu' &= -\xi\alpha\sin\omega + \eta\beta\cos\omega = 0,
\end{aligned}$$

it follows that $\xi\alpha = \eta\beta = 0$, and hence $\alpha = \beta = 0$.

The transformations $\omega = 0, \pi/2, \pi, 3\pi/2$ form the cyclic group of order 4 and have $\omega = \pi/2$ as a generator. It is evident from (2) that $\omega = \pi/2$ leaves (8) *formally* unaltered, and therefore the other rotations do the same. Hence no curve occurs in (7) *more* than four times.

4. $C_2 = 0$ **is a circle.** We now suppose that $4I_3 = I_2^2 \neq 0$. Then $2I_3 \neq I_2^2$ and the reduction to (8) holds with $\xi = \eta \neq 0$, i. e., with $k = -1$. From (2) we see that

$$(10) \quad c' - d' = (c+d)\cdot\tan^3\omega + 3(c-d)\cdot\tan^2\omega - 3(c+d)\cdot\tan\omega - (c-d).$$

If $c+d \neq 0$, the equation $c'-d' = 0$ has a real solution. If $c+d = 0$ it becomes

$$3(c-d)\tan^2\omega - (c-d) = 0$$

which also has a real solution. Hence we can always make $c' = d'$.

If $c = d$, then $c+d \neq 0$, for $c = d = 0$ implies $\xi = kc^2 - k^2d^2 = 0$ and equation (10) becomes $\tan^3\omega - 3\tan\omega = 0$.

Thus in order to retain the normalization $c'-d' = 0$, we require that $\omega = 0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6$. Just as in the first case, the conditions $\mu' = \nu' = 0$ with $\xi\eta \neq 0$ require that $\alpha = \beta = 0$. Since the rotation $\omega = \pi/6$ leaves our normalization *formally unaltered*, we have the canonical form

$$(11) \quad \varphi = -cx^3 - 3cx^2y + 3cxy^2 + cy^3 + 3gx^2 + 3gy^2 + 3hx + 3iy + j = 0, \quad cg \neq 0.$$

The group of this canonical form is the cyclic group of order 6, i. e., no curve occurs more than six times in this form.

5. $C_2 = 0$ **is a parabola.** When $2I_3 \neq I_2^2$, $I_3 = 0$, $C_2 = 0$ can be reduced to the form*

$$\xi x^2 + 2\nu y + \zeta = 0, \qquad \xi \neq 0.$$

We have $\xi = ac - b^2 \neq 0$, $\lambda = ad - bc = 0$, $\eta = bd - c^2 = 0$, $2\mu = ce + ag - 2bf = 0$. Since the first relation is the determinant of the coefficients of c and d in the second two, it follows that $c = d = 0$. Our conditions become merely

$$c = d = 0, \quad ag - 2bf = 0, \quad b \neq 0.$$

From (9) it follows that $\zeta' = \xi\alpha^2 + 2\nu\beta + \zeta$. If ζ is not already zero, it follows that $2\nu = bg - 2cf \neq 0$, for the determinant of the coefficients of g and f in $\nu = 0$ and $2\mu = ag - 2bf = 0$ is precisely -2ξ, which is not zero. The other alternative, $g = f = 0$, implies $\zeta = 0$. Hence a transformation $\omega = 0$, $\alpha = 0$, $\beta = -\zeta/2\nu$ will make $\zeta' = 0$ and not destroy the conditions already imposed.

* MacDuffee, loc. cit., p. 250.

The invariant I_4 now becomes $-\xi\nu^2$ and hence vanishes if and only if $\nu = 0$. We assume $I_4 \neq 0$, i. e., $g \neq 0$. From (9) we have

$$(12) \qquad \eta' = \xi \sin^2\omega, \quad \mu' = +\xi\alpha\cos\omega + \nu\sin\omega, \quad \zeta' = \xi\alpha^2 + 2\nu\beta.$$

Hence to preserve the normalization we must have $\omega = 0, \pi$, $\alpha = \beta = 0$. We therefore have a canonical form in which

$$c = d = 0, \quad a : 2b = f : g = k, \quad \zeta = eg - f^2 = 0, \quad bg \neq 0.$$

The condition $eg - f^2 = 0$ implies that the quadratic terms are $\pm$ a perfect square. Let us set

$$3ex^2 + 6fxy + 3gy^2 = \pm(px + qy)^2.$$

Since $f = gk$, $3g = \pm q^2 \neq 0$, we have $p = kq$. Then the quadratic terms may be written $\pm q^2(kx+y)^2$, or more simply, $r(kx+y)^2$. The canonical form may be then written

$$(13) \qquad \varphi = 2kbx^3 + 3bx^2y + r(kx+y)^2 + 3hx + 3iy + j = 0, \qquad br \neq 0$$

where the group is $\omega = 0, \pi$, $\alpha = 0$, $\beta = 0$.

6. $C_2 = 0$ **represents coincident straight lines.** A curious result of the last paragraph was the fact that we may always make $\zeta = 0$, i. e., there is no cubic curve whose covariant $C_2 = 0$ represents distinct parallel lines.

We now return to the point in the last paragraph where we saw that I_4 vanishes with ν, and assume $I_4 = 0$. We have $c = d = g = f = 0$, $b \neq 0$. We see from (12) that only β is at our disposal. From (2) we have, with $\omega = 0$ and $\alpha = 0$,

$$e' = b\beta + e.$$

We can thus make $e' = 0$, after which no further normalization is possible. The canonical form

$$(14) \qquad \varphi = ax^3 + 3bx^2y + 3hx + 3iy + j = 0, \qquad b \neq 0,$$

is left formally unaltered only by the transformations $\omega = 0, \pi$, $\alpha = \beta = 0$.

7. $C_2 = 0$ **is a straight line.** We now consider the cases for which $C_2 = 0$ is of less than the second degree, i. e., the invariant (5) is zero. The degree of C_2 is an arithmetic invariant, and we first assume that it is 1. That is

$$(15) \qquad \xi = \lambda = \eta = 0, \qquad (\mu, \nu) \neq (0, 0).$$

From (9)

$$\mu' = \mu\cos\omega + \nu\sin\omega, \qquad \nu' = -\mu\sin\omega + \nu\cos\omega,$$

so that we can make $\nu' = 0$. Geometrically this consists in rotating our axes until $C_2 = 0$ becomes parallel to the y-axis.

We now have

$$2\mu = ce + ag - 2bf \neq 0, \qquad 2\nu = de + bg - 2cf = 0.$$

By multiplying these relations by d and $-c$, respectively, and adding, we find

$$2\lambda g - 2\eta f = 2\mu d,$$

whence $d = 0$. It then follows from (15) that $b = c = 0$, and hence from $\mu \neq 0$ that $ag \neq 0$.

From (2) we see that $d' = -a\sin^3\omega$, so that only the two rotations $\omega = 0, \pi$, leave the condition $d' = 0$ unaltered. We now have

$$e' = a\alpha + e, \qquad a \neq 0,$$

so that we can make $e' = 0$, and in order to keep this condition we require that $\alpha = 0$. We now have

$$i' = \pm(i + 2g\beta), \qquad g \neq 0,$$

and, similarly, we can make $i' = 0$, after which β must also equal zero to preserve the normalization.

We thus have a canonical form

$$(16) \qquad \varphi = ax^3 + 6fxy + 3gy^2 + 3hx + j, \qquad ag \neq 0.$$

8. C_2 **is of degree** 0. Suppose that C_2 is of degree zero or vanishes identically. Then $\xi = \lambda = \eta = \mu = \nu = 0$. Since $I_2 = 0$, we know from (4) that $I_1 \neq 0$, and hence that

$$C_3 = (a + c)x + (b + d)y + (e + g);$$

is of the first degree. We choose our new axes so that the line $C_3 = 0$ is the y-axis and hence $(b + d) = (e + g) = 0$, $(a + c) \neq 0$.

If we substitute $-b$ for d in $2\lambda = ad - bc = 0$, we have $b(a+c) = 0$, and hence $b = d = 0$. From $\eta = bd - c^2 = 0$, we have $c = 0$ and therefore $a \neq 0$. From $2\mu = ce + ag - 2bf = 0$, we obtain $g = 0$ and hence $e = 0$.

From (2) we see that

$$a' = a\cos^2\omega, \qquad b' = -a\sin\omega\cos^2\omega, \qquad a \neq 0.$$

Hence to make $b' = 0$ and $a' = a$, we require that $\omega = 0$ or π. Then

$$e' = a\alpha, \qquad a \neq 0,$$

so that α must be zero to retain our normalization.

We now suppose that C_2 is not identically zero, i. e., that $\zeta = eg - f^2 \neq 0$ and therefore $f \neq 0$. From (2)

$$h' = \pm(h + 2f\beta), \qquad f \neq 0,$$

so that we can make $h' = 0$, after which no further normalization is possible. We have therefore the canonical form

$$\varphi = ax^3 + 6fxy + 3iy + j, \qquad af \neq 0. \tag{17}$$

c. C_2 **vanishes identically.** We return to the point where we assumed $\zeta \neq 0$, and consider the alternative possibility, viz., $f = 0$. The covariant C_1 now reduces to $-ai^2x$ and the covariant C_4 reduces to ax. Hence for this case the covariant $-C_1/C_4$ reduces to the constant i^2.

We first suppose that $-C_1/C_4 \neq 0$. From (2) we have

$$j' = 3i\beta + j, \qquad i \neq 0,$$

so that we can make $j' = 0$, after which no further normalization is possible, and we have the canonical form

$$\varphi = ax^3 + 3hx + 3iy, \qquad ai \neq 0, \tag{18}$$

whose group is $\omega = 0, \pi, \alpha = \beta = 0$.

Next, suppose that $-C_1/C_4 = 0$ so that $i = 0$. We have

$$\varphi = ax^3 + 3hx + j, \qquad a \neq 0, \tag{19}$$

which is unaltered in form only by the transformations $\omega = 0, \pi, \alpha = 0$, β arbitrary. From (2) we see that these transformations do not alter (19), as may also be seen from the fact that the locus of $\varphi = 0$ consists of three lines parallel to the y-axis, which are evidently unaffected by a translation parallel to this axis. Thus (19) is a canonical form.

10. **Summary.** We list the eight canonical forms for the cubic together with the invariantive conditions which define them and the groups of transformations which leave them unaltered. All are subject to the general condition that φ be of degree three.

Case I. C_2 of degree 2, $I_3 \neq 0$, $4I_3 \neq I_2^2$.

$$\varphi = kcx^3 + 3kdx^2y + 3cxy^2 + dy^3 - 3kgx^2 + 3gy^2 + 3hx + 3iy + j = 0,$$
$$\omega = 0, \pi/2, \pi, 3\pi/2, \qquad \alpha = \beta = 0, \qquad (c^3 - kd^2)k \neq 0, \; k \neq -1.$$

Case II. C_2 of degree 2, $I_3 \neq 0$, $4I_3 = I_2^2$.

$$\varphi = x^3 + 3x^2y - 3xy^2 - y^3 - 3gx^2 - 3gy^2 - 3hx - 3iy - j = 0, \quad g \neq 0;$$
$$\omega = 0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6, \quad \alpha = \beta = 0.$$

Case III. C_2 of degree 2, $I_3 = 0$, $I_4 \neq 0$.

$$\varphi = 2kx^3 + 3x^2y + r(kx+y)^2 + 3hx + 3iy + j = 0, \qquad r \neq 0;$$
$$\omega = 0, \pi, \qquad \alpha = \beta = 0.$$

Case IV. C_2 of degree 2, $I_3 = 0$, $I_4 = 0$.

$$\varphi = ax^3 + 3x^2y + 3hx + 3iy + j = 0; \qquad \omega = 0, \pi, \qquad \alpha = \beta = 0.$$

Case V. C_2 is of degree 1.

$$\varphi = x^3 + 6fxy + 3gy^2 + 3hx + j, \quad g \neq 0; \quad \omega = 0, \pi, \quad \alpha = \beta = 0.$$

Case VI. C_2 is of degree 0.

$$\varphi = x^3 + 6fxy + 3iy + j, \quad f \neq 0; \qquad \omega = 0, \pi, \qquad \alpha = \beta = 0.$$

Case VII. $C_2 \equiv 0$, $-C_1/C_4 \not\equiv 0$.

$$\varphi = x^3 + 3hx + 3iy, \quad i \neq 0; \qquad \omega = 0, \pi, \qquad \alpha = \beta = 0.$$

Case VIII. $C_2 \equiv 0$, $-C_1/C_4 \equiv 0$.

$$\varphi = x^3 + 3hx + j; \qquad \omega = 0, \pi, \qquad \alpha = 0, \beta \text{ arbitrary.}$$

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ON FORMS WHICH REPEAT UNDER MULTIPLICATION.*

By Claiborne G. Latimer.

1. Prof. E. T. Bell has recently shown that a certain form $\sum_{i=1}^{8} a_i x_i^2$ repeats under multiplication, the a_i being functions of six arbitrary parameters.† This form is an analogue of Lagrange's self-reproductive form $x^2+ay^2+bz^2+abw^2$. In his paper, Bell suggests that a self-reproductive octenary quadratic form might be found analogous to Hermite's quaternary form.‡ He also suggests that if such a form is found, a theory of composition of octenary forms might be developed, similar to the recent discussion of quaternary forms by Brandt.§

We shall find such an octenary form in this paper. This will be obtained as a consequence of a generalization of Hermite's quaternary quadratic to the corresponding Hermitian form. By proper specialization of the parameters in our octenary form we obtain Bell's form referred to above.

2. Prof. Dickson showed that every element of Caylay's eight-unit generalization of real quaternions A, may be written in the form $U = u_0+u_1E_1+u_2E_2+u_3E_3$, where the u_i are elements of a field F, equivalent to the field of complex numbers and the E_i are the ordinary quaternion units.‖ He also showed that if $u = r+se$ is an element of F, where r and s are real numbers and $e^2 = -1$, and $U = r-se$, then

$$E_i u = UE_i, \quad (uE_i)(vE_i) = -uV, \quad (uE_i)(vE_j) = (UV)(E_iE_j), \quad (i \neq j),$$

where v, V are conjugate elements of F. Let $E_1 = i$, $E_2 = j$, $E_3 = k$ where i, j, k are the ordinary quaternion units. Introduce the new units $E_1' = (bc)^{1/2}E_1$, $E_2' = (ac)^{1/2}E_2$, $E_3' = (ab)^{1/2}E_3$ where a, b, c are positive real numbers. Then it will be found that the product of two elements of A may be written in the form

$$\begin{aligned}(u_0+u_1E_1'+u_2E_2'+u_3E_3')(\pm V_0+v_1E_1'+v_2E_2'+v_3E_3') \\ = w_0'+w_1E_1'+w_2E_2'+w_3E_3'\end{aligned}$$

where the u_i, V_0, v_i are elements of F and

* Received February 27, 1928.

† Annals of Math., vol. 27, pp. 99–104.

‡ Bachmann, *Zahlentheorie*, vol. 4, pt. 1, p. 13.

§ Math. Ann., vol. 91, pp. 300–315.

‖ *Linear Algebras*, p. 15, footnote.

$$(1)\qquad \begin{aligned} \pm w_0' = w_0 &= u_0 V_0 \mp (bcu_1 V_1 + acu_2 V_2 + abu_3 V_3),\\ w_1 &= u_0 v_1 \pm u_1 v_0 + a(U_2 V_3 - U_3 V_2),\\ w_2 &= u_0 v_2 \pm u_2 v_0 + b(U_3 V_1 - U_1 V_3),\\ w_3 &= u_0 v_3 \pm u_3 v_0 + c(U_1 V_2 - U_2 V_1). \end{aligned}$$

It is understood that every large letter U_i, V_i is the conjugate with respect to F of the corresponding small letter and that the upper or lower sign is read throughout according as the upper or lower sign is taken in $\pm V_0$.

It is well known that the norm of the product of two elements of A is equal to the product of their norms. The norm of $U = u_0 + u_1 E_1' + u_2 E_2' + u_3 E_3'$ may be written $\varphi(u_i, U_j) = u_0 U_0 + bcu_1 U_1 + acu_2 U_2 + abu_3 U_3$ where U_i is the conjugate of u_i.* We have then

$$(2)\qquad \varphi(u_i, U_j) \cdot \varphi(v_i, V_j) = \varphi(w_i, W_j)$$

where the w_i are given by (1) and the W_j are obtained from (1) by replacing every small (large) letter by the corresponding large (small) letter. We shall hereafter disregard our hypotheses that a, b, c are positive numbers and that the large letters are the conjugates of the corresponding small letters since (2) is a formal identity in the letters involved.

3. If we set every large letter in (1) and (2) equal to the corresponding small letter, the resulting formulas are those used by the writer to show that Hermite's quaternary quadratic repeats under multiplication.† We shall show in a similar manner by means of (1) and (2) that the Hermitian form $f(x_i, X_j) = X_0 x_0 + \sum_{ij=1}^{3} A_{ij} X_i x_j$ also repeats under multiplication; where (A_{ij}) is the adjoint of the general square third order Hermitian matrix (a_{ji}). Let $|a_{ji}| = D$. In $f(x_i, X_j)$ let

$$(3)\qquad \begin{aligned} x_0 &= u_0,\\ x_1 &= u_1 - \frac{A_{12}}{A_{11}} u_2 + \frac{a_{31}}{a_{33}} u_3,\\ x_2 &= u_2 + \frac{a_{32}}{a_{33}} u_3,\\ x_3 &= u_3, \end{aligned}$$

and let the X_i be the corresponding functions of the new variables, U_i, except that in the coefficients of the transformation, the subscripts of the a_{ij}, A_{12} are interchanged. The resulting form will be found to be $\varphi(u_i, U_j)$

* It is understood that in the expression for U, the u_i represent elements of F and in φ they represent the corresponding complex numbers.

† Amer. Math. Monthly, vol. 34, pp. 363–64.

with $a = D/A_{11}$, $b = A_{11}/a_{33}$, $c = a_{33}$. Apply the same transformation to $f(y_i, Y_j)$, the new variables being v_i, V_j. Then

$$f(x_i X_j) \cdot f(y_i Y_j) = \varphi(u_i, U_j) \cdot \varphi(v_i, V_j) = \varphi(w_i, W_j)$$

where the w_i, W_i may be expressed in terms of the x_i, X_i, y_i, Y_i by means of (1), (3) and the three sets of equations similar to (3) in X_i, U_j; y_i, v_j; Y_i, V_j. If we apply the inverse of (3) to $\varphi(w_i, W_j)$, the new variables being z_i, Z_j the resulting form is of course $f(z_i, Z_j)$. The z_i are given in terms of the w_i by replacing in (3) the x_i, u_i by the corresponding z_i, w_j. We may then express the z_i in terms of the x_i, X_j, y_i, Y_j. It will be found that they are as given below. We have then the following generalization of Hermite's result.

THEOREM 1. *If (A_{ij}) is the adjoint of the general square third order Hermitian matrix (a_{ji}), then*

$$\left(X_0 x_0 + \sum_{ij=1}^{3} A_{ij} X_i x_j\right)\left(Y_0 y_0 + \sum_{ij=1}^{3} A_{ij} Y_i y_j\right) = Z_0 z_0 + \sum_{ij=1}^{3} A_{ij} Z_i z_j,$$

where

$$(4)\quad z_0 = x_0 Y_0 \mp \sum_{ij=1}^{3} A_{ij} Y_i x_i, \quad z_j = x_0 y_j \pm x_j y_0 + \sum_{i=1}^{3} a_{ij} S_i \quad (j = 1, 2, 3),$$

$$S_1 = X_2 Y_3 - X_3 Y_2, \quad S_2 = X_3 Y_1 - X_1 Y_3, \quad S_3 = X_1 Y_2 - X_2 Y_1,$$

and the Z_i are obtained by replacing in the above formula every small (large) letter by the corresponding large (small) letter except that the A_{ij}, a_{ij}, are unchanged.

4. In this paragraph we shall obtain a self-reproductive octenary form by making certain substitutions in the form given in the above theorem. In the matrix (a_{ji}) set

$$(5)\quad \begin{aligned} a_{ii} &= b_{ii}\,(i = 1, 2, 3), & a_{13} &= b_{13}(fh)^{1/2} + i c_{13}(dg)^{1/2}, \\ a_{12} &= b_{12}(fg)^{1/2} + i c_{12}(dh)^{1/2}, & a_{23} &= b_{23}(gh)^{1/2} + i c_{23}(df)^{1/2}, \end{aligned}$$

where the b_{ij}, c_{ij}, d, f, g, h are arbitrary real parameters, the last four assumed to be positive. Since (a_{ji}) is Hermitian, its remaining elements are determined. Let

$$(6)\quad \begin{aligned} B_{12} &= h b_{13} b_{23} + d c_{13} c_{23} - b_{12} b_{33}, & C_{12} &= f b_{13} c_{23} - g b_{23} c_{13} + b_{33} c_{12}, \\ B_{13} &= g b_{12} b_{23} - d c_{12} c_{23} - b_{13} b_{22}, & C_{13} &= -h c_{12} b_{23} - f c_{23} b_{12} + b_{22} c_{13}, \\ B_{23} &= f b_{12} b_{13} + d c_{12} c_{13} - b_{11} b_{23}, & C_{23} &= h b_{13} c_{12} - g b_{12} c_{13} + b_{11} c_{23}, \\ B_{11} &= b_{22} b_{33} - g h b_{23}^2 - d f c_{23}^2, & B_{22} &= b_{11} b_{33} - f h b_{13}^2 - d g c_{13}^2, \\ & & B_{33} &= b_{11} b_{22} - f g b_{12}^2 - d h c_{12}^2. \end{aligned}$$

It will be found that the A_{ij} are obtained from (5) and (6) by replacing in the former every small letter with subscripts by the corresponding large letter. In $f(x_i, X_j)$ let

$$(7)\qquad \begin{aligned} x_0 &= r_0 + i(dfgh)^{1/2} r_1, & x_1 &= (gh)^{1/2} r_2 + i(df)^{1/2} r_5, \\ x_2 &= (fh)^{1/2} r_3 + i(dg)^{1/2} r_6, & x_3 &= (fg)^{1/2} r_4 + i(dh)^{1/2} r_7, \end{aligned}$$

and let each X_i be the conjugate imaginary of x_i. It will be found that the resulting form is the octenary quadratic $F(r_i) = r_0^2 + dfgh\, r_1^2 + F_1(r_j)$ $(j = 2, 3, \cdots, 7)$, where the matrix of F_1 is

$$(8)\qquad \begin{pmatrix} B_{11}gh & B_{12}fgh & B_{13}fgh & 0 & -C_{12}dgh & -C_{13}dgh \\ B_{12}fgh & B_{22}fh & B_{23}fgh & C_{12}dfh & 0 & -C_{23}dfh \\ B_{13}fgh & B_{23}fgh & B_{33}fg & C_{13}dfg & C_{23}dfg & 0 \\ 0 & C_{12}dfh & C_{13}dfg & B_{11}df & B_{12}dfg & B_{13}dfh \\ -C_{12}dgh & 0 & C_{23}dfg & B_{12}dfg & B_{22}dg & B_{23}dgh \\ -C_{13}dgh & -C_{23}dfh & 0 & B_{13}dfh & B_{23}dgh & B_{33}dh \end{pmatrix} = (\lambda_{ij})$$
$$(ij = 2, \cdots, 7).$$

Let $F(s_i)$ be the form obtained in a similar manner from $f(y_i, Y_j)$. Then by (4)

$$F(r_i) \cdot F(s_i) = F(t_i)$$

where the t_i are expressible in terms of the r_i, s_j by means of (4), (7) and the two sets of equations obtained from (7) by replacing the x_i, r_j, by the corresponding y_i, s_j; and z_i, t_j. Let

$$\begin{aligned} T_{ij} &= r_i s_j - r_j s_i & K_1 &= dT_{76} + fT_{34} & K_3 &= dT_{65} + hT_{23} & L_2 &= fT_{54} + hT_{27}, \\ \tau_{ij} &= r_i s_j \pm r_j s_i & K_2 &= gT_{42} + dT_{57} & L_1 &= hT_{73} + gT_{46} & L_3 &= gT_{62} + fT_{35}. \end{aligned}$$

Also let K_i', L_i' $(i = 1, 2, 3)$ be the same as the corresponding unprimed letters except that the subscripts of the second T_{ij} are interchanged, and let $b_{ij} = b_{\bar{j}\bar{i}}$, $c_{ij} = -c_{\bar{j}\bar{i}}$ $(i, j = 1, 2, 3)$. It will be found that

$$(9)\qquad \begin{aligned} t_0 &= r_0 s_0 + dfgh\, r_1 s_1 \mp \sum_{ij=2}^{7} \lambda_{ij} r_i s_j, \\ t_1 &= T_{10} \mp [B_{11} T_{52} + B_{12} L_3' + C_{12} K_3' + B_{13} L_2' + C_{13} K_2' + B_{22} T_{68} \\ &\qquad + B_{23} L_1' + C_{23} K_1' + B_{33} T_{74}], \\ t_2 &= \tau_{02} - df\tau_{15} + b_{11} K_1 + b_{12} f K_2 + b_{13} f K_3 + c_{12} dL_2 + c_{13} dL_3, \\ t_3 &= \tau_{03} - dg\tau_{16} + b_{21} g K_1 + b_{22} K_2 + b_{23} g K_3 + c_{21} dL_1 + c_{23} dL_3, \\ t_4 &= \tau_{04} - dh\tau_{17} + b_{31} h K_1 + b_{32} h K_2 + b_{33} K_3 + c_{31} dL_1 + c_{32} dL_2, \\ t_5 &= \tau_{05} + gh\tau_{12} + b_{11} L_1 + b_{12} g L_2 + b_{13} h L_3 - c_{12} h K_2 - c_{13} g K_3, \\ t_6 &= \tau_{06} + fh\tau_{13} + b_{21} f L_1 + b_{22} L_2 + b_{23} h L_3 - c_{21} h K_1 - c_{23} f K_3, \\ t_7 &= \tau_{07} + fg\tau_{14} + b_{31} f L_1 + b_{32} g L_2 + b_{33} L_3 - c_{32} g K_1 - c_{32} f K_2. \end{aligned}$$

We have then

THEOREM 2. *If* $F_1(r_j)$ $(j = 2, 3, \cdots, 7)$ *is a quadratic form with the matrix* (8), *where the* B_{ij}, C_{ij} *are given by* (6), *and if* $F(r_i) = r_0^2 + dfghr_1^2 + F_1(r_j)$, *then*

$$F(r_i) \cdot F(s_i) = F(t_i)$$

where the t_i *are given by* (9).

If in F we set $f = g = h = 1$, $d = c_{ij} = 0$ $(i, j = 1, 2, 3)$ the identity of Theorem 2 becomes Hermite's identity. If we set one of d, f, g, h, b_{ii} $(i = 1, 2, 3)$ equal to unity and $b_{ij} = c_{ij} = 0$ $(i \neq j)$, the resulting form, aside from notation, is that found by Prof. Bell to be self-reproductive. It may be shown that the determinant of F_1 is equal to $(dfgh)^3 \cdot |A_{ij}|^2$ and hence the determinant of F is $(dfghD)^4$.

UNIVERSITY OF KENTUCKY.

ON THE MULTIPLE SOLUTIONS OF THE PELL EQUATION.*

By D. H. Lehmer.

1. Introduction. Most of the literature written on the Pell equation is concerned with the discovery and application of its fundamental solution. Less attention has been paid to the multiple solutions; in fact, no systematic discussion has been made of their many properties. The more fundamental of these have been established by means of the hyperbolic functions.† It is the purpose of this note to indicate a method by which a complete study of the multiple solutions of the Pell equation can be deduced from Lucas' theory of recurring series of the second order. A number of formulas and theorems of particular interest will be found in section 5 and 6. These are readily derived from the principle proved in section 4. The notation used is that of Lucas' classical memoir,‡ and numbers in square brackets [ ] refer to the equations of this article.

2. The general recurring series of the second order. Let a and b be the roots of the equation

$$x^2 - Px + Q = 0 \tag{1}$$

where P and Q are any integers prime to each other. We have then $a+b = P$, $ab = Q$.

Let

$$(a-b)^2 = \Delta = \delta^2 = P^2 - 4Q \tag{2}$$

so that

$$a = \frac{P+\delta}{2}, \qquad b = \frac{P-\delta}{2}. \tag{3}$$

Lucas considers two symmetric functions of a and b namely:

$$U_n = \frac{a^n - b^n}{a-b}, \qquad V_n = a^n + b^n$$

and shows that they are recurring series of the second order with (1) for scales of relation. That is, they differ only in the choice of initial values:

$$U_0 = 0, \quad U_1 = 1, \qquad V_0 = 2, \quad V_1 = P.$$

* Received February 9, 1928.

† Mathews: Theory of Numbers, pp. 93–95. D. H. Lehmer: Annals of Math., (2), vol. 27, pp. 471–476. Cf. also Cunningham: Brit. Assn. Rept. 1907, p. 462.

‡ Amer. Jour. of Math., vol. 1 (1877), pp. 184–240, 289–321.

In general let W_n be the nth term of the recurring series whose scale is (1) so that

(4) $$W_{n+2} = P W_{n+1} - Q W_n$$

and is determined uniquely by the choice of certain values for W_0 and W_1. Then it is easy to verify that:

(5) $$W_n = W_1 U_n - Q W_0 U_{n-1}.$$

In fact the series W_n thus defined satisfies the recurrence (4) and has for $n = 0$ and $n = 1$ the proper values namely W_0 and W_1. For example

(6) $$V_n = P U_n - 2 Q U_{n-1}.$$

3. **The Pellian case.** Let us consider two functions X_n and Y_n satisfying the recurrences:

$$X_{n+2} = 2 X_1 X_{n+1} - X_n, \qquad Y_{n+2} = 2 X_1 Y_{n+1} - Y_n,$$

with $X_0 = 1$, $Y_0 = 0$, and $(X_1 Y_1)$ to be determined later. In the notation of section 2,

(7) $$P = 2 X_1, \qquad Q = 1,$$

and (5) becomes:

$$X_n = X_1 U_n - U_{n-1}.$$

Comparing this with (6) and (7) and using (4) we have:

(8) $$X_n = \frac{1}{2} V_n, \qquad Y_n = Y_1 U_n.$$

Consider the expression:

[46] $$V_n^2 - \Delta U_n^2 = 4 Q^n.$$

From (8), (2) and (7) we have

(9) $$X_n^2 - \frac{X_1^2 - 1}{Y_1^2} Y_n^2 = 1.$$

Thus far X_1 and Y_1 have been left arbitrary. Now we impose the condition that

$$\frac{X_1^2 - 1}{Y_1^2} = D$$

where D is some integer, not a square. That is $(X_1 Y_1)$ are chosen so that

$$X_1^2 - D Y_1^2 = 1.$$

When this is done equation (9) becomes

$$X_n^2 - D Y_n^2 = 1, \tag{10}$$

which shows that $(X_n\, Y_n)$ are multiple solutions of the Pell equation. The initial values $(X_1\, Y_1)$ are taken as the fundamental solution of (10) and may be found by well known methods. From (3) we have:

$$a = \frac{2X_1 + \sqrt{4X_1^2 - 4}}{2} = X_1 + \sqrt{D}\, Y_1,$$

$$b = \frac{2X_1 - \sqrt{4X_1^2 - 4}}{2} = X_1 - \sqrt{D}\, Y_1.$$

4. **Principle of substitution.** Summing up the results of the preceding section we have the following principle:

For every relation in Lucas' theory there exists one in terms of the multiple solutions of the Pell equation in which:

$$U_n, V_n, P, Q, \delta^2 = \Delta, a, b$$

are replaced by

$$Y_n/Y_1,\ 2X_n,\ 2X_1,\ 1,\ 4DY_1^2,\ X_1 + \sqrt{D}\, Y_1,\ X_1 - \sqrt{D}\, Y_1$$

respectively.

Thus the equations:

$$[6] \qquad V_n + \delta U_n = 2a^n, \qquad V_n - \delta U_n = 2b^n$$

become the familar relations:

$$X_n + \sqrt{D}\, Y_n = (X_1 + \sqrt{D}\, Y_1)^n,$$

$$X_n - \sqrt{D}\, Y_n = (X_1 - \sqrt{D}\, Y_1)^n.$$

The formulas for negative arguments:

$$[50] \qquad U_{-n} = -U_n/Q^n, \qquad V_{-n} = V_n/Q^n.$$

become:

$$Y_{-n} = -Y_n, \qquad X_{-n} = X_n.$$

The addition formulas:

$$[49] \qquad 2U_{m+n} = U_m V_n + U_n V_m,$$

$$2V_{m+n} = V_m V_n + \Delta U_m U_n$$

become:

$$Y_{m+n} = Y_m X_n + Y_n X_m,$$

$$X_{m+n} = X_m X_n + D Y_m Y_n,$$

and so on.

5. **Algebraic theory of X_n and Y_n.** A very large number of relations may be written down by applying the principle of the foregoing section.

The following relations together with those preceding constitute the most important ones.

(11) [87] $$X_{2n} = 2X_n^2 - 1 = X_n^2 + DY_n^2,$$

[3] $$Y_{2n} = 2Y_n X_n,$$

[51] $$X_{m-n} = X_m X_n - DY_m Y_n,$$

$$Y_{m-n} = Y_m X_n - Y_n X_m,$$

$$Y_1 X_n = X_1 Y_n - Y_{n-1} = \frac{1}{2}(Y_{n+1} - Y_{n-1}) = Y_{n+1} - Y_n X_1,$$

$$X_{n+m} X_{n-m} + DY_{n+m} Y_{n-m} = X_{2n},$$

$$X_{n+m} X_{n-m} - DY_{n+m} Y_{n-m} = X_{2m},$$

[33] $$X_{n+m}^2 - X_n^2 = DY_m Y_{2n+m},$$

[32] $$Y_{n+m}^2 - Y_n^2 = Y_m Y_{2n+m},$$

[52] $$X_n + X_m = 2X_{(n+m)/2} X_{(n-m)/2},$$

$$Y_n + Y_m = 2Y_{(n+m)/2} X_{(n-m)/2},$$

[70] $$X_n = 2^{n-1} X_1^n + \frac{1}{2} \sum_{i=1}^{[\frac{n}{2}]} (-1)^i \frac{n}{i} \binom{n-i-1}{i-1} (2X_1)^{n-2i},$$

$$Y_{n+1} = Y_1 \sum_{i=0}^{[\frac{n}{2}]} (-1)^i \binom{n-i}{i} (2X_1)^{n-2i},$$

$$2^n X_n = \sum_{i=0}^{[\frac{n+1}{2}]} \binom{n}{2i} D^i (2Y_1)^{2i} (2X_1)^{n-2i},$$

$$2^{n-1} Y_n = Y_1 \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n}{2i+1} D^i (2Y_1)^{2i} (2X_1)^{n-2i-1},$$

[44] $$\frac{X_n}{X_1} = (-1)^{\frac{n-1}{2}} + 2 \sum_{i=0}^{\frac{n-1}{2}} (-1)^i X_{n-2i-1}, \qquad (n = 2k+1),$$

[42] $$\frac{Y_n}{Y_1} = -1 + 2 \sum_{i=0}^{\frac{n-1}{2}} X_{n-2i-1}, \qquad (n = 2k+1),$$

[41] $$\frac{Y_n}{Y_1} = 2 \sum_{i=0}^{\frac{n-2}{2}} X_{n-2i-1}, \qquad (n = 2k).$$

These are a very few of the hundreds of relations that exist between the functions X_n and Y_n. A glance at Lucas' memoir will indicate what

is possible in this direction. Relations in terms of determinants, continued fractions, binomial coefficients, continued radicals, logarithms, cyclotomic functions, infinite series etc. are included in the algebraic theory of X_n, Y_n.

Every formula in X_n, Y_n or D may be generalised by replacing these quantities by X_{nr}, Y_{nr}/Y_r, and DY_r^2 respectively. By replacing X_n by $\cos n\theta$, Y_n by $\sin n\theta/\sin\theta$ and D by $\sin^2\theta$, every relation in X_n and Y_n may be transformed into a formula in circular functions. If X_n is replaced by $\cosh ny$, Y_n by $\sinh ny/\sinh y$, and D by $\sin^2 ny$ the hyperbolic functions may be studied in like manner.

6. **The arithmetic theory of X_n and Y_n.** The equations (8) show how intimate the connection is between the number-theoretic properties of (X_n, Y_n) and (U_n, V_n). These equations are sufficient for the most part to establish the following fundamental properties of X_n and Y_n. As Carmichael† has pointed out, Lucas was inaccurate in certain of his theorems by not allowing for the singularity of the prime 2 in his theory. Fortunately 2 is not such an exception in our discussion. The theorems marked with a * cannot be deduced immediately from Lucas' memoir. The present writer in a paper which he hopes to publish shortly has considered an extension of Lucas' theory by which he has been able to strengthen many of Lucas' classical theorems. Some of the theorems marked * indicate the effects of this extension on the theory of the Pell equation.

THEOREM 1. *X_n and Y_n are relatively prime.*

THEOREM 2. *If the G. C. D. of m and n is d then the G. C. D. of Y_m and Y_n is Y_d.*

THEOREM 3. *Y_m is a divisor of Y_n if and only if m is a divisor of n.*

COROLLARY: *Every Y_n is a multiple of Y_1.*

THEOREM 4. *X_m is a divisor of X_n if and only if n/m is an odd divisor of n.*

THEOREM 5. *If Y_ω is the first Y to contain the factor m then Y_n is divisible by m if and only if $n = k\omega$. (The number ω is called the rank of apparition of m in the series Y_n.)*

* THEOREM 6. *The number of terms less than Y_n and prime to Y_n, with the exception of the ever present common factor Y_1, is Euler's $\varphi(n)$.*

* THEOREM 7. *If p is a prime factor of D prime to Y_1, then $Y_1 \cdot Y_2 \cdot Y_3 \cdots Y_{p-1} \equiv -(X_1/p) \pmod p$ where (X_1/p) is Legendre's symbol.*

This theorem is an extension of Wilson's theorem. The converse of this theorem is true and gives a theoretical test for primality. Theorems 3, 6 and 7 exhibit properties of Y_n similar to those of the natural numbers. Also compare theorems 9 and 10.

† Annals of Math. (2), vol. 15, pp. 30–70.

* THEOREM 8. *If p is an odd prime not dividing DY_1, then its rank of apparition is some divisor of* $\frac{1}{2}\left\{p-\left(\frac{D}{p}\right)\right\}$.

This is the law of apparition of a prime p and is an extension of Fermat's theorem. In what follows p is a prime.

THEOREM 9. *If p divides Y_1 it divides Y_n. If p divides D, but not Y_1, then the rank of apparition of p is p, and p occurs to the first power as a divisor of Y_p.*

THEOREM 10. *If ω is the rank of apparition of p^α and if $\varkappa$ is any number prime to p, then $Y_{\varkappa\omega p^\lambda}$ contains the factor $p^{\alpha+\lambda}$ but no higher power of p.*

This is the law of repetition of the prime p. Unlike Lucas' law it holds for $p=2$.

* THEOREM 11. *If $m=\prod P_i^{\alpha_i}$ and if we define a function ψ_m by*

$$\psi_m = \frac{\prod p_i^{\alpha_i-1}\left[p_i-\left(\frac{D}{p_i}\right)\right]}{2^\varkappa}$$

where Legendre's symbol is taken as zero if p divides D and where $\varkappa$ is the number of distinct prime factors of m not dividing D, then $Y_{\psi_m}\equiv 0 \pmod m$. This corresponds to Euler's φ-function and his generalisation of Fermat's theorem. Compare Mathews† who replaces ψ_m by the L. C. M. of its factors.

*THEOREM 12. *If m is prime to D the primitive odd prime factors of Y_m are of the form $2km\pm 1$, and those of X_m are of the form $4km\pm 1$.*

*THEOREM 13. (a) *If p is a prime of the form $4n+1$, then $4X_{pr}/X_r$ and $4Y_{pr}/Y_r$ may both be put in the form t^2-Du^2.* (b) *If p is a prime of the form $4n-1$, then $4X_{pr}/X_r$ may be put in the form t^2+Dpu^2 and $4Y_{pr}/Y_r$ may be put in the form Dt^2+pu^2.*

This is an extension of Gauss' theorem about the cyclotomic function $4(x^p-1)/(x-1)$. Attention should be called to certain inaccuracies in Lucas' results on this topic.

Finally we give three typical theorems for determining the primality or non-primality of an integer N prime to $2DY_2$. The first is of theoretical interest only, the second‡ is a practical test for a general integer N. The third is not as impractical as it would first appear. Taken with equation (10) it becomes a very effective test for the numbers in question.

THEOREM 14. *If $(N\pm 1)/2$ is the rank of apparition of N, then N is a prime.*

† Mathews, loc. cit., p. 94.

‡ Compare the writer's note in the Bull. Amer. Math. Soc., vol. 34 (1928), p. 54.

*THEOREM 15. *If* $Y_{N+1} \equiv 0 \pmod{N}$ *and if* $Y_{(N+1)/p} \equiv r \not\equiv 0 \pmod{N}$, *and if the G. C. D. of* N *and* r *is* ϱ, *then the prime factors of* N/ϱ *are of the form* $kp^{\alpha} \pm 1$ *where* α *is the highest power to which the prime* p *occurs as a factor of* $N \pm 1$.

*THEOREM 16. *The number* $2^n - 1$ *with* n *odd is a prime if and only if it divides the numerator of the convergent of order* 2^{n-1} *to the square root of three.*

In case the reader may wish to verify many of the above theorems we subjoin a table giving the first 30 terms Y_n of the most fundamental series namely $D = 2$ and also their prime factors.

n	Y_n	Non-primitive factors	Primitive factors
1	2	—	2
2	12	2^2	3
3	70	2	$5 \cdot 7$
4	408	$2^3 \cdot 3$	17
5	2378	2	$29 \cdot 41$
6	13860	$2^2 \cdot 3^2 \cdot 5 \cdot 7$	11
7	80782	2	$13^2 \cdot 239$
8	470832	$2^4 \cdot 3 \cdot 17$	577
9	2744210	$2 \cdot 5 \cdot 7$	$197 \cdot 199$
10	15994428	$2^2 \cdot 3 \cdot 29 \cdot 41$	$19 \cdot 59$
11	93222358	2	$23 \cdot 353 \cdot 5741$
12	543339720	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17$	1153
13	3166815962	2	$79 \cdot 599 \cdot 33461$
14	18457556052	$2^2 \cdot 3 \cdot 13^2 \cdot 239$	$113 \cdot 337$
15	107578520350	$2 \cdot 5^2 \cdot 7 \cdot 29 \cdot 41$	$31^2 \cdot 269$
16	627013566048	$2^5 \cdot 3 \cdot 17 \cdot 577$	665857
17	3654502875938	2	$103 \cdot 137 \cdot 8297 \cdot 15607$
18	21300003689580	$2^2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 197 \cdot 199$	13067
19	124145519261542	2	$37 \cdot 179057 \cdot 9369319$
20	723573111879672	$2^3 \cdot 3 \cdot 17 \cdot 19 \cdot 29 \cdot 41 \cdot 59$	$241 \cdot 5521$
21	4217293152016490	$2 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 239$	$4663 \cdot 45697$
22	24580185800219268	$2^2 \cdot 3 \cdot 23 \cdot 353 \cdot 5741$	$43 \cdot 89 \cdot 11483$
23	143263821649299118	2	$47 \cdot 229 \cdot 982789 \cdot 6771937$
24	835002744095575440	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 577 \cdot 1153$	$97 \cdot 13729$
25	4866752642924153522	$2 \cdot 29 \cdot 41$	$1549 \cdot 29201 \cdot 45245801$
26	28365513113449345692	$2^2 \cdot 3 \cdot 79 \cdot 599 \cdot 33461$	$22307 \cdot 66923$
27	165326326037771920630	$2 \cdot 5 \cdot 7 \cdot 197 \cdot 199$	$53 \cdot 146449 \cdot 7761799$
28	963592443113182178088	$2^3 \cdot 3 \cdot 13^2 \cdot 17 \cdot 113 \cdot 239 \cdot 337$	1535466241
29	5616228332641321147898	2	$4456048214 9 \cdot 63018038201$
30	32733777552734744709300	$2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 29 \cdot 31^2 \cdot 41 \cdot 59 \cdot 269$	$601 \cdot 2281$

ON CERTAIN HITHERTO UNSOLVED CASES OF THE COMPLEX MULTIPLICATION OF ELLIPTIC FUNCTIONS.*

By S. C. Mitra.

1. **Introduction.** The object of the present note is to give the solutions of a number of cases of the complex multiplication of elliptic functions which have hitherto remained unsolved.†

The procedure adopted by me is the same as that of Russell. I have obtained in several cases $\varkappa\varkappa' + \lambda\lambda'$ and $\varkappa\varkappa'\lambda\lambda'$ in terms of the roots of certain cubic equations, from which $\varkappa\varkappa'$ can be determined at once. In other cases, I have determined

$$J \equiv \frac{(1-16\,\varkappa^2\varkappa'^2)^3}{108\,\varkappa^2\varkappa'^2},$$

from which $\varkappa\varkappa'$ can be determined.

Notation.

$\varkappa$ = Modulus of the elliptic function.

$\varkappa' = \sqrt{1-\varkappa^2}$.

λ = modulus of the transformed elliptic function.

$\lambda' = \sqrt{1-\lambda^2}$.

J = Klein's absolute invariant.

J' = transformed value of J.

τ and τ' are two quantities such that, in the cubic transformation,

$$J = -\frac{(\tau-1)(9\tau-1)^3}{64\,\tau}, \qquad J' = -\frac{(\tau'-1)(9\tau'-1)^3}{64\,\tau'}$$

and $\tau\tau' = 1$.

$\varphi(\omega)$ and $\psi(\omega)$ are Hermite's functions such that $\sqrt[4]{\varkappa} = \varphi(\omega)$, $\sqrt[4]{\varkappa'} = \psi(\omega)$.

m, n are positive integers,

r, s are integers.

* Received February 16, 1928.

† Many cases of complex multiplication of elliptic functions have been worked out by Greenhill (*Proc. L. M. S.*, vol. 19), Weber (*Lehrbuch der Algebra*, 2nd Ed., vol. 3, pp. 720–26. See also *Acta Math.*, vol. 2, and *Math. Annalen*, vol. 33.), Russel (*Proc. L. M. S.*, vol. 21), Mathews (*Proc. L. M. S.*, vol. 21), Ramanujan (*Q. J. M.*, vol. 45) and Kronecker (*Berlin. Sitz.*, 1857, 1862). We have also the interesting treatise on the complex multiplication of elliptic functions by Fueter, viz., „Vorlesungen über die singulären Moduln und die komplexe Multiplikation der elliptischen Funktionen."

My best thanks are due to Prof. Ganesh Prasad, D. Sc., for his kind encouragement and interest.

If the modular equations of the mth and nth orders be written in the form

$$F_m(\Omega_1, \omega) = 0, \qquad F_n(\Omega, \Omega_1) = 0,$$

then

$$\Omega_1 = \frac{\omega + 16r}{m}, \qquad \Omega = \frac{\Omega_1 + 16s}{n}.$$

Putting $\Omega = -\omega^{-1} + 2t$ we get

$$\text{(A)} \qquad \omega = mnt - 8r - 8ms + \sqrt{-[mn - (mnt - 8r - 8ms)^2]}.$$

2. **Modulus corresponding to $\omega = \sqrt{-106}$.** If, in (A), we put $t = 9$, $m = 23$ and $n = 5$, then $\omega = 3 + \sqrt{-106}$. The modular equation of the 5th order is

$$-\varkappa'\lambda + \varkappa\lambda' - 2(4\varkappa\lambda\varkappa'\lambda')^{1/3} = 1.$$

Putting $(4\varkappa\lambda\varkappa'\lambda')^{1/12} = x$, we get

$$-\varkappa'\lambda + \varkappa\lambda' = 1 + 2x^4.$$

Then

$$\text{(a)} \qquad (\varkappa'\lambda + \varkappa\lambda')^2 = 1 + 4x^4 + 4x^8 + x^{12}.$$

The modular equation of the 23rd order gives

$$\sqrt[4]{\varkappa\lambda} + \sqrt[4]{\varkappa'\lambda'} = 1 - \sqrt{2}x,$$

whence

$$\text{(b)} \qquad \varkappa\lambda + \varkappa'\lambda' = 1 - 4\sqrt{2}x + 12x^2 - 10\sqrt{2}x^3 + 12x^4 - 4\sqrt{2}x^5 + x^6.$$

From (a) and (b) we have

$$\left(x^6 + \frac{1}{x^6}\right) - 8\sqrt{2}\left(x^5 + \frac{1}{x^5}\right) + 56\left(x^4 + \frac{1}{x^4}\right) - 116\sqrt{2}\left(x^3 + \frac{1}{x^3}\right) + 332\left(x^2 + \frac{1}{x^2}\right) - 344\sqrt{2}\left(x + \frac{1}{x}\right) + 554 = 0.$$

Putting $x + 1/x = \sqrt{2}y$, we get the equation

$$y(2y^2 - 4y + 3)(2y^3 - 12y^2 + 23y - 12) = 0.$$

The real root of the cubic equation is

$$y_0 = 2 - \left(\frac{1}{2} - \frac{\sqrt{53}}{6\sqrt{6}}\right)^{1/3} - \left(\frac{1}{2} + \frac{\sqrt{53}}{6\sqrt{6}}\right)^{1/3},$$

and its discriminant is 424. Also

$$(\varkappa\varkappa'+\lambda\lambda')^2 = 2x^{12}(8y^6-24y^4+26y^2-10)(2y^3-8y^2+9y-2)^2$$
$$= -16x^{12}(4131y^2-15876y+10373),$$

and $4\varkappa\varkappa'\lambda\lambda' = x^{12}$, whence $\varkappa\varkappa'$ can be determined in terms of y_0.

Since $\omega = 3+\sqrt{-106}$,

$$\varphi(\omega) \equiv \varphi(3+\sqrt{-106}) = e^{\pi i/8}\frac{\varphi(2+\sqrt{-106})}{\psi(2+\sqrt{-106})} = e^{3\pi i/8}\frac{\varphi(\sqrt{-106})}{\psi(\sqrt{-106})}$$

and

$$\psi(\omega) \equiv \psi(3+\sqrt{-106}) = \frac{1}{\psi(\sqrt{-106})}.$$

Therefore

$$\varphi^4(\omega)\,\psi^4(\omega) = i\frac{\varphi^4(\sqrt{-106})}{\psi^8(\sqrt{-106})}.$$

In the present notation $\sqrt[4]{\varkappa} = \varphi(\omega)$, $\sqrt[4]{\varkappa'} = \psi(\omega)$. Hence

$$\varkappa\varkappa' = i\frac{\varphi^4(\sqrt{-106})}{\psi^8(\sqrt{-106})}.$$

But in the notation usually employed

$$\sqrt[4]{\varkappa} = \varphi(\sqrt{-106}), \qquad \sqrt[4]{\varkappa'} = \psi(\sqrt{-106}),$$

Therefore

$$\frac{1}{\varkappa}-\varkappa = \frac{\psi^8(\sqrt{-106})}{\varphi^4(\sqrt{-106})}.$$

3. Some other moduli. In each of the following cases, I shall simply indicate the method and give the result. The process is the same as in the preceding section.

Let

$$(4\varkappa\lambda\varkappa'\lambda')^{1/12} = x, \qquad x+\frac{1}{x} = \sqrt{2}\,y,$$

for the cases (a) and (b).

(a) *Modulus corresponding to* $\omega = \sqrt{-109}$. This is obtained by combining the modular equation of the 11th order with that of the 23rd order. We obtain the equation

$$(2y^3-4y^2+9y-9)(2y^3-12y^2+9y+5) = 0;$$

of which the first factor corresponds to $\omega = \sqrt{-109}$, and its discriminant is 436.

We get

$$(\varkappa\varkappa' + \lambda\lambda') = (1 - 4x^2 + 4x^4 - x^6)(1 - 4\sqrt{2}x + 12x^2 - 10\sqrt{2}x^3 + 12x^4 - 4\sqrt{2}x^5 + x^6),$$

and the real root is

$$y_0 = \frac{2}{3} + \frac{1}{6}(18\sqrt{327} + 226)^{1/3} - \frac{1}{6}(18\sqrt{327} - 226)^{1/3}.$$

Thus

$$\varkappa = \varphi^4(12 + \sqrt{-109}) = \varphi^4(\sqrt{-109}),$$
$$\varkappa' = \psi^4(12 + \sqrt{-109}) = \psi^5(\sqrt{-109}).$$

In this way we can get $\varkappa\varkappa'$.

(b) *Modulus corresponding to* $\omega\sqrt{-517}$ This is obtained by combining the modular equation of the 11th order with that of the 47th order. We obtain the equation

$$(2y^3 + 4y^2 - 3y - 7)(2y^3 - 20y^2 - 71y - 55) = 0,$$

of which the second factor corresponds to $\omega = \sqrt{-517}$, and its discriminant is 2086. We get

$$\varkappa\varkappa' + \lambda\lambda' = \sqrt{2}(1 - 4x^2 + 4x^4 - x^6)(2x - 3x^3 + 2x^5),$$

and the real root is

$$y_0 = \frac{10}{3} + \frac{1}{3}\left(\frac{11875 + 9\sqrt{1551}}{4}\right)^{1/3} + \frac{1}{3}\left(\frac{11875 - 9\sqrt{1551}}{4}\right)^{1/3}.$$

(c) *Modulus corresponding to* $\omega = \sqrt{-235}$. Let $(4\varkappa\lambda\varkappa'\lambda')^{1/12} = x$ and $y = \sqrt{2}x$, for the cases (c) and (d).

In the case (c), the modulus is obtained by combining the modular equation of the 5th order with that of the 47th order. We obtain the equation

$$(y^3 - 2y - 2)(y^3 + 6y^2 + 10y - 2) = 0.$$

The second factor corresponds to $\omega = \sqrt{-235}$. It can be easily proved that

$$\sqrt[4]{2\varkappa\varkappa'} = \left(\frac{4y + 2y^3 - y^5}{8\sqrt{2}}\right)^{1/2} - \left(\frac{4y - 2y^3 - y^5}{8\sqrt{2}}\right)^{1/2};$$

where

$$y = -2 + \left(3 + \frac{\sqrt{235}}{3\sqrt{3}}\right)^{1/3} + \left(3 - \frac{\sqrt{235}}{3\sqrt{3}}\right)^{1/3}.$$

Thus we can obtain $2\varkappa\varkappa'$.

(d) *Modulus corresponding to* $\omega = \sqrt{-355}$. This is obtained by combining the modular equation of the 5th order with that of the 71st order. We obtain the equation

$$(y^3-2y^2+2y-2)\{y^3+(\sqrt{5}+1)y^2+(10+4\sqrt{5})y-2\}$$
$$\times\{y^3+(\sqrt{5}-1)y^2+(10-4\sqrt{5})y-2\} = 0.$$

The equations

$$y^3+(\sqrt{5}+1)y^2+(10+4\sqrt{5})y-2 = 0,$$
$$y^3-(\sqrt{5}-1)y^2+(10-4\sqrt{5})y-2 = 0$$

correspond to $\omega = \sqrt{-355}$ and $\omega = \sqrt{-71/5}$. We have

$$\sqrt[4]{2\varkappa\varkappa'} = \left(\frac{4y+2y^3-y^5}{8\sqrt{2}}\right)^{1/2} - \left(\frac{4y-2y^3-y^5}{8\sqrt{2}}\right)^{1/2},$$

and, for the two cubics, the real roots are:

$$y_0 = -\frac{(\sqrt{5}+1)}{3} + \frac{1}{3}\{(55\sqrt{5}+146)+(6+3\sqrt{5})(1065)^{1/2}\}^{1/3}$$
$$+\frac{1}{3}\{(55\sqrt{5}+146)-(6+3\sqrt{5})(1065)^{1/2}\}^{1/3};$$

$$y_0 = \frac{(\sqrt{5}-1)}{3} + \frac{1}{3}\{(146-55\sqrt{5})+(3\sqrt{5}-6)(1065)^{1/2}\}^{1/3}$$
$$+\frac{1}{3}\{(146-55\sqrt{5})-(3\sqrt{5}-6)(1065)^{1/2}\}^{1/3}.$$

4. **Modulus corresponding to** $\omega = \sqrt{-529}$. In the modular equation of the 23rd order, let us put $\lambda = \lambda' = 1/\sqrt{2}$. Let $(2\varkappa\varkappa')^{1/12} = x$; then we have

$$\left(\sqrt[4]{\frac{\varkappa}{\sqrt{2}}} + \sqrt[4]{\frac{\varkappa'}{\sqrt{2}}}\right) = 1-\sqrt{2}\,x,$$

whence it is not difficult to get the equation

$$\left(x^6+\frac{1}{x^6}\right)-16\sqrt{2}\left(x^5+\frac{1}{x^5}\right)+112\left(x^4+\frac{1}{x^4}\right)-232\sqrt{2}\left(x^3+\frac{1}{x^3}\right)$$
$$+656\left(x^2+\frac{1}{x^2}\right)-688\sqrt{2}\left(x+\frac{1}{x}\right)+1108 = 0.$$

Putting $x+1/x = \sqrt{2}\,y$, we get the equation

$$(2y^3-16y^2+12y-3)^2-23(2y^2-y)^2 = 0.$$

5. **Modulus corresponding to $\omega = \sqrt{-89}$.** Greenhill* has written that a 12-ic in $\sqrt[6]{2\varkappa\varkappa'}$ must be expected, but in the present paper, I shall show that the determination of the modulus can be made to depend upon the solution of a cubic equation.

In (A), let us put $m=31$ and $n=3$. Then in order that $mnt-8r-8ms=2$, we may have $t=2$ or -6.

Therefore $\omega = 2+\sqrt{-89}$ and $\omega = -6+\sqrt{-57}$ will come by the same transformation.

The modular equation of the 3rd order is $\sqrt{\varkappa\lambda'}-\sqrt{\varkappa'\lambda}=1$. Putting $(\varkappa\lambda\varkappa'\lambda')^{1/4}=x$, we have $\varkappa'\lambda+\varkappa\lambda'=1+2x^2$, and $(\varkappa\lambda+\varkappa'\lambda')^2=-4x^2$. Therefore $\varkappa\lambda+\varkappa'\lambda'=2ix$ and

$$\sqrt[4]{\varkappa\lambda}+\sqrt[4]{\varkappa'\lambda'}=\{(2ix+2x^2)^{1/2}+2x\}^{1/2}.$$

The modular equation of the 31st order is

$$(P^2-4Q)^2=4PR,$$

where

$$P=1+\{(2ix+2x^2)^{1/2}+2x\}^{1/2},$$
$$Q=x+\{(2ix+2x^2)^{1/2}+2x\}^{1/2},$$
$$R=x;$$

substituting in the above modular equation, we get after considerable reductions,

$$(16x^8+1)-(64+8i)(8x^7+x)+(1136-1600i)(4x^6+x^2)$$
$$-(2176-6384i)(2x^5+x^3)+(2248-9728i)x^4=0.$$

Putting $v=\sqrt{2}\,x$ the equation reduces to

$$\left(v^4+\frac{1}{v^4}\right)-(32+4i)\sqrt{2}\left(v^3+\frac{1}{v^3}\right)+(568-800i)\left(v^2+\frac{1}{v^2}\right)$$
$$-(544-1596i)\sqrt{2}\left(v+\frac{1}{v}\right)+(562-2432i)=0.$$

Again putting $v+1/v=\sqrt{2}\,zi$ we get the equation

$$(z+13)\{z^3-(17-32i)z^2-(61+16i)z-(11+16i)\}=0.$$

The linear factor corresponds to $\omega=-6+\sqrt{-57}$, while the cubic factor corresponds to $\omega=2+\sqrt{-89}$.

* *Proc. L. M. S.*, vol. 21.

Here

$$\varphi(\omega) = \varphi(2+\sqrt{-89}) = e^{\pi i/4}\varphi(\sqrt{-89}),$$
$$\psi(\omega) = \psi(2+\sqrt{-89}) = \psi(\sqrt{-89}).$$

6. **Modulus corresponding to $\omega = \sqrt{-171}$.** Here the simplest numerical result to obtain is Klein's *absolute invariant J*, where

$$J = -\frac{(1-16x^2x'^2)^3}{108x^2x'^2}.$$

This is obtained by performing the cubic transformation on $J' = 2^9$ corresponding to $\omega = \sqrt{-19}$. We have

$$J' = -\frac{(\tau'-1)(9\tau'-1)^3}{64\tau'} = -2^9$$

and $\tau\tau' = 1$. Putting $\tau' = 27/x$, and $x - 27 = y^3$, we get

$$y(y^3 - 216) = 96(y^3+27).$$

Writing $y = 6v$, we have

$$v^4 - 16v^3 - v - 2 = 0,$$

or

$$\left(v^2 - 8v - \frac{7}{2}\right)^2 - 57\left(v+\frac{1}{2}\right)^2 = 0,$$

whence

$$2v = (8+\sqrt{57}) + (135+18\sqrt{57})^{1/2}.$$

Therefore

$$\tau = (8199+1086\sqrt{57}) + (135+18\sqrt{57})^{1/2}(498+66\sqrt{57}).$$

7. (a) **Modulus corresponding to $\omega = \sqrt{-387}$.** This is obtained by performing the cubic transformation on $\omega = \sqrt{-43}$.

Here the equation for v is

$$\left(v^2 - 80v - \frac{79}{2}\right)^2 - 7^2\cdot 129\left(v+\frac{1}{2}\right)^2 = 0$$

whence

$$2v = (80+7\sqrt{129}) + 9(159+14\sqrt{129})^{1/2},$$
$$\tau = (8192007+721266\sqrt{129}) + (459378+40446\sqrt{129})(159+14\sqrt{129})^{1/2}.$$

(b) *Modulus corresponding to* $\omega = \sqrt{-603}$. The equation for v is

$$v^4 - 880v^3 - v - 110 = 0$$

and

$$2v = (440 + 31\sqrt{201}) + 3(43071 + 3038\sqrt{201})^{1/2}.$$

(c) *Modulus corresponding to* $\omega = \sqrt{-243}$. Here

$$J' - 1 = -\frac{11^2 \cdot 23^2}{3^2} = -\frac{64009}{9}.$$

Therefore

$$\frac{(27\tau'^2 - 18\tau' - 1)^2}{64\tau'} = \frac{64009}{9}.$$

It can be seen by solving this equation that

$$\tau' = \left\{\frac{4}{3}(3^{1/3} - 3^{-1/3}) - 1\right\}^2.$$

Therefore

$$\tau = \left(1 - \frac{2}{3^{2/3}}\right)^{-4}.$$

UNIVERSITY OF CALCUTTA.
January 12, 1928.

ON THE COMPLEX ROOTS OF A TRANSCENDENTAL EQUATION OCCURRING IN THE ELECTRON THEORY.*

By James Pierpont.

1. In a recent paper, Prof. G. A. Schott† has considered the equation

$$(a+bz)\,e^z - z = 0 \tag{1}$$

in which he takes

$$a = 1{,}6\cdot 10^{-15}, \quad b = 1+\beta, \quad \beta = 7{,}5\cdot 10^{-16}. \tag{2}$$

It is important for him to show that (1) has no root $z = x+iy$ in which $x > 0$. His method appears to be as follows. Setting $z = x+iy$ in (1) gives, on equating reals and imaginaries

$$\begin{aligned} [(a+bx)\cos y - by \sin y]e^x - x &= 0, \\ [(a+bx)\sin y + by \cos y]e^x - y &= 0. \end{aligned} \tag{3}$$

If these equations are plotted for the values of a, b given in (2) we get two curves whose intersections are the roots of (1). No method of simply calculating these curves being known by the present writer, I was led to consider the question *de novo*. I naturally turned to various reference works for information but got very little help. Further efforts have led to the results incorporated in this paper which I believe will prove of interest to others.

It is well known that the real roots of a numerical algebraic equation may be calculated by Horner's method. Many theorems as Descartes' and Sturm's give us valuable information as to the location of these roots. A still more general method is that devised by Graeffe‡ which gives both real and complex roots. This method has been developed by Fourier and especially by Carvallo.§

Turning to transcendental equations we find quite a different situation. In the first place the equation may have no root real or imaginary as the

* Received April 13, 1928.

† G. A. Schott F. R. S. The theory of the linear oscillator and its bearing on the electron theory. Phil. Mag., 3 (1927), p. 739.

‡ C. H. Graeffe. Die Auflösung der Höheren numerischen Gleichungen, Zürich 1837.

§ E. Carvallo. Méthode pratique pour la résolution numerique complète des équations algébriques ou transcendantes, 3rd ed., Paris 1910.

equation $e^z = 0$ shows. Secondly we have no algorithm like Sturm's for locating approximately the roots. A short time ago I would have said that there was no known method of expeditiously finding the complex roots to any assigned degree of accuracy. I now know better but no thanks to the current hand books as will transpire later.

2. Let us now turn to equation (1); we will not take the values of a, b given in (2) as this choice of the constants materially simplifies the numerical solution. Instead we will choose less special values and consider for definiteness the equation

$$(A) \qquad (1+2z)e^z - z = 0.$$

Our problem is to find *all* its roots. The methods here employed are partly entirely general and partly of special nature.

We begin by showing that (A) *has no real roots.* For if $z = x$ is a real root of (A) we have

$$e^x = \frac{x}{1+2x}.$$

If we set

$$y = e^x, \quad \text{and} \quad y = \frac{x}{1+2x},$$

we have two well known curves (exponential and hyperbola) whose graphs, we see at once, do not meet in a real point.

We show next that (A) *has no purely imaginary roots.* For setting $z = iy$, y real, in (A) we get

$$(1+2iy)e^{iy} = iy.$$

The square of the modulus of both sides gives

$$1+4y^2 = y^2, \quad \text{or} \quad 3y^2 = -1,$$

which is impossible as y is real. Hence all the roots of (A) are complex, if any exist. Setting $z = x+iy$ in (A) gives $u+iv = 0$ where

$$u = [(1+2x)\cos y - 2y\sin y]e^x - x,$$
$$v = [(1+2x)\sin y + 2y\cos y]e^x - y.$$

If z is a root, $u = 0$, $v = 0$. Replacing z by $\bar{z} = x - iy$ we ge $\bar{u} + i\bar{v} = 0$ where $\bar{u} = u$, $\bar{v} = -v$. Hence if z is a root, so is $\bar{z}$, i. e., as in algebraic equations the roots occur in conjugate pairs. We may thus *in the following regard* $y > 0$.

We show next that *no root* $z = x+iy$ *has* $x \geqq 0$. We have seen $x = 0$ is impossible, hence take $x > 0$. If z satisfies (A),

(B) $$e^{-z} = 2 + \frac{1}{z},$$

since $z \neq 0$. Let us set

$$T = e^{-z}, \qquad t = 2 + \frac{1}{z}.$$

Then

$$|T| = e^{-x} < 1,$$

$$t = 2 + \frac{x}{r^2} - \frac{iy}{r^2}, \qquad r^2 = x^2 + y^2 \;\therefore\; |t^2| > 4 \;\therefore\; |t| > 2.$$

Thus the two sides of (B) have different moduli and the assumption that $z = x + iy$, $x > 0$ is a root of (A) leads to a contradiction.

If $z = -x + iy$, $x > 0$ is a root, then $x < 1$.

For then $T = e^{-z} = e^{x-iy}$ gives $|T| = e^x > e = 2{,}718 \cdots$ for $x > 0$.

$$t = 2 + \frac{1}{z} \text{ gives } |t^2| = 4 - \frac{4x}{r^2} + \frac{1}{r^2} < 4 + \frac{1}{r^2} < 4 + \frac{1}{x}.$$

If $x \geqq 1$,

$$|t| \leqq \sqrt{5} < 2{,}24.$$

We are thus led to a contradiction if $x > 1$.

We have thus the result:

All the roots of (A) *lie within the band* $-1 < x < 0$ *parallel to the imaginary y axis.*

This band we will call B, which we limit however to $y \geqq 0$.

3. To get further information regarding the location of the roots we will consider the graphs afforded by the two members of (B).

If we set $w = e^{-z}$ the graph of the band B is the ring about $w = 0$ whose radii are 1 and $e = 2{,}718 \cdots$.

Let us set

$$\omega = 2k\pi, \quad \alpha = \omega + \frac{\pi}{2}, \quad \beta = \omega + \pi,$$

$$\gamma = \omega + \frac{3\pi}{2}, \quad \delta = \omega + 2\pi,$$

where k is an arbitrary positive integer.

The straights $y = \omega$, $y = \alpha$, $y = \beta$, $y = \gamma$, $y = \delta$, give rise to four rectangles in the band B marked I, II, III, IV, in Fig. 1, to which correspond the 4 segments of the ring similarly marked in Fig. 2.

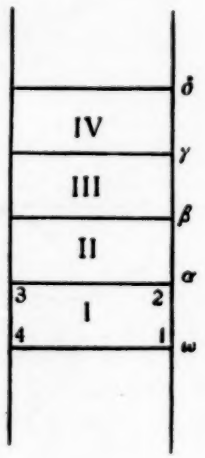

Fig. 1.

We now consider the map afforded by $t = 1/z$. To the straight $z = -a + iy$, $a > 0$ corresponds the circle

$$\left(u + \frac{1}{2}a\right)^2 + v^2 = \frac{1}{4}a^2.$$

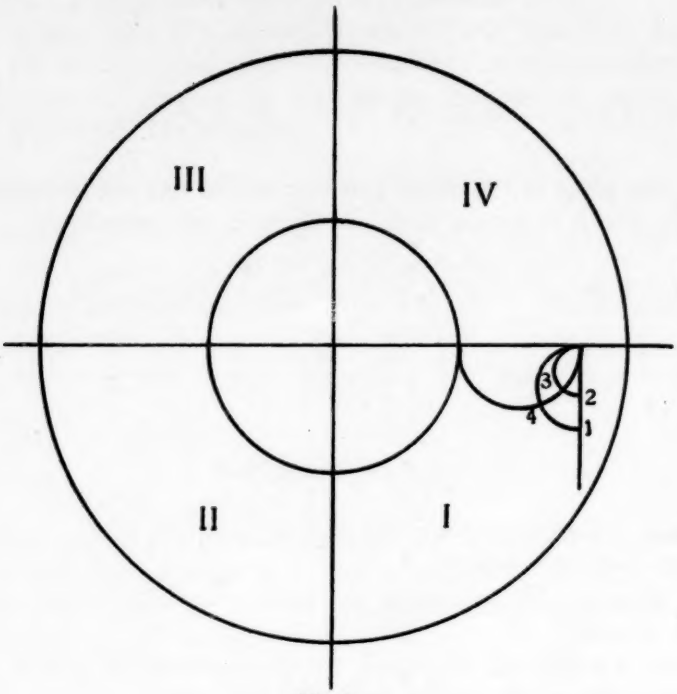

Fig. 2.

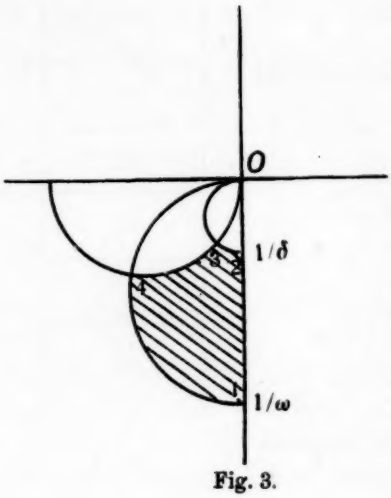

Fig. 3.

To the straight $z = x + ib$, corresponds the circle

$$u^2 + \left(v + \frac{1}{2}b\right)^2 = \frac{1}{4}b^2.$$

As the band B lies entirely in the 2° quadrant, its image afforded by $1/z$ lies entirely in the 3° quadrant. Thus the graph of the rectangles I, II, III, IV, $k > 0$ afforded by $1/z$ is the shaded region in Fig 3. The graph afforded by $s = 2 + 1/z$ is obtained by moving Fig. 3, 2 units to the right, parallel to the x axis. We have inserted it in Fig. 2. Now for z to be a root of (A) it is

necessary that w and s should represent the same z in Fig. 2. This shows that for $k>0$ there is no root z falling in the rectangles II, III, IV.

There are an ∞ of rectangles of the type I corresponding to $k=1, 2, \cdots$. *We show there is one and only one root in each of them.*

To this end we use Cauchy's theorem. Setting $g=2+1/z-e^{-z}$ we have

$$\frac{1}{2\pi i}\int_C d\log g(z) = \frac{1}{2\pi i}\Delta \log z = n,$$

the number of zeros of g in the rectangle I whose edge we call C. Here $\log \cdot g$ is the change in $\log g$ after z runs over C. Thus to show that g has one and only one root in I, we have only to show that the radius vector joining 0 with the point corresponding to g, describes a rotation of 360° about 0.

Now on the segment (1,2) in Fig. 1, $x=0$

$$u = \cos y - 2y\sin y,$$
$$v = \sin y + 2y\cos y - y.$$

At 1, $y=\omega$, $\therefore u=1$, $v=\omega$; at 2, $y=\alpha$, $u=-2\alpha$, $v=1-\alpha$. Hence as z describes the segment (1,2) g describes a curve which we have indicated schematically by (1,2) in Fig. 4. The point 1 is in the 1° quadrant and 2 in the 3°. We show that g crosses the u axis once in this interval. For if $u=0$ we have $\cos y = 2y\sin y$, or $2y = \cot y$. The two curves $Y=2y$, $\eta=\cot y$ obviously cut once and only once in the interval $\omega \leqq y \leqq \alpha$.

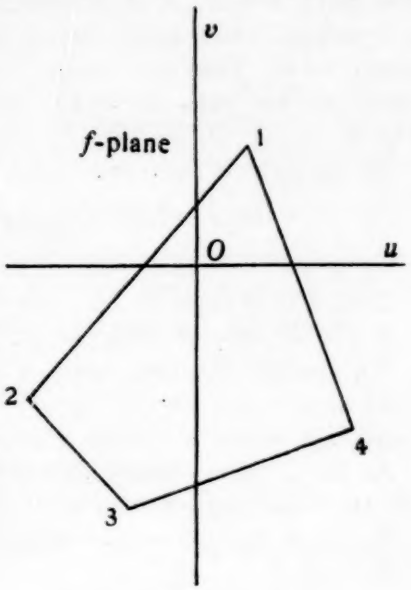

Fig. 4.

If $v=0$,

$$2y\cos y = y\sin y.$$

Let us consider the two curves

$$Y = 2y\cos y, \qquad \eta = y - \sin y$$

in the interval (ω, α). For $y=\omega$, $Y=2\omega$, $\eta=\omega$; for $y=\alpha$, $Y=0$, $\eta=\alpha-1$. These two curves therefore cross and as Y is steadily decreasing and η steadily increasing, the curves cross just once in (ω, α). On the segment (2,3) in Fig. 1, $y=\alpha$, hence

$$u = -2\alpha e^x - x, \qquad v = e^x(1+2x) - \alpha.$$

At 2, $x=0$; $u=-2\alpha$, $v=1-\alpha$; at 3, $x=-1$, $u=1-2\alpha/e$, $v=-\alpha-1/e$.

Thus f at 3 lies in the 3° quadrant in Fig. 4.

We show that as g goes from 2 to 3 it cuts neither the u nor v axis. For if $u=0$, $e^x=-x/2\alpha$. Obviously the two curves

$$Y=e^x, \quad \text{and} \quad \eta=-x/2\alpha$$

do not meet in the interval $(-1, 0)$. Hence $u\neq 0$. Similarly if $v=0$ we have

$$e^x=\frac{\alpha}{1+2x}.$$

The curve $Y=e^x$ is the exponential curve, the curve $\eta=\alpha/(1+2x)$ is a hyperbola whose graph shows at once that it does not meet the Y curve. Hence $v\neq 0$. Thus as z ranges over (2, 3) in Fig. 1, g describes a curve which neither cuts the u nor the v axis, as schematically indicated in Fig. 4.

On (3, 4), $x=-1$, and

$$u=-(\cos y+2y\sin y)e^{-1}+1, \qquad v=(2y\cos y-\sin y)e^{-1}-y.$$

At 3, $y=\alpha$; $u=1-2\alpha/e$, $v=-\alpha-1/e$.

At 4, $y=\omega$; $u=1-1/e$, $v=-\omega(e-2)/e$, a point in the 4° quadrant.

If $u=0$, $\cos y+2y\sin y=e$ or $2y\sin y=e-\cos y$.

We consider the two curves $Y=2y\sin y$, $\eta=e-\cos y$.

At 4, $y=\omega$, $Y=0$, $\eta=e-1$.

At 3, $y=\alpha$, $Y=2\alpha$, $\eta=e$.

As the Y curve steadily increases and the η increases, the two curves cut once and only once in (ω, α). Hence $u=0$ once only in (3, 4).

If $v=0$, $2y\cos y=ey+\sin y$.

Set

$$Y=2y\cos y, \qquad \eta=ey+\sin y.$$

At 4, $y=\omega$, $Y=2\omega$, $\eta=e\omega$.

At 3, $y=\alpha$, $Y=0$, $\eta=1+e\alpha$.

The Y curve steadily decreases, the η curve steadily increases hence they do not meet. Thus g moves along the segment indicated schematically in Fig. 4.

Finally on (4, 1), $y=\omega$

$$u=e^x(1+2x)-x, \qquad v=2\omega e^x-\omega.$$

If $u=0$,

$$e^x=\frac{x}{1+2x}.$$

The graph of the two curves $Y = e^x$, $\eta = x/(1+2x)$ shows that they do not meet. Hence $u \neq 0$ in (4, 1).

If $v = 0$, $e^x = 1/2$. Hence $v = 0$ once only in (4, 1).

We have thus shown that the radius vector Og turns just once around as z runs over the edge of the rectangle I. Hence

THEOREM. *In each rectangle* I *for* k $1, 2, \cdots$ *there lies one and only one root of* (A). *To each of these roots* $x+iy$ *there corresponds a conjugate root* $x-iy$.

Let z_n be the root of (A) in the nth rectangle I. Then

$$z_n = x_n + (2n\pi + t_n)i, \quad -1 < x_n < 0, \quad 0 \leqq t_n < \pi/2.$$

Also $\lim_{n=\infty} 1/z_n = 0$ $\therefore$ limit of the right side of (B) is 2. Hence

$$\lim_{n=\infty} e^{-z_n} = 2 = \lim e^{-x_n} e^{it_n}.$$

Hence

THEOREM. *An asymptotic expression for the roots of* (A) *is*

$$z = -\log_e 2 + 2m\pi i$$

where m *is a large positive or negative integer.*

In the foregoing we have excluded the rectangle corresponding to $k = 0$. This rectangle has $z = 0$ as one vertex and this would require a division by 0 in our work. Let us therefore return to equation (A) and set

$$f = (1+2z)e^z - z.$$

If we allow z to range over the rectangle, i. e., the rectangle R_0, whose opposite vertices are $z = 0$, $z = -1 + \pi i/2$, we find by rough plotting that f describes a simple closed curve enclosing the origin. Hence equation (A) has a single root in this rectangle.

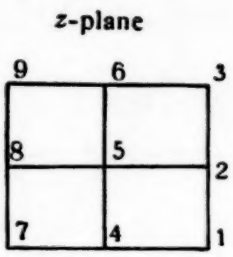

Fig. 5.

4. We now show how any one of the roots of (A) or (B) may be calculated with an assigned degree of accuracy. To fix our ideas we seek the root lying in the rectangle R_0.

A rough calculation shows this root lies in the rectangle T whose opposite vertices are $z = -1/4 + i/2$ and $z = -1/2 + 7/10\,i$. This rectangle we now divide into 9 equal rectangles whose vertices we call $1, 2, \cdots, 9$, as in Fig. 5.

We now let z run over T and calculate its image as given by

$$g = 2 + \frac{1}{z} - e^{-z} = u + iv,$$

where, setting $x' = -x$,

$$u = 2 - x'/r^2 - e^{x'}\cos y, \qquad v = e^{x'}\sin y - y/r^2.$$

To calculate the values of g when z is at the points $1, 2, \cdots, 9$ we form the following table, using a sliderule. Here

$$a = e^{x'}\cos y, \qquad b = e^{x'}\sin y.$$

	x'	y	$\cos y$	$\sin y$	r^2	x'/r^2	y/r^2	a	b	u	v
1	.25	.5	.878	.479	.3125	.800	1.60	1.12	.613	.08	−.99
2	.25	.6	.825	.565	.4225	.592	1.42	1.06	.723	.35	−.70
3	.25	.7	.765	.644	.5525	.453	1.27	.979	.824	.57	−.45
4	.375	.5	.878	.479	.391	.959	1.27	1.27	.694	−.23	−.58
5	.375	.6	.825	.565	.501	.748	1.20	1.20	.819	.05	−.38
6	.375	.7	.765	.644	.631	.595	1.11	1.11	.934	.30	−.18
7	.5	.5	.878	.479	.5	1.—	1.00	1.45	.790	−.45	−.21
8	.5	.6	.825	.565	.61	.82	.984	1.36	.932	−.18	−.05
9	.5	.7	.765	.644	.74	.676	.946	1.26	1.06	.06	+.11

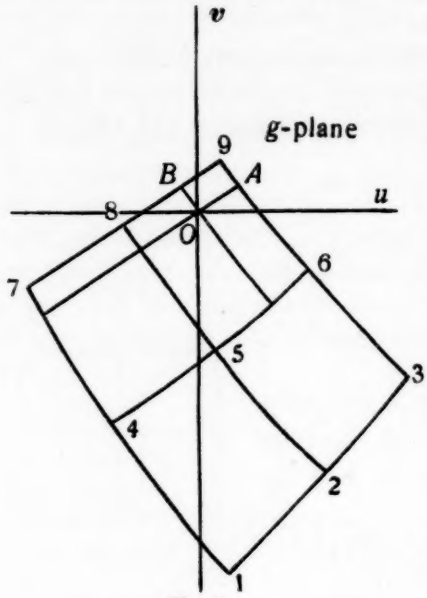

Fig. 6.

From the graph Fig. 6 we see that the root lies in the rectangle $(5689) = T'$. If in the g graph we draw through O the curves roughly parallel to the nearest sides we find by measurement that $OA = 0{,}35$ of the side (8,9) and $OB = 0{,}15$ of the side (6,9). As g moves nearly proportional to z, we may divide the rectangle (5689) in the z-plane in the same proportion and take as approximation of the root

$$a = -0{,}48 + 0{,}65\,i.$$

5. To get closer approximations to the root we extend Newton's method to complex roots. We set

$$u_1 = -\frac{f(a)}{f'(a)}, \qquad a_1 = a + u_1, \qquad |u_1| = \varrho_1$$

$$u_2 = -\frac{f(a_1)}{f'(a_1)}, \qquad a_2 = a_1 + u_2, \qquad |u_2| = \varrho_2 \quad \text{etc.}$$

We have to show that the sequence

$$a_1, a_2, a_3 \cdots, \tag{1}$$

has a limit ζ and that $f(\zeta) = 0$. It is desirable also to know the degree of approximation at any stage a_n of the calculation.

The first to treat this question was Cauchy. In the note to his Leçons sur le Calcul Différentiel, Paris, 1829, = Oeuvres 2nd Series, vol. 4, pp. 573–609, we find a complete solution of this problem. This note seems to have been buried in oblivion until resurrected by J. Helo in a memoir, Über die Newtonsche Näherungsmethode und eine Verallgemeinerung derselben. Helsingfors, 1914. This in turn seems to have met a similar fate, as I find no reference to it in the literature. This is to be regretted as Helo's paper not only treats the approximation obtained by setting the first two terms of the development

$$f(a)+uf'(a)+\frac{u^2}{2}f''(a)+\frac{u^3}{6}f'''(a)+\cdots \tag{2}$$

equal to 0, but also that obtained by setting the first three terms equal to 0. Also by generalizing the notion of the osculating parabola considered by Fourier,* Helo has obtained closer limits of approximation wether using two or three terms of (2).

Without going into these refinements we wish to show how Cauchy's results may be obtained using only Darboux's mean value theorem.

$$f(z) = f(a)+(z-a)f'(a)+\frac{\alpha}{2}(z-a)^2 f''(a, z). \tag{3}$$

Here $|\alpha| \leqq 1$ and (a, z) is a point on the segment $\overline{az}$.

This theorem is fundamental in an important paper by Faber.† To simplify our formulae we set

$$A = |f(a)|, \quad A' = |f'(a)|; \qquad A_1 = |f(a_1)|, \quad A_1' = |f'(a_1)|, \text{ etc.}$$

C_n denotes the circle $|z-a_n| \leqq \varrho_n \quad n = 1, 2, 3 \cdots$

$$m \leqq |f'(z)|, \quad |f''(z)| \leqq M \text{ in } C_1.$$

We have now the following

THEOREM 1. *Let $f(z)$ be a one value analytic function in C_1. If*

$$2\varrho_1 M < A', \tag{4}$$

$f(z) = 0$ has one and one root in C_1.

* J. Fourier. Analyse des équations déterminées. Paris, 1831.

† G. Faber. Über die Newtonsche Näherungsformel. Crelle's Journal 138 (1910), as well as in the paper by Helo. By its means, Cauchy's proof may be greatly simplified.

For let z be a point on the circumference of C_1. Then (3) gives replacing a by a_1

$$f(z) = (z-a_1)\left[f'(a_1)+\frac{f(a_1)}{z-a_1}+\frac{\alpha}{2}(z-a_1)f''\right] = (z-a_1)\,g(z).$$

Hence

$$\log f = \log(z-a)+\log g.$$

Hence by Cauchy's theorem, f has only one root in C_1 if $|g|>0$ on C_1. This is so if

$$A_1' > \frac{A_1}{\varrho_1}+\frac{1}{2}\varrho_1 M \text{ on } C_1. \tag{5}$$

Now

$$f(a_1) = f(a)+u_1 f'(a)+\frac{\beta}{2}u_1^2 f''(a, a_1), \qquad |\beta|\leqq 1.$$

As $f(a)+u_1 f'(a) = 0$ this gives

$$A_1 \leqq M\frac{\varrho_1^2}{2}. \tag{6}$$

Also

$$f'(a_1) = f'(a)+\gamma u_1 f''(a, a_1), \qquad |\gamma|\leqq 1.$$

Hence

$$A' \geqq |A'-\varrho_1 M|,$$

or using (4)

$$A_1' > |2\varrho_1 M-\varrho_1 M|,$$

$$\therefore \quad A_1' > \varrho_1 M. \tag{7}$$

Thus by (6)

$$\frac{A_1}{\varrho_1}+\frac{1}{2}\varrho_1 M < \varrho_1 M < A_1', \quad \text{by (7)}.$$

Hence when (4) holds so does (5), which proves the theorem.

THEOREM 2. *The condition of theorem* 1 *holding, we have*

$$\varrho_2 < \frac{1}{2}\varrho_1. \tag{8}$$

For by definition

$$\varrho_2 = \frac{A_1}{A_1'}. \tag{9}$$

By (6)

$$A_1 < \varrho_1^2\frac{M}{2}.$$

By (7)

$$A_1' > \varrho_1 M.$$

These in (9) give (8).

THEOREM 3. *If*

$$2\varrho_1 M < m, \tag{10}$$

$f(z)$ has a root in C_2.

For since (10) holds so does (4). Hence (8) holds

$$\therefore\ \varrho_2 < \varrho_1; \qquad \therefore\ 2\varrho_2 M < 2\varrho_1 M < m < A_1'.$$

Thus

$$2\varrho_2 M < A_1'. \tag{11}$$

This relation stands in the same relation to C_2 that (4) does to C_1. Thus the reasoning of theorem 1 applies to theorem 3. We see also that theorem 3 holds for each of the circles C_n, $n = 2, 3, \cdots$.

THEOREM 4. *The degree of approximation is afforded by the inequality*

$$\varrho_n < \varrho_1 Q^{(2^{n-1}-1)}, \tag{12}$$

where $Q = \varrho_1 M/2m < 1$. For

$$\varrho_2 = \frac{A_1}{A_1'} < \frac{A_1}{m} \leqq \frac{\varrho_1^2 M}{2m} = \varrho_1 Q.$$

Similarly

$$\varrho_3 = \frac{A_2}{A_2'} < \frac{\varrho_2^2 M/2}{m} < \frac{\varrho_1^2 Q^2 M}{2m} = \varrho_1 Q^3,$$

$$\varrho_4 < \varrho_1 Q^7, \cdots.$$

We apply these results to our equation (A). Here $f(z) = (1+2z)e^z - z$. If we take

$$a = -0{,}48 + 0{,}65\,i,$$

we find

$$u_1 = -.000535 + .00944\,i, \quad \varrho_1 = .00946,$$
$$a_1 = -0{,}48405 + 0{,}6594\,i.$$

Instead of the circle C_1, I have taken the rectangle S whose opposite vertices have the coördinates

$$(0{,}46;\ 0{,}63), \qquad (0{,}50;\ 0{,}67).$$

I find that in S

$$1{,}4 < |f'(z)|, \qquad |f''(z)| < 2{,}8.$$

Hence à fortiori relative to C

$$1{,}4 < m, \qquad M < 2{,}8.$$

The relation (10) holds here since

$$2\varrho_1 M < 0{,}063 < 1{,}4 < m.$$

Here

$$Q < \frac{1{,}12 \cdot 10^{-2} \cdot 2{,}8}{2 \cdot 1{,}4} = 1{,}12 \cdot 10^{-2}.$$

Hence

$$\varrho_3 < 1{,}33 \cdot 10^{-8}.$$

EXTENDED RESULTS IN ELIMINATION.*

By Thomas W. Moore.†

The primary purpose of this paper is to offer a means of securing an eliminant condition for three double binary forms in digredient variables. The method followed is a generalization of one employed by Coble and Morley‡ for the case of j different forms in j variables, applicable to a certain definite range of cases. The earlier paragraphs will illustrate the method of Coble and Morley applied to the case of two binary forms of arbitrary order, with the particular object of showing that the eliminant condition may be obtained in more than one way. The remainder of the paper will be devoted to the consideration of the three double binary forms in digredient variables, and a determinant representing the eliminant will be established for a limited range of cases.

The initial problem in elimination with which we are concerned may be stated as follows: Given j forms, homogeneous and of orders $m_1, m_2, \cdots, m_j$ in the j variables y, to find the polynomial in the coefficients of the various forms whose vanishing is the necessary and sufficient condition that these j forms have a common solution $x_1, x_2, \cdots, x_j$. It is desired to find this eliminant or resultant as a single determinant containing no extraneous factors and not vanishing identically. The method of Coble and Morley is not the only means of effecting this for certain cases of forms in more than two homogeneous variables, but it does offer a more general method than any heretofore devised. The eliminant may sometimes be obtained by a dialytic method of elimination of pairs of variables from successive pairs of equations, where the resultant would be secured multiplied by some extraneous factor. But a different form would arise for every different way in which this could be done, and the resultant would appear as the G. C. D. of all these results.

The method of Morley and Coble may be stated briefly as follows. Given the forms:

$$\alpha_{1y}^{m_1},\ \alpha_{2y}^{m_2},\ \cdots,\ \alpha_{jy}^{m_j}, \qquad m_1 \leqq m_2 \leqq \cdots \leqq m_j,$$

* Received April 25, 1928. Presented to the American Mathematical Society, April 7, 1928.

† National Research Fellow in Mathematics.

‡ F. Morley and A. B. Coble, New results in elimination, Amer. Journ. Math., 49, 4 (1927), pp. 463–488.

then they define the covariant J_k:

$$J_k = \sum_k (\alpha_1, \alpha_2, \cdots, \alpha_j)\ \alpha_{1x}^{m_1-k_1-1}\ \alpha_{1y}^{k_1} \cdots \alpha_{jx}^{m_j-k_j-1}\ \alpha_{jy}^{k_j},$$
$$k_1+k_2+\cdots+k_j = k,$$

where it is assumed that x is known as a common solution of the j forms. Next they consider the modulus

$$F = \beta_{1y}^{k-m_1}\ \alpha_{1y}^{m_1} + \beta_{2y}^{k-m_2}\ \alpha_{2y}^{m_2} + \cdots + \beta_{jy}^{k-m_j}\ \alpha_{jy}^{m_j}$$

where $\beta_{1y}^{k-m_1}, \cdots, \beta_{jy}^{k-m_j}$, are arbitrary forms, and then prove the theorem: If the equations $\alpha_{1y}^{m_1} = 0, \cdots, \alpha_{jy}^{m_j} = 0$ have a common solution x, then all the covariants J_k for $k = 0, 1, \cdots, m_1 + m_2 + \cdots + m_j - j$ are contained in the modulus F. For $k = 0, 1, \cdots, m_1 - 1$, $J_k \equiv 0$. This theorem will furnish an identity in y, and from the number of equations resulting, one has to eliminate the products of the unknown root x and the unknown parameters β. By multiplying the forms $\alpha_{1x}^{m_1}, \cdots, \alpha_{jx}^{m_j}$ by products of x sufficient to raise them to the order $\sum^{j} m_i - j - k$ of J_k in x, one secures an added number of equations from which to eliminate the products in x. For further details of this method in the general case the reader may refer to the article already cited.

It is of particular interest here, in view of further applications, to consider the case of two binary forms of orders m and n respectively, $m \leqq n$. This case can be handled fully by other methods, notably Sylvester's dialytic method of elimination which gives a determinant of order $m+n$, and Bézout's method which affords a determinant of order n. It will be shown that the resultant determinant can be secured in several different forms and different orders. It may be remarked that for two forms of equal order, the determinant of lowest order obtained is just the determinant of Bézout.

Suppose the two binary forms are α_y^m, β_y^n; $m \leqq n$. Then we have $J_k = (\alpha\beta)\sum \alpha_x^{m-k_1-1}\ \alpha_y^{k_1}\ \beta_x^{n-k_2-1}\ \beta_y^{k_2}$, $k_1 + k_2 = k$, and also $J_k(x^{m+n-k-2}; y^k) \underset{y}{\equiv} \lambda_y^{k-m}\ \alpha_y^m + \mu_y^{k-n}\ \beta_y^n$. For $k < m$, $J_k \underset{y}{\equiv} 0$; for $m \leqq k < n$, only the term in α_y will appear on the right, and for $k \geqq n$, both terms will appear. k ranges in value from 0 to $m+n-2$.

For $k = 0$, J_k is the Jacobian of the forms α_x^m, β_x^n. We multiply α_x^m by $x_1^{n-2}, \cdots, x_2^{n-2}$; β_x^n by $x_1^{m-2}, \cdots, x_2^{m-2}$ securing $m+n-2$ equations in this way. The addition of the Jacobian equation gives $m+n-1$ equations from which to eliminate the $m+n-1$ products $x_1^{m+n-2}, \cdots, x_2^{m+n-2}$ and

consequently the eliminant results as a determinant of order $m+n-1$. Coble has shown that in the general case this determinant does not vanish identically, and it is obvious that it is of degree m in the coefficients β, n in the coefficients α.

When $k = m+n-2$, J_k contains no x and we have $m+n-1$ equations from the identity in y, from which to eliminate the quantities l, $n-1$ λ's and $m-1$ μ's, $m+n-1$ altogether. Thus the eliminant in this case is again a determinant of order $m+n-1$. This is the maximum order of the determinant secured, and for all other values of k in this range the determinants will be of lower order, the minimum order being n. The $m+n-1$ determinants obtained in this way may be arranged in pairs of conjugates, that is, with their rows and columns interchanged, for $k = k_j$ and $k = m+n-2-k_j$, $k_j = 0, \cdots, m+n-2$, when $m+n$ is odd, and with one unpaired determinant when $m+n$ is even.

It may happen that several different values of k will give determinants of minimum order. $k = (m+n-2)/2$, $m+n$ even, $k = (m+n-3)/2$, $m+n$ odd, will give a determinant of least order n, but not necessarily the only one. For example, for $m = 3$, $n = 7$, the orders of the nine determinants for successive values of k from 0 to 8 are 9, 8, 7, 7, 7, 7, 7, 8, 9, where the determinant of order 7 for $k = 4$ is the only one unpaired. For $m = n$, $k = m-1$ gives the single determinant of lowest order n, which is the determinant of Bézout.

We propose to extend this method to the case of three double binary forms in digredient variables, namely:

$$A\colon\ \alpha_t^m a_\tau^\mu;\quad B\colon\ \beta_t^n b_\tau^\nu;\quad C\colon\ \gamma_t^p c_\tau^\pi$$

where t and τ are the digredient variables. Since there is one degree of freedom in each variable two of these forms will be satisfied simultaneously for a finite number of value pairs t and τ, while it will be a condition for a third to be satisfied by the same pair of values. Consequently we wish to find the condition, in the form of a determinant, on the coefficients of these three forms that they have a common solution. The simplest case is the one of three bilinear forms, where all the indices are one.

One can see that the eliminant will be of degree $n\pi+\nu p$ in the coefficients of A; of degree $m\pi+p\mu$ in those of B; and of degree $m\nu+n\mu$ in those of C. For, if we eliminate τ from B and C, the coefficients of B will appear to the degree π, those of C to the degree ν. But the coefficients of B are already of degree n in t, and those of C of degree p in t. Hence the resultant of B and C will be of degree $n\pi+p\nu$ in t, and consequently the coefficients of A will enter in the resultant

to this degree. But we cannot pursue this method to get the eliminant itself very successfully, for if we took the pair A and B and eliminated τ once more, and then eliminated t from these two intermediate forms, we would find the coefficients of the three forms entering to too high degree. Furthermore we should have a different result for each of the three ways in which this could be done, and the resultant would have to appear as the G. C. D. of these results, an unsatisfactory process at best. So we wish to devise a more practical means of elimination.

Let us assume s, σ as the value pair satisfying the three equations $A = 0$, $B = 0$, $C = 0$, and use t, τ as current variables. Then in place of Coble and Morley's form J_k we define:

$$\begin{aligned}
J^{(1)}_{k,n} &= (bc)(\gamma\alpha)\sum_{k,\varkappa}\alpha_s^{m-k_1-1}\alpha_t^{k_1}\beta_s^{n-k_2}\beta_t^{k_2}\gamma_s^{p-k_3-1}\gamma_t^{k_3} \\
&\quad\times a_\sigma^{\mu-\varkappa_1}a_\tau^{\varkappa_1}b_\sigma^{\nu-\varkappa_2-1}b_\tau^{\varkappa_2}c_\sigma^{\pi-\varkappa_3-1}c_\tau^{\varkappa_3} \\
&\quad-(ca)(\beta\gamma\sum_{k,\varkappa}\alpha_s^{m-k_1}\alpha_t^{k_1}\beta_s^{n-k_2-1}\beta_t^{k_2}\gamma_s^{p-k_3-1}\gamma_t^{k_3} \\
&\quad\times a_\sigma^{\mu-\varkappa_1-1}a_\tau^{\varkappa_1}b_\sigma^{\nu-\varkappa_2}b_\tau^{\varkappa_2}c_\sigma^{\pi-\varkappa_3-1}c_\tau^{\varkappa_3}, \\
&\quad k_1+k_2+k_3 = k; \qquad \varkappa_1+\varkappa_2+\varkappa_3 = \varkappa, \\
J^{(2)}_{k,\varkappa} &= (ca)(\alpha\beta)\sum_{k,\varkappa}(\qquad\qquad)-(ab)(\gamma\alpha)\sum_{k,\varkappa}(\qquad\qquad), \\
J^{(3)}_{k,\varkappa} &= (ab)(\beta\gamma)\sum_{k,\varkappa}(\qquad\qquad)-(bc)(\alpha\beta)\sum_{k,\varkappa}(\qquad\qquad),
\end{aligned}$$

where the last four summations can be written down in a way similar to the first two, since they are defined completely by the symbolic factors standing outside the summation signs. These three forms of $J_{k\varkappa}$ are essentially distinct and all will be used at times. Each of these forms $J_{k\varkappa}$ is of order k in t, $\varkappa$ in τ, $m+n+p-2-k$ in s and $\mu+\nu+\pi-2-\varkappa$ in σ, andof degree one in the coefficients of each of the forms A, B, C. The form J_{00} will be denoted shortly by J.

In a manner analogous to that employed by Morley and Coble for the forms in a single set of variables, we shall make use of the forms $J_{k\varkappa}$ and the three forms A, B, C to set up a determinant which shall represent the eliminant in any given case. That implies that we must have a number of independent equations equal in number to the quantities to be eliminated. Next the determinant must contain the coefficients of each form to the proper degree and must not vanish identically. The forms $J_{k\varkappa}$ or some linear expression in them must be satisfied by the common solution s, σ.

It has not been proved so far, save in two or three isolated cases, that $J_{k\varkappa}$ can be expressed as a syzygy in terms of the three forms A, B, C

when they have a common solution, as Coble was able to do in the case of a single set of variables. That is, in symbolic notation, we have not been able yet to show the existence of the relation

$$J^{kx}(s^{m+n+p-2-k}, \sigma^{\mu+\nu+\pi-2-x}; t^k, \tau^x) \equiv \sum_{t,\tau} \lambda_t^{k-m} l_\tau^{x-\mu} \cdot A + \mu_t^{k-n} m_\tau^{x-\nu} B + \nu_t^{k-p} n_\tau^{x-\pi} \cdot C.$$

Obviously, since a $1:1$ form is the double binary form of lowest orders, this syzygy would first appear when using a J_{11} form. So we shall confine ourselves in the present paper to the types of forms to which J, J_{10} and J_{01} are applicable, where each of these must be identically zero in t and τ if the existence of a common pair of solutions is assumed

I. **Cases to which J is applicable.** Each one of the three forms $J^{(1)}$, $J^{(2)}$, $J^{(3)}$ will vanish for all values of the indices $m:\mu$, $n:\nu$, $p:\pi$ in case A, B, C are satisfied by a common value pair.

Type 1.

$$m:1, \quad n:1, \quad p:1, \quad m \leqq n \leqq p.$$

J is a form of order $m+n+p-2$ in s, 1 in σ. The following determinant of order $2(m+n+p-1)$ results.

	$2(m+n+p-1)$
1	$\lvert A\,B\,C \rvert$
$n+p-1$	$\lvert A \rvert$
$m+p-1$	$\lvert B \rvert$
$m+n-1$	$\lvert C \rvert$

This skeleton notation is the same as that used by Coble and Morley in their article (Cf. loc. cit., pp. 473–474). The numbers at the top of the determinant give the number of products in s and σ to be eliminated, or in other words the number of columns. The numbers at the left give in succession the number of equations resulting from the identity $J_{k,x}$ in t and τ, and the number of equations resulting from multiplying A, B, and C by products in s and σ of order sufficient to raise them to the orders of $J_{k,x}$ in these variables. For instance, in this particular case, there are $2(m+n+p-1)$ products in s and σ to be eliminated. J does not contain t and τ and thus only one equation arises from the identity, containing the coefficients of A, B, and C linearly. A is of orders $m:1$ and we multiply it by $s_1^{n+p-2}, \cdots, s_2^{n+p-2}$, getting in all $n+p-1$ equations involving only coefficients of A. Similar remarks hold for B and C.

The determinant should be of degrees $(n+p, m+p, m+n)$ respectively in the coefficients of A, B, and C. This is apparent from inspection.

The question of its vanishing under conditions other than the existence of a common solution for the three forms will be considered later.

Type 2.

$$m:1,\ n:1,\ 1:\pi,\ m \leqq n,$$
$$R(n\pi+1,\ m\pi+1,\ m+n), \qquad J(s^{m+n-1}, \sigma^{\pi}) \underset{t,\tau}{\equiv} 0.$$

The determinant is of order $(m+n)(\pi+1)$.

$$\begin{array}{r|c|} & (m+n)(\pi+1) \\ 1 & |ABC| \\ n\pi & |A| \\ m\pi & |B| \\ m+n-1 & |C| \end{array}.$$

Type 3.

$$1:\mu,\ 1:\nu,\ 1:\pi,\ \mu \leqq \nu \leqq \pi,$$
$$R(\nu+\pi,\ \mu+\pi,\ \mu+\nu), \qquad J(s,\ \sigma^{\mu+\nu+\pi-2}) \underset{t,\tau}{\equiv} 0.$$

The determinant is of order $2(\mu+\nu+\pi-1)$.

Type 4.

$$1:\mu,\ 1:\nu,\ p:1,\ \mu \leqq \nu,$$
$$R(p\nu+1,\ p\mu+1,\ \mu+\nu), \qquad J(s^{p},\ \sigma^{\mu+\nu-1}) \underset{t,\tau}{\equiv} 0.$$

The determinant is of order $(\mu+\nu)(p+1)$.

Types 3 and 4 are the same as 1 and 2 respectively when the variables t and τ are interchanged.

II. **Cases to which $J_{10}(J_{01})$ is applicable.** Since the form $J_{10}(J_{01})$ affects only the variable $s(t)$, the indices $\mu, \nu, \pi\ (m, n, p)$ may be considered perfectly arbitrary when we seek whether the various forms of $J_{10}(J_{01})$ will vanish when A, B, C have a common solution. Then consider $m:\mu$, $n:\nu$, $p:\pi$, and assume a common solution.

If

$$m = n = p = 1, \quad J_{10}^{(1)} = 0, \quad J_{10}^{(2)} = 0, \quad J_{10}^{(3)} = 0.$$

If

$$m = n = 1, \quad p \geqq 2, \quad J_{10}^{(1)} - J_{10}^{(2)} - J_{10}^{(3)} = 0,$$

but the separate forms fail to vanish of themselves. The case 1:1, 1:1, 2:2 is special and for it $J_{10}^{(1)} + J_{10}^{(2)} - J_{10}^{(3)} = 0$.

If

$$m = 1, \quad n \geqq 2, \quad p \geqq 2, \quad J_{10}^{(1)} - J_{10}^{(2)} + J_{10}^{(3)} = 0.$$

If

$$m \geqq 2, \quad n \geqq 2, \quad p \geqq 2, \quad J_{10}^{(1)} = 0, \quad J_{10}^{(2)} = 0, \quad J_{10}^{(3)} = 0,$$

The same conclusions hold for the choices of the indices μ, ν, π in the alternate case.

Thus it appears that we must use different forms of the covariant $J_{10}(J_{01})$ as the indices m, n, p (μ, ν, π) vary, in order to get a form which will vanish when A, B, C have a common solution.

Then the form $J_{10}(J_{01})$ may be used to secure a resultant in the following cases.

Type 1.

$m:1,\ n:1,\ 2:2,\ m \leqq n,$
$R(2n+2, 2m+2, m+n),$
$J_{10}(s^{m+n-1}, \sigma^2; t) \underset{t}{\equiv} 0,$

$$\begin{array}{r|c|} & 3(m+n) \\ 2 & |ABC| \\ 2n & |A| \\ 2m & |B| \\ m+n-2 & |C| \end{array},$$

$1:\mu,\ 1:\nu,\ 2:2,\ \mu \leqq \nu,$
$R(2\nu+2, 2\mu+2, \mu+\nu),$
$J_{01}(s^2, \sigma^{\mu+\nu-1}; \tau) \underset{\tau}{\equiv} 0,$

$$\begin{array}{r|c|} & 3(\mu+\nu) \\ 2 & |ABC| \\ 2\nu & |A| \\ 2\mu & |B| \\ \mu+\nu-2 & |C| \end{array}.$$

Type 2.

$m:1,\ 2:\nu,\ 2:\pi,\ \nu \leqq \pi,$
$R(2\nu+2\pi, m\pi+2, m\nu+2),$
$J_{10}(s^{m+1}, \sigma^{\nu+\pi-1}; t) \underset{t}{\equiv} 0,$

$$\begin{array}{r|c|} & (m+2)(\nu+\pi) \\ 2 & |ABC| \\ 2(\nu+\pi-1) & |A| \\ m\pi & |B| \\ m\nu & |C| \end{array},$$

$1:\mu,\ n:2,\ p:2,\ n \leqq p,$
$R(2n+2p, \mu p+2, \mu n+2),$
$J_{01}(s^{n+p-1}, \sigma^{\mu+1}; \tau) \underset{\tau}{\equiv} 0,$

$$\begin{array}{r|c|} & (\mu+2)(n+p) \\ 2 & |ABC| \\ 2(n+p-1) & |A| \\ p\mu & |B| \\ n\mu & |C| \end{array}.$$

Type 3.

$2:2,\ 2:2,\ 2:\pi,$
$R(2\pi+4,\ 2\pi+4,\ 8),$
$J_{10}(s^3, \sigma^{\pi+2}; t) \underset{t}{\equiv} 0,$

$$\begin{array}{r|c|} & 4(\pi+3) \\ 2 & |ABC| \\ 2(\pi+1) & |A| \\ 2(\pi+1) & |B| \\ 6 & |C| \end{array},$$

$2:2,\ 2:2,\ p:2,$
$R(2p+4,\ 2p+4,\ 8),$
$J_{01}(s^{p+2}, \sigma^3; \tau) \underset{\tau}{\equiv} 0,$

$$\begin{array}{r|c|} & 4(p+3) \\ 2 & |ABC| \\ 2(p+1) & |A| \\ 2(p+1) & |B| \\ 6 & |C| \end{array}.$$

The question whether any one of these determinants vanishes also when the forms A, B, C do not have a common solution is still to be answered.

This is shown not to be the case by taking special forms of A, B, C which do not possess a common solution and setting up the corresponding determinants. This has been done for every type considered, and for the J_{10} and J_{01} types for all the different covariant forms used. A particular example will serve to illustrate this.

Let us consider the type $2:2,\ 2;2,\ 2:\pi$, for which $J_{10}^{(1)} \underset{t}{\equiv} 0$ may be used. We may take the particular forms:

$$A\colon\ t_1^2\,\tau_1^2 = 0;\quad B\colon\ t_2^2\,\tau_2^2 = 0;\quad C\colon\ (t_1^2 - t_2^2)\,(\tau_1^\pi - \tau_2^\pi) = 0,$$

where A has the solution $0, 0$; B has the solution ∞, ∞; and C $1, 1$, and the three do not have a common solution. Then $J_{10}^{(1)}$ formed for A, B, C is

$$J_{10}^{(1)} = -s_2^3\,\sigma_1^{\pi+1}\,\sigma_2\,t_1 - 2s_1 s_2^2\,\sigma_1^{\pi+1}\,\sigma_2\,t_2 + 2s_1^2\,s_2\,\sigma_1\,\sigma_2^{\pi+1}\,t_1 + s_1^3\,\sigma_1\,\sigma_2^{\pi+1}\,t_2,$$

and when this is equated to zero identically in t, we have the two equations

$$2s_1^2\,s_2\,\sigma_1\,\sigma_2^{\pi+1} - s_2^2\,\sigma_1^{\pi+1}\,\sigma_2 = 0,$$
$$s_1^3\,\sigma_1\,\sigma_2^{\pi+1} - 2s_1\,s_2^2\,\sigma_1^{\pi+1}\,\sigma_2 = 0.$$

The determinant corresponding to the resultant is of order $4(\pi+3)$. When we arrange the columns first in powers of s, s_1^3 to s_2^3, and then under each product in s, according to the powers in σ, $\sigma_1^{\pi+2}$ to $\sigma_2^{\pi+2}$, we find that the first $4(\pi+1)$ rows, $2(\pi+1)$ from A and $2(\pi+1)$ from B, each contain one element 1 and all others zero. The next six rows coming from C have two elements $+1$, two -1, and the others zero. The last two rows are formed from J_{10}, with two elements other than zero. There are eight columns of the determinant whose elements found in the first $4(\pi+1)$ rows are all zero. Then suppressing the remaining $4(\pi+1)$ columns and these rows, we have left the determinant of eighth order:

$$\begin{vmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & -1
\end{vmatrix}$$

whose numerical value is 9. Consequently the determinant fails to vanish other than in the case of a common solution of A, B, C, and is of proper degree and therefore must actually be the resultant.

This method of elimination offers hope of extension in several ways. The immediate matter is to learn whether the general form $J_{k,x}$ can be expressed in terms of the three given forms by means of a syzygy of the type already mentioned. It is not particularly difficult to determine the types of forms to which a given $J_{k,x}$ would apply, once we know the form of the covariant expression we may use, for we have seen already that this is not always the same. It will be noted that this method affords a means of eliminating two sets of variables, where no one has ever handled more than one set before. The domain of the variables can possibly be extended to more than the binary field. For example, a pencil of curves in the plane affords a binary-ternary form. The most general form would combine these things, namely the elimination of several sets of variables in domains of arbitrary dimension from a given number of forms.

BALTIMORE,
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UNTERSUCHUNGEN ÜBER GESTALT UND LAGE ABGESCHLOSSENER MENGEN BELIEBIGER DIMENSION.*

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Einleitung.†

1. In einigen in den letzten zwei Jahren erschienenen Arbeiten[1] habe ich gezeigt, daß jeder kompakte metrisierbare Raum auf bestimmte Weisen mit elementar-geometrischen Gebilden (Komplexen) approximiert werden kann, und zwar so, daß bei dieser Approximation die Dimension erhalten bleibt. Diese Approximationen können dabei auf zwei prinzipiell ver-

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† In dieser Einleitung wird nicht nur eine Übersicht der in der Arbeit enthaltenen Resultate gegeben, sondern auch eine Reihe von Definitionen aufgestellt, die im folgenden ohne weiteres als bekannt vorausgesetzt werden.

[1] "Simpliziale Approximationen in der allgemeinen Topologie", "Über den allgemeinen Dimensionsbegriff usw.", Math. Ann., 96 (1926) und 98 (1928), SS. 489–511 und 617–637, ferner Comptes rendus Paris, 183 (1926), SS. 640 und 722, sowie 184 (1927), SS. 317, 425, 575 (séances du 7 et 21 février et du 7 mars 1927). Diese Arbeiten werden als "Simp. Appr.", "Dimensionsbegriff" und "C. R.", 1, 2, 3, 4, 5 zitiert.

Die Kenntnis dieser Arbeiten wird im folgenden nicht vorausgesetzt. Dagegen wird die Kenntnis der Hauptbegriffe der Dimensionstheorie (so wie sie z. B. in meiner "Darstellung der Grundzüge der Urysohnschen Dimensionstheorie", Math. Ann., 98, SS. 31–63 dargestellt sind) angenommen. Die letztgenannte Arbeit wird kurz als "Darstellung" zitiert; dort wird auch auf die, die Dimensionstheorie eröffnende Brouwersche Note (Crelle, 1913) und die grundlegenden Arbeiten von Urysohn und Menger hingewiesen.

Aus der kombinatorischen Topologie wird in den beiden ersten Kapiteln nur der Begriff des n-dimensionalen Komplexes benutzt: darunter wird eine aus endlich vielen Simplexen zusammengesetzte Figur verstanden, wobei angenommen wird, daß alle diese Simplexe von Dimensionen $\leqq n$ und mindestens eins genau n-dimensional ist; ferner haben zwei Simplexe des Komplexes entweder keinen gemeinsamen Punkt, oder eine gemeinsame Seite, die dann ihren Durchschnitt bildet. Auch sämtliche Seiten dieser Simplexe werden als Simplexe des Komplexes betrachtet. Ein Simplex, der nicht eine Seite eines anderen Simplexes des Komplexes ist, soll ein *Grundsimplex* heißen. Vgl. hierzu Simpl. Appr., §§ 1–2, sowie Dimensionsbegriff, Fußnote 2).

In den Kapiteln 3–4 wird überdies der Begriff der Bettischen Zahl im Sinne der Definitionen "modulo 2" benutzt. Vgl. hierzu z. B. Veblen und Alexander "Manifolds of n dimensions" (Annals of math., 14 (1913), 163–178), Veblen, Analysis Situs (Colloquium Lectures), Alexander, "A proof and generalisation of the Jordan-Brouwer separation theorem", (Trans. Amer. Math. Soc., 23 (1922), SS. 333–349), und insbesondere Alexander, "Combinatorial Analysis Situs" (Transactions, 28 (1926), SS. 301–329); die beiden letztgenannten Arbeiten von Alexander werden als J-B-Th. bzw. CAS zitiert; sie enthalten reichlich alles, was im folgenden aus der kombinatorischen Topologie benutzt wird.

schiedene und dennoch miteinander nahe verwandte Weisen aufgefaßt werden.

Die eine ergibt sich daraus, daß man zu jedem kompakten metrisierbaren Raum eine Folge von (abstrakt zu denkenden) Komplexen definieren kann, die miteinander nach bestimmten Regeln verbunden werden; eine solche Folge von Komplexen, betrachtet mit den zugehörigen "Bindungsrelationen" (auf deren nähere Struktur es im Augenblicke nicht ankommt), bildet das, was ich ein *Spektrum* des Raumes nenne; dabei kann man stets voraussetzen, daß (im Falle eines endlich-dimensionalen Raumes) die Komplexe des Spektrums sämtlich von der Dimension des Raumes sind.

Auch umgekehrt (und das ist wesentlich) *approximiert* auf eine bestimmte Weise jede vermöge gewisser abstrakt zu formulierender Relationen als ein Spektrum zu betrachtende Folge von Komplexen einen kompakten metrisierbaren Raum; dabei tritt diese Folge von Komplexen als Spektrum dieses (durch sie approximierten) Raumes auf; ferner definiert ("approximiert") ein *n-dimensionales* (das heißt aus lauter n-dimensionalen Komplexen bestehendes) Spektrum immer einen höchstens n-dimensionalen Raum. Es sei noch hinzugefügt, daß, falls der gegebene Raum von einer unendlichen Dimension ist, jedes diesen Raum approximierende Spektrum in dem Sinne *unendlich-hoch dimensional* ist, daß alle Komplexe des Spektrums bis auf höchstens endlich viele ihrer Dimension nach jede, noch so große Zahl überschreiten.

Das ist die *abstrakte Simplizialapproximation.*

2. Es gibt aber auch eine andere, rein geometrische Art, eine beliebige abgeschlossene[2] Menge F mit Komplexen zu approximieren und diese besteht in den sogenannten *kleinen Transformationen* bzw. *kleinen Abbildungen.*

Es sei zunächst ε eine beliebig kleine (aber feste) positive Zahl und F eine (in einem metrischen Raume R liegende) abgeschlossene Menge. Ich sage, daß die eindeutige stetige Abbildung f der Menge F auf eine Teilmenge desselben Raumes R eine *ε-Transformation* ist, wenn jeder Punkt x von F von seinem Bildpunkte $f(x)$ höchstens um ε entfernt ist. Es ist leicht, diese Definition von der Voraussetzung, daß F sowie die Bildmenge $f(F)$ im selben metrischen Raume R liegen, zu befreien: das geschieht gerade durch Einführung des Begriffes einer *ε-Abbildung.* Es sei f eine eindeutige stetige Abbildung des kompakten metrischen Raumes F,

[2] "Abgeschlossene Menge" = "*kompakte* abgeschlossene Teilmenge des Hilbertschen Raumes" = "kompakter metrisierbarer Raum" (Vgl. z. B. Urysohn, "Der Hilbertsche Raum usw.", Math. Ann., 92 (1924), S. 302, oder auch "Darstellung", § 1). Für nicht kompakte Mengen wird das Wort "abgeschlossene Menge" *nie* gebraucht; es könnte höchstens von (in bezug auf einen gegebenen Raum) abgeschlossenen *Teil*mengen (dieses Raumes) die Rede sein (so z. B. in Kap. I, § 1).

wobei die Bildmenge $f(F)$ in irgendeinem (metrischen oder sogar nur topologischen) Raume liegt. Ich nenne diese Abbildung eine *ε-Abbildung*, wenn, wie der Punkt ξ von $f(F)$ auch gewählt sein möge, die Originalmenge $f^{-1}(\xi)$ dieses Punktes von einem Durchmesser $\leqq 2\varepsilon$ ist.

3. Es sei nun irgendeine Klasse $\mathfrak{K}$ von (als Punktmengen aufzufassenden) geometrischen Figuren gegeben, deren Struktur uns aus irgendwelchen Gründen einfach und bequem erscheint, so daß wir gerne z. B. jede abgeschlossene Menge in irgendeinem Sinne durch die Figuren der Klasse $\mathfrak{K}$ beschreiben, durch diese Figuren ausdrücken möchten. Dann ist es zweckmäßig, folgende Definition einzuführen: Ich sage, daß die abgeschlossene Menge F durch eine *kleine Abbildung* (kurz: durch eine *k. A.*) in eine Figur der Klasse $\mathfrak{K}$ übergeht, wenn man zu jedem ε eine ε-Abbildung der Menge F finden kann, so daß die Bildmenge von F bei dieser Abbildung eine Figur der Klasse $\mathfrak{K}$ ist. Wenn dabei sowohl die Menge F als auch sämtliche Figuren der Klasse $\mathfrak{K}$ in einem festen metrischen Raume R liegen, so kann man (in einem völlig analogen Sinne) von *kleinen Transformationen* (kurz: *k. T.*) sprechen.

Die geometrische Approximation einer abgeschlossenen Menge ist nun durch den Satz gegeben: *Jeder kompakte metrische Raum von der Dimension n geht durch kleine Abbildungen in n-dimensionale Komplexe, aber nicht in Komplexe niedrigerer Dimension über.*[3]

4. Es gilt sogar mehr. Vermöge eines Urysohnschen Satzes ist jeder kompakte metrisierbare Raum einer abgeschlossenen Teilmenge des Hilbertschen Raumes homöomorph und kann also als eine solche betrachtet werden; es ergibt sich nun, daß man den soeben formulierten Satz in folgender schärferen Form aussprechen kann: *Jede abgeschlossene kompakte Teilmenge des Hilbertschen Raumes kann mittels kleiner Transformationen in Komplexe, und zwar eine n-dimensionale Menge in Komplexe derselben Dimension n (und keiner niedrigeren) übergeführt werden; unendlich hoch dimensionale Mengen können zwar bei jedem ε in einen Komplex K_ε ε-transformiert werden, dabei aber wächst die Dimension von K_ε mit $\frac{1}{\varepsilon}$ notwendig über jede Grenze.*[3]

Das sind die Resultate, die die Grundlage und den Ausgangspunkt der vorliegenden Arbeit bilden. Sie werden aber nicht als bekannt vorausgesetzt: vielmehr werden sie im wesentlichen auf eine neue, einfachere und zweckmäßigere Weise abgeleitet und zu weitergehenden Betrachtungen benutzt.

5. Wenn wir uns fragen: worin liegt der Grund der Tatsache, daß die kompakten metrischen Räume eine so nahe Verwandtschaft mit den elementar

[3] Bewiesen zum erstenmal in C. R., 1, und im "Dimensionsbegriff".

geometrischen Gebilden haben, daß man (im Einklang mit den in den vorigen Paragraphen formulierten Sätzen) sagen kann, daß jeder dieser Räume im wesentlichen die Gestalt eines Komplexes derselben Dimension hat? — so lautet die Antwort auf diese Frage in aller Eindeutigkeit:

"Im Borel-Lebesgueschen Überdeckungssatze".

Es folgt in der Tat aus diesem Satze, daß ein kompakter metrischer Raum bei jedem $\varepsilon > 0$ als Vereinigungsmenge von endlich vielen abgeschlossenen Mengen dargestellt werden kann, deren Durchmesser sämtlich $< \varepsilon$ sind. Solche Systeme von abgeschlossenen Mengen wollen wir kurz *ε-Überdeckungen* des gegebenen Raumes nennen. Eine ε-Überdeckung der abgeschlossenen Menge F ist demgemäß ein endliches System von abgeschlossenen Teilmengen von F

$$F_1, F_2, \cdots, F_s, \tag{1}$$

die den beiden Bedingungen

$$F_1 + F_2 + \cdots + F_s = F,$$
$$\delta(F_i) < \varepsilon \qquad (i = 1, 2, \cdots, s)$$

genügen.

Zu jeder ε-Überdeckung (1) einer Menge gehört eine natürliche Zahl, und zwar die größte Zahl n von der Eigenschaft, daß es Punkte von F gibt, die zu n verschiedenen Elementen des Mengensystems (1) gehören. Diese wohlbestimmte natürliche Zahl heißt die *Ordnung*[4] des Mengensystems (der Überdeckung) (1). Eine ε-Überdeckung von der Ordnung n werde ich kurz als eine *(ε, n)-Überdeckung* bezeichnen.

Nun definiert aber jedes endliche Mengensystem

$$\mathfrak{P} = F_1, F_2, \cdots, F_s \tag{2}$$

folgendermaßen einen (abstrakt zu denkenden) Komplex $N(\mathfrak{P})$, den ich den *Nerv*[5] des Mengensystems (2) nenne und der eigentlich das einzige neue Hilfsmittel darstellt, welches die hier zu behandelnde Theorie zu dem bekannten Werkzeug der elementaren mengentheoretischen Topologie beiträgt. Die Konstruktion des Komplexes $N(\mathfrak{P})$ geschieht folgendermaßen. Jeder Menge F_i des Systems (2) lassen wir einen Eckpunkt a_i des zu konstruierenden Komplexes entsprechen. Wir machen ferner ein Eckpunktsystem

$$a_{i_1}, a_{i_2}, \cdots, a_{i_{r+1}}$$

dann und nur dann zum Eckpunktsystem eines (in den Komplex $N(\mathfrak{P})$ aufzunehmenden) r-dimensionalen Simplexes, wenn die Durchschnittsmenge $F_{i_1} \cdot F_{i_2} \cdot \cdots \cdot F_{i_{r+1}}$ nicht leer ist. Auf diese Weise werden der Reihe nach alle Simplexe von $N(\mathfrak{P})$ definiert. Die höchste Dimension eines Simplexes

[4] Lebesgue, Fund. Math., 2, SS. 257 u. f.

[5] Vgl. Simpl. Appr., § 13, und C. R., 3.

aus $N(\mathfrak{P})$ (die also zugleich die Dimension von $N(\mathfrak{P})$ selbst ist) ist somit $n-1$ (wenn n die Ordnung des Mengensystems $\mathfrak{P}$ war).

Diese, als Nerven von immer feiner werdenden Überdeckungen der abgeschlossenen Menge F auftretenden Komplexe $N(\mathfrak{P})$ sind es, die die Approximation von F liefern und die für die Gestalt dieser Menge maßgebend sind. Um das Übereinstimmen der Dimension von F (im allgemeinen Sinne) mit der Dimension dieser Komplexe einzusehen, genügt es, sich daran zu erinnern, daß die Dimension einer abgeschlossenen Menge nichts anderes ist, als die kleinste Zahl λ von der Beschaffenheit, daß es zu jedem ε eine $(\varepsilon, \lambda+1)$-Überdeckung dieser Menge gibt.[6]

6. Das erste Kapitel der vorliegenden Arbeit ist im wesentlichem gerade der Herstellung des Zusammenhanges zwischen den "kleinen Transformationen" und den Nerven von ε-Überdeckungen einer abgeschlossenen Menge gewidmet. Um von vornherein einen deutlichen Umriß des Anwendungsbereiches dieser Betrachtungen zu geben, fange ich mit dem Beweise der Tatsache an, daß *eine abgeschlossene Teilmenge des Hilbertschen Raumes dann und nur dann kompakt ist, wenn sie sich durch k. T. in Komplexe überführen läßt.*[7] Im Anschluß daran wird ein, wie mir scheint, endgültig einfacher Beweis des Überführungssatzes des § 4 (und somit auch des Satzes des § 3) gegeben.

7. Ferner wird im ersten Kapitel die Beziehung zwischen Nerven von ε-Überdeckungen und Komplexen, in die eine gegebenen Menge F durch k. T. (bzw. k. A.) übergeht, in die Form folgendes Satzes gebracht:

Jeder Komplex, der als Bild von F bei einer ε-Transformation auftritt, kann (nach eventueller Unterteilung) als Nerv einer 3ε-Überdeckung von F aufgefaßt werden.

Die Umkehrung dieses Satzes gilt ohne weiteres für zusammenhängende abgeschlossene Mengen (Kontinua); was allgemeine nicht zusammenhängende abgeschlossene Mengen betrifft, so kann man nur behaupten, daß eine solche Menge durch eine ε-Transformation in den Nerv einer beliebigen *irreduziblen* ε-Überdeckung übergeht.

Dabei heißt eine ε-Überdeckung $\mathfrak{P}$ der Menge F *irreduzibel,* wenn es keine ε-Überdeckung $\mathfrak{p}$ gibt, deren Nerv ein echter Teilkomplex[8] des Nerves von $\mathfrak{P}$ ist.

[6] "Darstellung", Kap. II.

[7] Wie mir Herr Fréchet mitteilt, befindet sich dieser (gleichzeitig mit mir auch von Niemytzki bewiesene) Satz im wesentlichen bereits in seiner Arbeit aus Pal. Rend., 30 (1910), S. 18; vgl. auch Fréchet, Fund. Math., 12, S. 304.

[8] Unter einem *Teilkomplex* eines Komplexes K wird (wenn nichts anderes explizit gesagt ist) stets ein *kombinatorischer* Teilkomplex verstanden, d. h. ein Komplex, dessen Simplexe sich sämtlich unter den Simplexen von K befinden; ein Teilkomplex von K ist echt, wenn er nicht mit K identisch ist.

Jedenfalls kann man aber eine beliebige F in einen echten oder unechten Teilkomplex des Nerves einer beliebigen ε-Überdeckung ε-überführen, so daß man folgende Behauptung aussprechen kann:

Wenn eine Folge von Komplexen gegeben ist, in die ein kompakter metrischer Raum F durch k. A. (bzw. k. T.) übergeht, so ist dadurch auch eine Folge von beliebig fein werdenden ε-Überdeckungen von F definiert, die die gegebenen Komplexe als ihre Nerven haben; wenn umgekehrt eine Folge von beliebig fein werdenden ε-Überdeckungen gegeben ist, so ist dadurch eine Folge von Komplexen definiert, in welche F durch k. T. bzw. k. A. übergeht; diese Komplexe sind dabei echte oder unechte Teilkomplexe der Nerven der gegebenen Überdeckungen.

8. Beim Beweise des letztgenannten Satzes spielt folgende (leicht zu bestätigende) Tatsache eine wesentliche Rolle: *Jeder Komplex kann als Nerv des Systems seiner baryzentrischen Sterne*[9] *betrachtet werden.* Diese Tatsache ist für uns von zweierlei Wichtigkeit. Erstens bildet sie insofern die Grundlage der ganzen Approximationstheorie, als sie zeigt, daß im einfachen, sozusagen rationalen Falle — im Falle, wo der gegebene Raum ein Komplex ist — die approximierenden Komplexe, wie es sich auch gehört, mit dem gegebenen Komplexe bzw. mit seinen Unterteilungen identisch sind, wodurch die ganze Approximation zu einer Trivialität wird. Zweitens wird durch diese Tatsache ein Zusammenhang zwischen der hier vertretenen Ansicht, die Gestalt jeder Menge sei durch die Gestalt ihrer Überdeckungsnerven im hohen Maße definiert, und dem Poincaréschen Dualitätsgesetz d. h. der Existenz von dualen Zellenzerlegungen einer Mannigfaltigkeit hergestellt. In der Tat sind die Elemente der zu einer gegebenen Simplizialzerlegung einer M^n dualen Zellenzerlegung nichts anderes als die baryzentrischen Sterne der Simplizialzerlegung; die Simplizialzerlegung der Mannigfaltigkeit ist somit mit dem Nerv des (als System seiner Elemente betrachteten) "reziproken Polyeder" identisch. Im allgemeinen Falle eines beliebigen Raumes hat man statt der "reziproken Polyeder" ε-Überdeckungen (die im Falle von Komplexen noch immer durch die baryzentrischen Sterne gegeben sind); die Nerven dieser Überdeckungen können aber mit dem Raume selbst nur dann zusammenfallen, wenn dieser Raum ein Komplex ist (dann aber tun sie es auch); in übrigen Fällen beschränken sich diese Nerven darauf, den Raum zu *approximieren.*

9. Das zweite Kapital ist den abstrakten Approximationen gewidmet. Wie schon erwähnt, erhält man ein Spektrum des gegebenen Raumes, wenn man die (als Nerven von unendlich fein werdenden Überdeckungen

[9] Die baryzentrische Unterteilung ist unter dem Namen *regular subdivision* von Veblen eingeführt worden; (siehe z. B. Colloquium Lectures, SS. 41–44 und 85–88). Die Definition ist in Kap. I, § 10, wiedergegeben.

auftretenden) Komplexe einer unendlichen Folge untereinander mittels gewisser Relationen verbindet. Diese Relationen sind in meiner ursprünglichen Darstellung[10] axiomatisch beschrieben und bildeten den kompliziertesten Teil des ganzen Approximationsbegriffes. In der vorliegenden Arbeit wird dieser Begriff auf eine, wie mir scheint, prinzipiell viel einfachere Grundlage gestellt, indem die Komplexe zu einem Spektrum dadurch verbunden werden, daß für jedes m eine simpliziale Abbildung[11] des $m+1$-ten Komplexes der Folge auf den m-ten Komplex vorgeschrieben wird. Auf diese Weise läßt sich eine besondere Art eines Spektrums, das sogenannte Projektionsspektrum, einführen, dessen Definition wir schon jetzt in aller Ausführlichkeit aussprechen.

10. Definition des Projektionsspektrums. Es sei eine Folge von Komplexen

$$K_1, K_2, \cdots, K_m, \cdots \tag{3}$$

und für jedes m eine simpliziale Abbildung $\pi(K_{m+1})$ des ganzen Komplexes K_{m+1} auf den ganzen Komplex K_m gegeben: $\pi(K_{m+1}) = K_m$; diese Abbildungen sollen die *Projektionen* (von K_{m+1} auf K_m) heißen.

Eine Folge von Simplexen

$$T_1, T_2, \cdots, T_m, \cdots \tag{T}$$

wobei für jedes m T_m ein Simplex von K_m und dabei T_m die Projektion von T_{m+1} ist, soll eine *Projektionsfolge* heißen. Wenn zwei Projektionsfolgen (T) und

$$t_1, t_2, \cdots, t_m, \cdots \tag{t}$$

so beschaffen sind, daß für jedes m entweder T_m mit t_m identisch, oder t_m eine Seite von T_m ist, so sagen wir, daß die Folge (t) von der Folge (T) *umfaßt* wird.

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[10] Simpl. Appr., §§ 6–8.

[11] Unter einer *simplizialen Abbildung* (Brouwer, Math. Ann., 70 (1911), S. 162) eines Komplexes A auf einen Komplex B versteht man eine folgendermaßen entstehende Abbildung: jedem Eckpunkt a von A ist ein Eckpunkt b von B zugeordnet, und zwar so, daß, wenn $a_1, a_2, \cdots, a_{r+1}$ die Eckpunkte eines Simplexes S^r von A sind, so sind die entsprechenden (evtl. zusammenfallenden) Eckpunkte $b_1, b_2, \cdots, b_{r+1}$ Eckpunkte eines Simplexes T von B. Jeder Punkt x von S^r kann sodann als Schwerpunkt gewisser in den Punkten $a_1, a_2, \cdots, a_{r+1}$ konzentrierter Massen aufgefaßt werden; wenn man dieselben Massen der Reihe nach in den Punkten $b_1, b_2, \cdots, b_{r+1}$ anbringt, so bestimmen sie eindeutig einen Schwerpunkt $y = f(x)$, der als Bildpunkt von x erklärt wird. Auf diese Weise wird jedes Simplex S von A auf ein gewisses Simplex T von B abgebildet, wobei diese Abbildungen sich zu einer stetigen — "*simplizialen*" — Abbildung des ganzen Komplexes A auf einen (echten oder unechten) Teilkomplex B' von B zusammenschließen. Bei der Definition des Projektionsspektrums wird stets angenommen, daß $B' = B$ ist.

Zwei Projektionsfolgen heißen *benachbart*, wenn jedes Simplex der einen mit dem entsprechenden Simplex der anderen mindestens einen gemeinsamen Eckpunkt hat.

Wir sagen nun, daß die Folge der Komplexe K_m und der zugehörigen Abbildungen $K_m = \pi(K_{m+1})$ ein Projektionsspektrum bildet, wenn folgende Bedingung erfüllt ist:

je zwei benachbarte Projektionsfolgen werden von einer dritten Projektionsfolge umfaßt.

11. **Definition des durch ein Projektionsspektrum approximierten Raumes.** Eine Projektionsfolge heißt eine *Kette*, wenn sie von keiner *anderen* Folge umfaßt wird.

Ein Projektionsspektrum definiert somit folgendermaßen einen Raum. *Die Ketten des Projektionsspektrums sind Punkte des Raumes*; die Simplexe, aus denen die Kette besteht, heißen *Koordinaten* des Punktes. Sodann besteht die m-te *Umgebung* eines (beliebigen) Raumpunktes ξ aus allen denjenigen Raumpunkten, deren erste m Koordinaten echte oder unechte Teilsimplexe der entsprechenden Koordinaten des gegebenen Punktes ξ sind.

12. **Bemerkung.** Aus den soeben formulierten Definitionen folgt, daß für jedes Paar natürlicher Zahlen m und r, $m < r$, eine simpliziale Abbildung $\pi^r_m(K_r)$ von K_r auf K_m nach der Vorschrift

$$K_m = \pi^r_m(K_r) = \pi \cdots \pi\pi(K_r)$$

definiert ist; diese Abbildung wollen wir die *Projektion von K_r auf K_m* nennen. Die identische Abbildung von K_r auf sich selbst soll gelegentlich auch als Projektion $\pi^r_r(K)$ bezeichnet werden, so daß das Projizieren des Komplexes K_r auf jeden Komplex K_m, $m \leqq r$, für jedes r eine eindeutig bestimmte Operation ist.

13. Aus dem soeben Gesagten folgt ohne weiteres, daß jede unendliche Teilfolge der ein Projektionsspektrum bildenden Folge von Komplexen, wenn man sie mit den zugehörigen Projektionen betrachtet, selbst zu einem Projektionsspektrum wird, wobei dieses "Teilspektrum" denselben Raum approximiert wie das ursprünglich gegebene Spektrum. Diese Bemerkung legt es nahe, sich auf die Betrachtung folgender beiden Typen von Projektionsspektren zu beschränken:

a) Projektionsspektra, deren sämtliche Komplexe dieselbe Dimension n haben — *solche Spektra heißen n-dimensional*;

b) Projektionsspektra, deren Komplexe K_m eine mit m ins Unendliche wachsende Dimension haben — solche Spektra heißen *unendlich-hochdimensional.*

14. Im zweiten Kapitel wird bewiesen, daß jeder kompakte metrisierbare Raum durch ein Projektionsspektrum *von der Dimension des Raumes*

approximiert werden kann, und daß umgekehrt, jedes Projektionsspektrum einen kompakten metrisierbaren Raum approximiert, dessen Dimension diejenige des Spektrums nicht übertrifft.

Ferner wird in diesem Kapitel der *Homöomorphiesatz* (§ 7) bewiesen, der Bedingungen aufstellt, unter welchen zwei Projektionsspektra denselben Raum approximieren; ohne eine solche Bedingung könnte die Theorie nicht als ein logisch-abgeschlossenes Ganzes angesehen werden.

Die soeben geschilderten Betrachtungen bilden den Inhalt des Hauptteiles des zweiten Kapitels. In einigen einleitenden Paragraphen werden irreduzible ε-Überdeckungen und deren Folgen untersucht, die für weitere Entwicklungen dieses Kapitels notwendig sind und auch ein selbständiges Interesse beanspruchen dürfen. Endlich wird (im § 18) ein wichtiger dimensionstheoretischer Satz von Urysohn[12] mit Hilfe der hier gemachten Betrachtungen aufs neue bewiesen.

15. Die Resultate der beiden ersten Kapitel erlauben es, die Frage der Zurückführung einer gestaltlichen Untersuchung von abgeschlossenen Mengen auf eine ebensolche Untersuchung von Komplexen wenigstens im Prinzip als gelöst zu betrachten. Es handelt sich nun um eine analoge Frage bezüglich der topologischen Eigenschaften, die der *Lage* einer abgeschlossenen Menge im n-dimensionalen Raum zukommen können. Um diese Frage in einer vernünftigen Weise stellen zu können, bemerken wir zunächst, daß der *Nerv einer beliebigen ε-Überdeckung* einer (im Euklidischen n-dimensionalen Raume R^n) liegenden abgeschlossenen Menge F immer dadurch *geometrisch realisiert werden kann*, daß man für seine Eckpunkte wirkliche Punkte der diesen Eckpunkten entsprechenden Mengen (Elemente der Überdeckung) nimmt und dann diese Eckpunkte zu Eckpunkten wirklicher elementar-geometrischer Simplexe macht, so daß der ganze Nerv zu einem geometrisch vorliegenden Komplex wird. Als auf diese Weise konstruierte Komplexe treten insbesondere die Komplexe auf, in die sich die Menge mittels k. T. überführen läßt. Indem man das auf alle Komplexe eines Spektrums anwendet, erhält man die *geometrische Realisation dieses Spektrums*. Das allgemeine Problem über die Lage einer abgeschlossenen Menge kann sodann folgendermaßen formuliert werden: *Welche Eigenschaften des Komplementärraumes $R^n - F$ lassen sich durch Eigenschaften der zu den geometrisch realisierten Komplexen eines Spektrums von F komplementären Mengen aussprechen?* Besonders interessant sind natürlich diejenigen Eigenschaften von $R^n - F$, die sich als Eigenschaften der im Spektrum auftretenden Komplexe selbst deuten lassen und dadurch einen Zusammenhang zwischen der Lage der Menge und ihrer Gestalt herzustellen erlauben.

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[12] "Darstellung", Kap. III, §§ 21–23.

Um ein konkretes Problem aus diesem Fragenkreis zu haben, erwähne ich nur das nächstliegende folgende: *Ist es möglich, die Zusammenhangszahlen von $R^n - F$ als Eigenschaften der approximierenden Komplexe zu deuten?* Diese Frage ist in ihrer allgemeinen Form noch nicht beantwortet. Nur in einem Spezialfall besitzen wir eine endgültige Lösung des Problems, nämlich in dem Falle, wo es sich um die einfachste (allerdings auch wichtigste) Lageeigenschaft einer Menge handelt, um ihre Eigenschaft, den Raum zu zerlegen oder nicht zu zerlegen. Der Behandlung dieser Eigenschaft ist das dritte Kapitel gewidmet, dessen wesentlicher Inhalt im Beweise der folgenden beiden Formen eines allgemeinen Zerlegungssatzes[13] besteht:

Eine abgeschlossene Menge F zerlegt den R^n dann und nur dann in eine endliche Anzahl $k+1$ von zusammenhängenden Gebieten, wenn k die kleinste Zahl von der Beschaffenheit ist, daß man für jedes ε eine ε-Überdeckung von F finden kann, deren Nerv die Zahl k als seine $n-1$-te Bettische Zahl hat. Die Menge F zerlegt den R^n dann und nur dann in unendlich viele Gebiete, wenn die $n-1$-te Bettische Zahl des Nerves einer beliebigen ε-Überdeckung mit Notwendigkeit über alle Grenzen wächst.

Nahezu denselben Satz kann man auch so ausdrücken:

F zerlegt den R^n dann und nur dann in höchstens $k+1$ Gebiete, wenn man durch kleine Transformationen F in Komplexe überführen kann, die den Raum ebenfalls in höchstens $k+1$ Gebiete zerlegen; (k ist dann die $n-1$-te Bettische Zahl dieser Komplexe).

16. Der Zerlegungssatz erlaubt unter anderem, diejenigen Kontinua invariant zu charakterisieren, die als gemeinsame Grenzen *aller* Komponenten ihres Komplementärraumes auftreten — solche Kontinua mögen *absolute* Gebietsgrenzen des R^n heißen. Auf diese Weise ist ein Weg gegeben, als Verallgemeinerung der Schönfliesschen geschlossenen Kurve und deren von Brouwer gegebenen invarianten Charakterisierung auch die Begriffe der *geschlossenen Fläche*, ferner überhaupt des *geschlossenen Gebildes* beliebiger Dimension topologisch-invariant zu begründen.[14] Diesen Weg wollen wir aber — wenigstens einstweilen — nicht gehen: eine wirklich befriedigende Definition solcher geschlossener Gebilde bedarf einer solchen Erklärung der Zusammenhangszahlen von F, die nicht nur für die Beschreibung der Zerlegungseigenschaften der gegebenen abgeschlossenen Menge in bezug

[13] C. R., 4; das Zerlegungsproblem ist für den Fall der Ebene ($n = 2$) in einer für alle späteren Untersuchungen auf diesem Gebiete grundlegenden Weise von Brouwer 1912 gelöst worden (Math. Ann., 72, SS. 422–425). Die in der vorliegenden Arbeit (und noch früher in der soeben zitierten Comptes-Rendus-Note) gegebene Lösung dieser Frage für den n-dimensionalen Raum schließt sich unmittelbar an die in meiner Arbeit "Kombinatorische Eigenschaften allgemeiner Kurven" (Math. Ann., SS. 512–555, insbesondere 527–534) gemachten Betrachtungen an. Vgl. auch Fußnote [16].

[14] C. R., 3 und 4.

auf den sie umfassenden R^n ausreicht, sondern zugleich sämtliche Zusammenhangszahlen des Komplementarraumes zu charakterisieren vermag. Diesen Bedingungen genügen *die Brouwerschen Zyklosenzahlen*,[15] welche unter dem kürzeren und sachlich angebrachten Namen der *Brouwerschen Zahlen* im Kapitel IV eingehend untersucht werden. Zunächst werden die Brouwerschen Zahlen in die in den früheren Kapiteln dargestellte allgemeine Approximationstheorie eingeordnet, was uns zugleich die Möglichkeit gibt, sie *ohne* jegliche Benutzung metrischer Begriffe zu definieren. Wenngleich metrisch formulierte Begriffe eine technisch bequemere Handhabe gestatten und deswegen auch nach Aufstellung ihrer topologischen Äquivalente benutzt werden können und sollen — wie es auch heute — trotz aller Metrisationssätze — mit dem Begriffe des metrischen Raumes, und in der vorliegenden Arbeit mit der ursprünglichen Form der Definition der Brouwerschen Zahl geschieht — so scheint mir eine rein topologische Fassung des Begriffes der Brouwerschen Zahl, wie ich sie gegeben habe, doch von einer *prinzipiellen* Wichtigkeit zu sein.

In den weiteren Paragraphen des vierten Kapitels wird zunächst mein früherer Beweis des Dualitätssatzes,[16] d. h. der Tatsache, daß die r-te Brouwerschen Zahl einer abgeschlossenen Menge $F \subset R^n$ gleich der $n-r-1$-ten Bettischen Zahl von $R^n - F$ ist, in einer verschärften Form erbracht.

Als Anwendung dieses Satzes wird eine topologisch-invariante Charakterisierung derjenigen Mengen $F \subset R^n$ gegeben, welche eine irreduzible Verschlingung[17] von einer gegebenen Dimension zulassen; als wichtigster Spezialfall dieses Verschlingungssatzes ist wohl die in ihm enthaltene Charakterisierung der gemeinsamen Grenzen von zwei oder mehreren Gebieten im R^n zu betrachten. Die als solche Gebietsgrenzen auftretenden abgeschlossenen Mengen bilden somit eine topologisch invariante Klasse von Kontinuen.

17. Erst der Beweis des Dualitätssatzes und der daran anschließenden Resultate ermöglicht den Begriff des n-dimensionalen geschlossenen Gebildes in der Gestalt der *geschlossenen Cantorschen Mannigfaltigkeiten* (im § 26 des Kap. IV) zu formulieren und eingehend zu begründen.

Zum selben Gedankenkreis gehören auch Sätze, die dimensionstheoretische Eigenschaften verschiedener kombinatorischer Konstruktionen in

[15] Brouwer, Math. Ann., 72 (1912), SS. 422–425; in voller Allgemeinheit zuerst bei Vietoris, Math. Ann., 97 (1927), SS. 454–472, insbesondere S. 464, Zeilen 5–10 von oben.

[16] Gött. Nachr. (Math.-Phys. Kl., 25. Nov. 1927); wie ich nachträglich erfahre, wurde derselbe Satz auch von Frankl (Wien. Ber., Dez. 1927, S. 689) bewiesen; er folgt auch aus einer von Lefschetz in Proc. Nat. Acad., 13 (1927), SS. 614–618 und 805–807 ausgesprochenen und in Annals of Math., 29 (1928), S. 232, bewiesenen allgemeineren Formel. Aus dem Dualitätssatz folgt eine neue Form des Zerlegungssatzes, da die Komponentenzahl von $R^n - F$ infolge des Dualitätssatzes um 1 größer als die $n-1$-te Brouwersche Zahl von F ist.

[17] Die Definition der irreduziblen Verschlingungen ist im Kap. III, § 13 gegeben.

abgeschlossenen Mengen feststellen und dadurch diese Konstruktionen in Einklang mit ihrem der Anschauung entnommenen Sinn bringen (§§ 18–25).

Ferner wird als Verallgemeinerung des Phragmén-Brouwerschen Satzes eine notwendige und hinreichende Bedingung dafür, daß zwei abgeschlossene Mengen, von denen keine den R^n zerlegt, eine diesen Raum zerlegende Vereinigungsmenge haben, als Spezialfall des entsprechenden Verschlingungssatzes im § 29 bewiesen.

Das Kapitel IV und mit ihm die ganze Arbeit schließen mit der Definition der *irreduziblen Membran*, welche als das mehrdimensionale Analogon des zwischen zwei Punkten irreduziblen Kontinuums zu betrachten ist.

Inhaltsübersicht.

ERSTES KAPITEL.

Kleine Transformationen (k. T.) und ε-Überdeckungen.

Abgeschlossene Mengen des Hilbertschen Raumes ..... §§ 1– 3
Der Überführungssatz und anschließende Sätze ..... §§ 4–11
Kleine Transformationen und ε-Überdeckungen ..... §§ 12–15
ε-Netze ..... §§ 16–17

ZWEITES KAPITEL.

Abstrakte Simplizialapproximationen.

Irreduzible ε-Überdeckungen ..... § 1
Folgen irreduzibler Überdeckungen. Unterteilungen ..... § 2
Konstruktion der approximierenden Spektra ..... §§ 3– 6
Der Homöomorphiesatz ..... §§ 7–17
Der Urysohnsche Zerlegungssatz ..... § 18

DRITTES KAPITEL.

Der Zerlegungssatz.

Allgemeine Bemerkungen über topologische Invarianten abgeschlossener Mengen. §§ 1– 4
Verschiedene Formen und Beweis des Zerlegungssatzes ..... §§ 5–12
Absolute Gebietsgrenzen ..... §§ 13–16

VIERTES KAPITEL.

Die Brouwerschen Zahlen.

Die Brouwerschen Zahlen vom Standpunkte der Approximationstheorie ..... §§ 1– 7
Der Dualitätssatz und seine Verschärfungen ..... §§ 8–13
Irreduzible Verschlingungen und Gebietsgrenzen ..... §§ 14–17
Definitionen und Sätze über Zyklen- und Homologieträger ..... §§ 18–25
Geschlossene Cantorsche Mannigfaltigkeiten und Phragmén-Brouwersche Sätze .. §§ 26–29
Irreduzible Membranen ..... §§ 30–32

Anhang I.

Bemerkungen über den Begriff eines Vollzyklus und seines Trägers.

Anhang II.

Der geometrische Dimensionsbegriff.

Anhang III.

Geschlossene Cantorsche Mannigfaltigkeiten und irreduzible Membranen.

ERSTES KAPITEL.

Kleine Transformationen (k. T.) und ε-Überdeckungen.

1. SATZ I. *Eine abgeschlossene Teilmenge F des Hilbertschen Raumes*[18] *ist dann und nur dann kompakt, wenn sie sich durch k. T. in Komplexe überführen läßt.*

Es sei in der Tat F eine kompakte abgeschlossene Teilmenge des Hilbertschen Raumes, und $\varepsilon > 0$ beliebig. Man wähle eine $\frac{\varepsilon}{6}$-Überdeckung

$$F_1, F_2, \cdots, F_s$$

der Menge und betrachte für jedes i, $1 \leqq i \leqq s$, einen beliebig, aber fest gewählten Punkt

$$a_i = (x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)}, \cdots)$$

von F_i. Man kann sodann die Zahl m so groß wählen, daß für alle i, $1 \leqq i \leqq s$, die Ungleichungen

$$\sum_{n=m+1}^{\infty} (x_n^{(i)})^2 < \frac{\varepsilon^2}{36}$$

gelten.

Man bilde jetzt die Menge F auf eine Teilmenge F' des durch die Gleichungen

$$x_{m+1} = x_{m+2} = \cdots = \cdots = 0$$

definierten Euklidischen Teilraumes R_m des Hilbertschen Raumes dadurch ab, daß man jedem Punkt

$$x = (x_1, x_2, \cdots, x_m, \cdots)$$

von F den Punkt

$$x' = (x_1, x_2, \cdots, x_m, 0, 0, 0, \cdots)$$

von R_m zuordnet. Diese Abbildung ist eine $\frac{\varepsilon}{2}$-Transformation, wie man folgendermaßen leicht zeigt. Zunächst gibt es für jeden Punkt x von F einen Punkt a_i, so daß die Entfernung $\varrho(x, a_i) < \frac{\varepsilon}{6}$ ist. Ferner ist

$$\varrho(a_i, a_i') = \sqrt{\sum_{n=m+1}^{\infty} (x_n^i)^2} < \frac{\varepsilon}{6}$$

und

$$\varrho(a_i', x') \leqq \varrho(a_i, x) < \frac{\varepsilon}{6},$$

[18] Siehe wegen der Definition des Hilbertschen Raumes z. B. Hausdorff, Mengenlehre (Berlin, 1927), S. 98.

also

$$\varrho(x, x') \leqq \varrho(x, a_i) + \varrho(a_i, a_i') + \varrho(a_i', x') < 3 \cdot \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.$$

2. Die Menge F läßt sich also durch eine $\frac{\varepsilon}{2}$-Transformation in eine beschränkte abgeschlossene Teilmenge F' des Euklidischen Teilraumes R^m des Hilbertschen Raumes überführen. Nun kann man aber F' innerhalb von R^m mittels folgenden "Ausfegungsverfahrens" in einen Komplex $\frac{\varepsilon}{2}$-transformieren.[19] Man betrachte eine Simplizialzerlegung $\mathfrak{Z}$ von R^m, deren sämtliche Simplexe einen Durchmesser $< \frac{\varepsilon}{2}$ haben und bezeichne durch Q die Gesamtheit aller Simplexe dieser Zerlegung, die im Innern oder auf dem Rande Punkte von F' enthalten. Wenn ein solches Simplex aus Punkten von F' besteht, fangen wir mit ihm nichts an. Es sei T^m ein Simplex von Q, das im Innern einen zu F' fremden Punkt enthält; man kann dann im Innern von T^m ein zu T^m homothetisches Simplex t^m finden, welches keinen Punkt von F' enthält; indem man den Rand von t^m homothetisch in den Rand von T^m transformiert, wird das ganze Zwischengebiet $T^m - t^m$ auf den Rand von T^m wegtransformiert. Nachdem wir das in allen Simplexen T^m gemacht haben, ist jedes der in Q vorkommenden Simplexe höchster (also m-ter) Dimension, insofern es nicht in F' enthalten war, ganz von Punkten der Menge F' befreit; diese Menge selbst wird aber infolge dieser Operation sich in eine abgeschlossene Menge F'' verwandelt haben, die jedes T^m aus Q enthält, sobald sie zu seinem Innengebiet nicht fremd ist. Auf dieselbe Weise fegt man alle T^{m-1} (insofern sie nicht zu F'' gehören) aus, und geht so zu Simplexen immer niedrigerer Dimensionen über, bis man schließlich zu den in Q vorkommenden Eckpunkten gelangt und als Endresultat eine Menge $F^{(m+1)}$ erhält, die zu einem beliebigen in Q vorkommenden Simplexe T^r $(0 \leqq r \leqq m)$ in einer der beiden folgenden Beziehungen steht: entweder ist F zum Innengebiet von T^r fremd, oder ist T^r in $F^{(m+1)}$ enthalten. Das bedeutet aber, daß $F^{(m+1)}$ ein (echter oder unechter) Teilkomplex von Q ist. Der ganze Übergang von F' zu $F^{(m+1)}$ ist aber eine $\frac{\varepsilon}{2}$-Transformation, weil während dieses Überganges jeder Punkt von F' in demselben Simplex Q verbleibt, zu dem er ursprünglich gehörte. Somit geht F in $F^{(m+1)}$ durch eine ε-Transformation über, und die erste Hälfte des Satzes I ist bewiesen.

3. Um zu zeigen, daß eine nicht kompakte Menge unmöglich durch kleine Transformationen in Komplexe übergeht, benutzen wir die *Voll-*

[19] Vgl. "Dimensionsbegriff", § 7.

ständigkeitseigenschaft[20] des Hilbertschen Raumes. Auf Grund eines Satzes von Hausdorff ist eine Teilmenge eines vollständigen Raumes dann und nur dann kompakt, wenn sie total beschränkt ist, d. h. sich für jedes ε als Vereinigungsmenge endlich vieler Teilmengen vom Durchmesser $<\varepsilon$ darstellen läßt. Falls nun die abgeschlossene Teilmenge F des Hilbertschen Raumes nicht kompakt ist und man ε so klein gewählt hat, daß F in keiner endlichen Summe von Mengen enthalten ist, deren Durchmesser $<\varepsilon$ ist, so ist es leicht, eine abzählbare Teilmenge D von F zu finden, deren Punkte paarweise voneinander Entfernungen $\geqq\varepsilon$ haben. Es sei jetzt $\varepsilon'=\frac{\varepsilon}{4}$ und eine ε'-Transformation von F beliebig gewählt. Wenn man die Bildmenge von D bei dieser Transformation durch D' bezeichnet, so haben die Punkte von D' paarweise eine Entfernung $\geqq\varepsilon-2\varepsilon'=\frac{\varepsilon}{2}$ voneinander, so daß D' eine divergente Menge ist; somit kann aber die Bildmenge von F bei unserer ε'-Transformation gewiß keine kompakte Menge und also erst recht kein Komplex sein, w. z. b. w.

4. Definition I. Es sei F eine abgeschlossene Teilmenge des R^n und δ eine positive Zahl; ein im Raume R^n liegender Komplex heißt ein *δ-Komplex in F*, wenn seine Eckpunkte sämtlich Punkte der Menge F und seine Simplexe sämtlich kleiner als δ sind.

Satz II. *Es sei F eine beliebige abgeschlossene Teilmenge des R^n und $\mathfrak{P}$ eine beliebige ε-Überdeckung von F, deren geometrisch realisierter Nerv durch N bezeichnet sein möge. Es existiert dann eine positive Zahl $\delta=\delta(F,\varepsilon,\mathfrak{P})$, so daß jeder δ-Komplex in F in einen echten oder unechten Teilkomplex von N durch eine ε-Transformation überführt werden kann.*

5. Der Beweis stützt sich auf folgenden

Hilfssatz.[21] *Es sei $\mathfrak{P}=(F_1, F_2, \cdots, F_s)$ eine ε-Überdeckung von F und $\varepsilon'<\varepsilon$ die größte unter den Zahlen $\delta(F_i)$, $i=1,2,\cdots,s$. Es existiert eine positive Zahl $d<\frac{1}{2}(\varepsilon-\varepsilon')$, die die folgende Eigenschaft hat: wenn irgendein Punkt z von F von gewissen Mengen des Systems $\mathfrak{P}$, etwa von $F_{i_1}, F_{i_2}, \cdots, F_{i_m}$, eine Entfernung $<d$ hat, so ist $F_{i_1}\cdot F_{i_2}\cdot\cdots\cdot F_{i_m}\neq 0$.*

Beweis des Hilfssatzes. Zu jedem Punkt x von F gehört eine positive Zahl r_x, die Entfernung von x bis zur Vereinigungsmenge aller den Punkt x nicht enthaltenden Mengen $\mathfrak{F}_i$ des Systems $\mathfrak{P}$. Man betrachte die aus allen Sphären $S(x,\frac{1}{2}r_x)$ gebildete Überdeckung des Raumes F und ersetze sie vermöge des Borel-Lebesgueschen Satzes durch eine endliche Teilüberdeckung

$$U_1, U_2, \cdots, U_q,$$

[20] Hausdorff, Mengenlehre, SS. 103, 104, 107–109.

[21] Vgl. Lebesgue, Fund. Math., 2, S. 259.

wobei $U_k = S(x_k, \frac{1}{2}r^k)$ und $r^k = r_{x_k}$ gesetzt ist. Man bezeichne sodann durch d irgendeine positive Zahl, die kleiner als die kleinste unter den Zahlen $r^k (k = 1, 2, \cdots, q)$ und $\frac{1}{2}(\varepsilon - \varepsilon')$ ist. Ich behaupte, daß d den Bedingungen des Hilfssatzes genügt. Es sei in der Tat $\varrho(z, F_{i_h}) < d$ für $h = 1, 2, \cdots, m$. Da wenigstens für ein k $z \subset U_k$ ist, so ist

$$\varrho(z, x_k) < \frac{1}{2}r^k, \text{ also } \varrho(x_k, F_{i_h}) \leqq \varrho(x_k, z) + \varrho(z, F_{i_h}) < \frac{1}{2}r^k + d < r^k,$$

und folglich $x_k \subset F_{i_h}$ (für $h = 1, 2, \cdots, m$), w. z. b. w.

Bemerkung I. Offenbar besitzt die Zahl d folgende zwei Eigenschaften, die wir oft gebrauchen werden:

1. Wenn M eine Teilmenge von F, die einen Durchmesser $< d$ hat und mit den Mengen $F_{i_1}, F_{i_2}, \cdots, F_{i_m}$ gemeinsame Punkte hat, so ist $F_{i_1} \cdot F_{i_2} \cdot \cdots \cdot F_{i_m}$ nicht leer.
2. Die Mengen $S(F_i, d)$, $i = 1, 2, \cdots, s$, sind sämtlich von Durchmessern $< \varepsilon$; sie bilden ein Mengensystem, welches dem System aller F_i *ähnlich*[22] ist.

Bemerkung II. Um eine Zahl d eindeutig festlegen zu können, definiere man sie ein für allemal etwa so: *die erste Rationalzahl d, die den Bedingungen des Hilfssatzes genügt, heiße $d(\mathfrak{P})$*, wenn $\mathfrak{P}$ die gegebene ε-Überdeckung ist.[23]

6. Der Beweis des Satzes II bedarf jetzt nur weniger Worte: es genügt in der Tat $\delta = d(\mathfrak{P})$ zu setzen. Es sei nun K ein δ-Komplex in F, dessen Eckpunkte $b_1, b_2, \cdots, b_t$ heißen mögen. Man bezeichne ferner durch $a_1, a_2, \cdots, a_s$ die Eckpunkte von N und nehme an, daß diese Bezeichnungen so gewählt sind, daß dem Elemente F_i von $\mathfrak{P}$ der Eckpunkt a_i von N entspricht. Jetzt lassen wir jedem Eckpunkt b_j von K eine bestimmte unter den *den Punkt b_j enthaltenden* Mengen F_i entsprechen; es sei $F_{i(j)}$ diese Menge; die soeben gemachte Zuordnung kann auch so aufgefaßt werden, daß dem Endpunkt b_j von K der Endpunkt $a_{i(j)}$ von N zugeordnet ist; auf diese Weise wird die Menge aller Eckpunkte von K eindeutig auf eine (echte oder unechte) Teilmenge der Menge aller Eckpunkte von N abgebildet. Diese Abbildung hat folgende Eigenschaft: wenn $b_{j_1}, b_{j_2}, \cdots, b_{j_r}$ Eckpunkte eines Simplexes von K sind, so sind die Punkte $a_{i(j_1)}, a_{i(j_2)}, \cdots, a_{i(j_r)}$ (unter denen manche zusammenfallen können) Eckpunkte eines Simplexes von N. Die Mengen $F_{i(j_1)}, F_{i(j_2)}, \cdots, F_{i(j_r)}$ haben, in der Tat, gemeinsame Punkte mit der aus allen Punkten $b_{j_1}, b_{j_2}, \cdots, b_{j_r}$ bestehenden Menge M; da aber der Durchmesser letzterer Menge $< d(\mathfrak{P})$ ist, so müssen die Mengen $F_{i(j_1)}, F_{i(j_2)}, \cdots, F_{i(j_r)}$ einen nicht leeren Durchschnitt haben, d. h. im Komplexe N

[22] Zwei Mengensysteme heißen *ähnlich*, wenn sie denselben Nerv haben.

[23] Dabei wird angenommen, daß die rationalen Zahlen in einer festen Folge vorliegen.

tritt ein Simplex mit den Eckpunkten $a_{i(j_1)}, a_{i(j_2)}, \cdots, a_{i(j_r)}$ auf. Unsere Eckpunktzuordnung definiert somit eine simpliziale Abbildung[11] von K auf einen (echten oder unechten) Teilkomplex von N, welche offenbar eine ε-Transformation ist, w. z. b. w.

Korollar I. *Zu jeder ε-Überdeckung $\mathfrak{P}$ von F kann man eine so kleine positive Zahl d' wählen, daß jeder d'-Komplex Q in $S(F, d')$ durch eine ε-Transformation in einen (echten oder unechten) Teilkomplex des Nerves N von $\mathfrak{P}$ übergeht.*

Es genügt in der Tat $d' < \frac{1}{3} d(\mathfrak{P})$ zu wählen und zu jedem Eckpunkt c_h des gegebenen Komplexes Q einen Punkt b_h von F unter der Bedingung $\varrho(c_h, b_h) = \varrho(c_h, F)$ zu bestimmen und zu jedem Simplex $(c_{h_1}, c_{h_2}, \cdots, c_{h_r})$ das Simplex $(b_{h_1}, b_{h_2}, \cdots, b_{h_r})$ zu konstruieren. Der aus letzteren Simplexen bestehende Komplex möge K heißen. K ist ein $3d'$-, also ein d-Komplex in F und Q geht in K durch eine d'-Transformation über. Aus der Beweisführung des Satzes II ergibt sich, daß K in einen Teilkomplex von N nicht nur mittels einer ε-, sondern sogar mittels einer ε'-Transformation übergeht, wobei ε' das Maximum der Durchmesser der Elemente von $\mathfrak{P}$ bildet; somit geht Q in denselben Teilkomplex von N durch eine $\varepsilon' + d$-, also durch eine ε-Transformation über.

Aus dem Korollar I ergibt sich unmittelbar:

Korollar II. *Es sei F eine beliebige λ-dimensionale abgeschlossene Teilmenge des R^n und ε eine beliebige positive Zahl; es existieren beliebig kleine Polyederumgebungen von F, die mittels ε-Transformationen in höchstens λ-dimensionale Komplexe übergehen.*

Es sei in der Tat $\mathfrak{P}$ eine $(\varepsilon, \lambda+1)$-Überdeckung von F; man betrachte eine Simplizialzerlegung $\mathfrak{Z}$ des ganzen R^n, die beliebig, aber jedenfalls so fein ist, daß die Simplexe dieser Zerlegung sämtlich kleiner als $d'(\mathfrak{P})$ sind. Die Gesamtheit aller Simplexe von $\mathfrak{Z}$, die im Innern oder auf dem Rande Punkte von F enthalten, bilden eine Polyederumgebung U von F, die als Komplex betrachtet, allen Bedingungen des Korollars I genügt und somit in einen (echten oder unechten) Teilkomplex des Nerves von $\mathfrak{P}$ durch ε-Transformation übergeführt werden kann. Da aber der Nerv von $\mathfrak{P}$ höchstens λ-dimensional ist, ist das Korollar II bewiesen.

7. Aus dem soeben bewiesenen Satze folgt, daß eine λ-dimensionale F im R^n bei jedem ε durch eine ε-Transformation in eine (echte oder unechte) abgeschlossene Teilmenge F' eines λ-dimensionalen Komplexes K übergeht. Wenn man nun auf diesen Komplex (indem man ihn eventuell in einer hinreichend feinen Unterteilung betrachtet) das Ausfegeverfahren des § 2 anwendet, kann man F' weiter so transformieren, daß als Endresultat ein Komplex (und zwar ein Teilkomplex des Komplexes K) entsteht, in welchen F durch eine 2ε-Transformation übergeht. Damit ist aber folgender *Spezialfall des Überführungssatzes* bewiesen:

Jede λ-dimensionale abgeschlossene Teilmenge des R^n geht durch k. T. in λ-dimensionale Komplexe über.

8. Es sei jetzt F eine λ-dimensionale kompakte abgeschlossene Teilmenge des Hilbertschen Raumes und $\mathfrak{P} = (F_1, F_2, \cdots, F_s)$ eine $(\varepsilon, \lambda+1)$-Überdeckung von F. Man wähle $\varepsilon' < \frac{1}{2} d(\mathfrak{P})$ und führe F nach dem Verfahren des § 1 mittels einer ε'-Transformation in eine Teilmenge F' des (durch die Gleichungen $x_n = x_{n+1} = x_{n+2} \cdots = 0$ bei passend gewähltem n bestimmten) Euklidischen Teilraum R^{n-1} über; dabei gehen die Mengen $F_1, F_2, \cdots, F_s$ in $F_1', F_2', \cdots, F_s'$ über. Aus der Ungleichung $\varepsilon' < \frac{1}{2} d(\mathfrak{P})$ folgt sodann, daß das Mengensystem $\mathfrak{P}' = (F_1', F_2', \cdots, F_s')$ dem System $\mathfrak{P}$ ähnlich [22] ist. Wenn N' der geometrisch realisierte Nerv von $\mathfrak{P}'$ ist, so kann man (kraft des Korollars I) eine hinreichend kleine Polyederumgebung U' von F' so wählen, daß diese Umgebung in einen echten oder unechten Teilkomplex von N' ε-überführt werden kann, woraus man (wie am Ende des vorigen Paragraphen) mühelos schließt, daß F' (und somit auch F) durch eine 2ε-Transformation in einen λ-dimensionalen Komplex übergeht. Dadurch ist bewiesen, daß *jede λ-dimensionale kompakte abgeschlossene Teilmenge des Hilbertschen Raumes durch k. T. in λ-dimensionale Komplexe übergeht.* Mithin geht jeder kompakte metrische Raum von der Dimension λ durch k. A. in Komplexe von derselben Dimension über.

9. Was sind die Komplexe, in die eine abgeschlossene Menge mittels kleiner Abbildungen (bzw. kleiner Transformationen) übergeht? Die Antwort auf diese Frage leistet der

Satz III. *Jeder Komplex K, in den sich eine abgeschlossene Menge F ε-abbilden läßt, kann nach eventueller Unterteilung als Nerv einer 3ε-Überdeckung dieser Menge aufgefaßt werden.*

10. Ich erinnere zunächst an den Begriff der *baryzentrischen Unterteilung* [9] eines gegebenen Komplexes K. Man versteht darunter bekanntlich folgendes: Jedem r-dimensionalen Simplex

$$T^r = (a_{i_1}, a_{i_2}, \cdots, a_{i_{r+1}})$$

von K sei der Schwerpunkt $b_{i_1 i_2 \cdots i_{r+1}}$ dieses Simplexes zugeordnet; die *Grundsimplexe*[1] der baryzentrischen Unterteilung von K sind nun definitionsgemäß alle Simplexe von der Form

$$t^m = (b_{i_1}, b_{i_1 i_2}, \cdots, b_{i_1 i_2 \cdots i_{m+1}}),$$

wobei $(a_{i_1}, a_{i_2}, \cdots, a_{i_{m+1}})$ ein Grundsimplex von K (in dessen ursprünglicher Zerlegung) und selbstverständlich $b_{i_1} = a_{i_1}$ ist.

Die Gesamtheit aller Simplexe der baryzentrischen Unterteilung, die an einen gegebenen Eckpunkt a_i der ursprünglichen Zerlegung von K an-

schließen, bildet den durch den Eckpunkt a_i bestimmten *baryzentrischen Stern* B_i. Das System aller baryzentrischen Sterne von K bildet eine Überdeckung von K, *die baryzentrische Überdeckung*: $\mathfrak{B}(K)$. Der Beweis des Satzes III stützt sich auf den

Hilfssatz II. Jeder Komplex K kann als der Nerv seiner baryzentrischen Überdeckung aufgefaßt werden.

Es seien, wie immer, $a_1, a_2, \cdots, a_s$ die Eckpunkte von K, $B_1, B_2, \cdots, B_s$ die (diesen Eckpunkten eindeutig zugeordneten) baryzentrischen Sterne. Das, was zu beweisen ist, ist einfach folgendes:

$B_{i_1} \cdot B_{i_2} \cdot \cdots \cdot B_{i_r}$ ist dann und nur dann $\neq 0$, wenn $a_{i_1}, a_{i_2}, \cdots, a_{i_r}$ Eckpunkte eines Simplexes von K sind.

Zunächst ist, falls $a_{i_1}, a_{i_2}, \cdots, a_{i_r}$ Eckpunkte eines T^{r-1} aus K sind, $B_{i_1} \cdot B_{i_2} \cdot \cdots \cdot B_{i_r}$ gewiß von Null verschieden, weil dann der Schwerpunkt $b_{i_1 i_2 \cdots i_r}$ von T^{r-1} in sämtlichen B_{i_r} enthalten ist. Es handelt sich also nur um den Beweis der umgekehrten Behauptung. Zu diesem Zweck sei zuerst auf folgende (elementare und mittels Induktion leicht zu beweisende) Eigenschaft der baryzentrischen Zerlegung aufmerksam gemacht:

Es sei $t^{m-1} = (b_{j_1}, b_{j_1 j_2}, \cdots, b_{j_1 j_2 \cdots j_m})$, $b_{j_1} = a_{j_1}$, ein baryzentrisches Teilsimplex des Simplexes $T^{m-1} = (a_{j_1}, a_{j_2}, \cdots, a_{j_m})$; dann ist t_{m-1} zur (dem Eckpunkt a_{j_1} von T^{m-1} gegenüberliegenden) Seite $T^{m-2} = (a_{j_2}, a_{j_3}, \cdots, a_{j_m})$ von T^{m-1} fremd.

Aus dieser Bemerkung ergibt sich sofort folgende Behauptung:

Wenn x ein Punkt des baryzentrischen Grundsimplexes $t^{m-1} = (b_{j_1}, b_{j_1 j_2}, \cdots, b_{j_1 j_2 \cdots j_m})$ und T^n irgendein den Punkt x enthaltendes Simplex der ursprünglichen Simplizialzerlegung von K ist, so ist $a_{j_1} = b_{j_1}$ ein Eckpunkt von T^n. Das Simplex t^{m-1} ist in der Tat ein baryzentrisches Teilsimplex eines wohldefinierten Simplexes T_0^{m-1} von K und a_{j_1} ist gewiß ein Eckpunkt von T_0^{m-1}; wenn nun x ein innerer Punkt von T_0^{m-1} ist, so ist T^n (welches ja auch den Punkt x enthält) notwendig mit T_0^{m-1} identisch, folglich ist a_{j_1} Eckpunkt von T^n; wenn aber x ein Randpunkt von T_0^{m-1} ist, so haben T_0^{m-1} und T^n eine gemeinsame r-dimensionale ($0 \leqq r \leqq m-1$) Seite T^r; diese Seite kann nicht in der dem Punkte a_{j_1} gegenüberliegenden Seite T_0^{m-2} von T_0^{m-1} enthalten sein (weil dann t^{m-1} gleichzeitig den Punkt a_{j_1} und den zu $T^r \subset T_0^{m-2}$ gehörenden Punkt x im Widerspruch zu unserer obigen Bemerkung enthalten würde), folglich ist $a_{j_1} \subset T^r \subset T^n$, womit unsere Behauptung bewiesen ist.

Es sei jetzt $B_{i_1} \cdot B_{i_2} \cdot \cdots \cdot B_{i_r} \neq 0$ und x ein Punkt letzterer Durchschnittsmenge. Wenn t_h ein den Punkt x enthaltendes unter den den Stern B_{i_k} bildenden Simplexen und T ein unter der Bedingung $T \supset x$ gewähltes Grundsimplex der ursprünglichen Zerlegung von K ist, so folgt aus der

Inklusion $a_{i_h} \subset t_h$ und der soeben bewiesenen Behauptung, daß a_{i_h} für jedes h ein Eckpunkt von T ist, womit der Beweis des Hilfssatzes II erbracht ist.

Um jetzt den Satz III zu beweisen, bemerke man vor allem folgendes:

Es sei eine ε-Abbildung $\varphi = f(F)$ gegeben; zu jedem $\eta > 0$ existiert dann ein $\delta > 0$ von der Eigenschaft, daß, wenn φ_0 eine Teilmenge von φ und[24] $\delta(\varphi_0) < \delta$ ist, die Originalmenge $F_0 = f^{-1}(\varphi_0)$ einen Durchmesser $< 2\varepsilon + \eta$ hat.

Man wähle jetzt eine so feine Unterteilung $\mathfrak{z}$ von K, daß die Originalmenge jedes dazugehörigen baryzentrischen Sternes $< 3\varepsilon$ ist. Somit bilden diese Originalmengen eine 3ε-Überdeckung von F und da der Nerv dieser Überdeckung, wie leicht ersichtlich, mit dem Nerv der baryzentrischen Überdeckung des (in der Simplizialzerlegung $\mathfrak{z}$) gegebenen Komplexes K, also mit K selbst (immer in der Unterteilung $\mathfrak{z}$ betrachtet) identisch ist, so ist der Satz III bewiesen.

11. Eine unmittelbare Folge des Satzes III ist die Behauptung:

Wenn F durch k. A. in n-dimensionale Komplexe übergeht, so ist dim $F < n$. Letzteres Ergebnis[25] zusammen mit dem Resultate des § 7 ergibt sodann den allgemeinen

ÜBERFÜHRUNGSSATZ. *Die Dimension einer abgeschlossenen Menge F ist die kleinste Zahl λ von der Eigenschaft, daß die Menge F durch kleine Abbildungen in λ-dimensionale Komplexe übergeht.*

12. Es gibt triviale Beispiele von Mengen, welche bei jedem ε ε-Überdeckungen zulassen, deren Nerven überhaupt nicht als stetiges Bild der gegebenen Menge aufgefaßt werden können. Es sei in der Tat F die aus der Zahl Null und allen Zahlen $\frac{1}{n}$, $n = 1, 2, 3, \cdots$ in inf., bestehende Menge; es sei $\varepsilon > 0$ beliebig, m so groß, daß $\frac{1}{m} < \varepsilon$ ist; man bezeichne ferner durch F_n, $1 \leqq n \leqq m$, die aus dem einzigen Punkte $\frac{1}{n}$ bestehende Menge, durch F_0 dagegen die aus dem Nullpunkt und allen $\frac{1}{n}$, $n \geqq m$,

[24] Wenn M eine beliebige Punktmenge eines metrischen Raumes ist, so bedeutet $\delta(M)$ den Durchmesser von M (Hausdorff, Mengenlehre, S. 10).

[25] Dieses Resultat (das ich im "Dimensionsbegriff" als das Brouwersche Invarianzprinzip bezeichnet habe) ist eine Verallgemeinerung eines grundlegenden Satzes von Brouwer (Math. Ann., 70 (1911), S. 164), der besagt, daß ein n-dimensionaler Würfel (und also auch ein beliebiger n-dimensionaler Komplex) mittels k. T. unmöglich in Komplexe niedrigerer Dimensionen übergehen kann; dieser Satz folgt aus dem im § 11 ausgesprochenen, *wenn man bewiesen hat*, daß der n-dimensionale Koordinatenraum die (allgemeine) Dimension n hat; da man aber letztere Behauptung am besten eben auf Grund des Brouwerschen Satzes für den Würfel beweist, so behält letzterer auch jetzt seine volle Bedeutung.

bestehende Menge. Das auf diese Weise gewonnene Mengensystem bildet offenbar eine ε-Überdeckung von F deren Nerv aus den "Eckpunkten" $a_1, a_2, \cdots, a_m$ und aus der "Strecke" $\overline{a_0 a_m}$ besteht und infolgedessen nicht als stetiges Bild der abzählbaren Menge F aufgefaßt werden kann. Eine *direkte* Umkehrung des Satzes III (für alle F und alle Überdeckungen) ist demnach unmöglich; es wird sich jedoch sofort ergeben, daß in sehr allgemeinen Fällen (ja sogar in allen Fällen, die ein Interesse beanspruchen dürfen) die Umkehrung des Satzes III dennoch richtig ist. Es gelten in der Tat folgende Sätze:

SATZ IV. *Es sei F eine (evtl. in den Hilbertschen oder in einen Euklidischen Raum eingebettete) abgeschlossene Menge und $\mathfrak{P}$ eine irreduzible ε-Überdeckung*[26] *derselben; F geht dann in den Nerv von $\mathfrak{P}$ mittels einer 2ε-Abbildung (bzw. einer 2ε-Transformation) über.*

SATZ IV'. *Wenn F ein Kontinuum ist, gilt die Behauptung des Satzes IV auch ohne die Voraussetzung über die Irreduzibilität von $\mathfrak{P}$.*

Ich erwähne noch ohne Beweis den

SATZ IV''. *Wenn F eine im Euklidischen R liegende abgeschlossene Menge, $\mathfrak{P}$ eine ganz beliebige ε-Überdeckung und δ eine willkürliche, kleine positive Zahl ist, so läßt sich eine δ-Umgebung von F durch eine ε-Transformation in den (geometrisch realisierten) Nerv von $\mathfrak{P}$ überführen.*[27]

13. Um diese Sätze zu beweisen, führen wir für einen Augenblick folgende Bezeichnung ein: ein Teilkomplex Q eines gegebenen Komplexes K soll ein *normaler* Teilkomplex heißen, wenn Q alle Eckpunkte von K enthält und wenn überdies jeder nicht isolierte Eckpunkt von K auch in Q nicht isoliert ist. Wir beweisen zunächst, *daß, wenn $\mathfrak{P}$ eine irreduzible ε-Überdeckung von F ist, letztere Menge sich in einen normalen Teilkomplex des Nerves N von $\mathfrak{P}$ ε-überführen läßt.* Beim Beweise dieser Tatsache kann man offenbar voraussetzen, daß F eine Teilmenge des R^n ist.

Es seien $F_1, F_2, \cdots, F_s$ die Elemente des Mengensystems $\mathfrak{P}$, $a_1, a_2, \cdots, a_s$ die entsprechenden Eckpunkte von N und $d = d(\mathfrak{P})$; man unterziehe R^n einer Simplizialzerlegung $\mathfrak{z}$, deren Simplexe kleiner als $\frac{1}{6}d$ sind und die insbesondere alle Punkte a_i unter ihren Eckpunkten zählt; Simplexe der Zerlegung $\mathfrak{z}$, die im Innern oder auf dem Rande Punkte von F enthalten, mögen *ausgezeichnete* Simplexe heißen. Wir wenden auf jedes ausgezeichnete Simplex das Ausfegeverfahren des § 2 an, was eine $\frac{1}{6}d$-Transformation der Menge F darstellt, deren Endresultat ein (aus ausgezeichneten Simplexen verschiedener Dimension aufgebauter) Komplex Q ist. Jedem Eckpunkt b_j von Q lassen wir in bekannter Weise einen Eckpunkt $a_i = f(b_j)$ von N, den *zugeordneten* Eckpunkt entsprechen, indem wir zuerst einen

[26] Die Definition einer irreduziblen ε-Überdeckung ist im § 7 der Einleitung gegeben.

[27] "Dimensionsbegriff", § 7 (insbesondere Fußnote [26]) und § 9.

bestimmten Punkt $c_j \subset F$ unter der Bedingung $\varrho(b_j, F) = \varrho(b_j, c_j)$ und dann eine natürliche Zahl i unter der Bedingung $c_j \subset F_i$ wählen. Wenn dabei $b_j = a_i$, so ist auch $f(b_j) = a_i$. Dadurch entsteht eine simpliziale Abbildung f von Q auf einen Teilkomplex N' von N. *Wenn F zusammenhängend ist, so gilt dasselbe auch von N', also ist in diesem Falle N' gewiß ein normaler Teilkomplex von N*; es soll jetzt gezeigt werden, daß auch im Falle, wenn $\mathfrak{P}$ eine irreduzible ε-Überdeckung ist, N' ein normaler Teilkomplex von N ist. Es sei in der Tat N' kein normaler Teilkomplex von N und a_i ein Eckpunkt, der in N als ein nicht isolierter, in N' dagegen als ein isolierter Punkt auftritt. Man bezeichne mit Q_i die Gesamtheit aller derjenigen Simplexe von Q, die wenigstens einen Eckpunkt besitzen, dem der Punkt a_i zugeordnet ist. Man überzeugt sich leicht davon, daß $Q_i \subset S(F_i, \frac{1}{3}d)$ ist. Nun aber geht F in Q mittels einer $\frac{1}{3}d$-Transformation φ über; man definiere demgemäß $\Phi_i = \varphi^{-1}(Q_i)$. Da

$$\Phi_i \subset S\left(Q_i, \frac{1}{3}d\right) \subset S(F_i, d)$$

ist, so bilden die Φ_i eine ε-Überdeckung von F, deren Nerv N'' ein echter oder unechter Teilkomplex von N ist. Kraft unserer Voraussetzung ist aber Q_i zu allen Q_h, $h \neq i$, also Φ_i zu allen Φ_h fremd; daraus folgt, daß der Eckpunkt a_i in N'' als ein isolierter Punkt auftritt, so daß N'' ein *echter* Teilkomplex von N, also $\mathfrak{P}$ keine irreduzible ε-Überdeckung ist.

14. Um jetzt die beiden Sätze IV und IV′ zu beweisen, genügt es, folgendes zu zeigen:

Es sei K ein Komplex, dessen Simplexe sämtlich kleiner als $\delta > 0$ sind, und K' ein normaler Teilkomplex von K; es existiert eine stetige δ-Abbildung von K auf sich selbst, bei der die Bildmenge von K' mit dem ganzen Komplex K identisch ist.

15. Um diese Behauptung (und somit die beiden Sätze IV und IV′) zu beweisen, verfahren wir folgendermaßen. Es sei $\delta' < \delta$ der Maximaldurchmesser eines Simplexes von K und $2h < \delta - \delta'$ eine positive Zahl; wir unterteilen K' so fein, daß die Simplexe dieser Unterteilung kleiner als h und die Hälfte der kleinsten Entfernung zwischen zwei verschiedenen Eckpunkten von K sind. An jeden nicht isolierten Eckpunkt a_i von K schließt mindestens eine Kante des Komplexes K', also mindestens eine Kante L_i' der erwähnten Unterteilung dieses Komplexes. Die Gesamtheit aller L_i' und aller isolierten Punkte von K bildet einen Komplex L', der in K' enthalten ist.

Die Behauptung des § 14 wird bewiesen, wenn gezeigt wird, daß man eine δ-Abbildung des Komplexes K auf sich selbst so finden kann, daß dabei die Bildmenge von L' der ganze Komplex K ist.

Eine solche Abbildung entsteht durch Zusammensetzung zweier Abbildungen φ' und φ'' von K auf sich selbst, die folgendermaßen zu definieren sind. Wenn a_i in K isoliert ist, ist $\varphi'(a_i) = \varphi''(a_i) = a_i$. Es sei a_i irgendein nicht isolierter Eckpunkt von K. In jedem an den Punkt a_i anschließenden Grundsimplex T_j von $K (j = 1, 2, \cdots, s)$ wähle man eine Strecke $\overline{a_i b_j} = L_i^j$, die bis auf den Punkt a_i im Innern von T_j verläuft und dieselbe Länge wie L_i' hat. Die Gesamtheit aller auf diese Weise in a_i angebrachten Strecken bildet den Streckenkomplex L_i, wobei für $i \neq k$ $L_i \cdot L_k = 0$ ist. Man bilde jetzt L_i' unter Festhaltung von a_i stetig auf L_i ab und mache dies für jedes i, dadurch entsteht eine h-Abbildung von L' auf den aus sämtlichen L_i bestehenden Streckenkomplex L; diese Abbildung läßt sich aber leicht zu einer h-Abbildung von K auf sich selbst erweitern, die alle Punkte außerhalb einer beliebig kleinen Umgebung von L' fest läßt. Diese Abbildung soll φ' heißen; sie führt L' in den aus L und den isolierten Punkten bestehenden Komplex über. Jetzt bilden wir jede Strecke L_i^j auf das ganze Simplex T_j unter Festhaltung des Punktes a_i eindeutig und stetig ab. Diese Abbildung läßt sich zu einer den ganzen Rand von T_j punktweise festhaltenden Abbildung von T_j auf sich erweitern; diese in sämtlichen Grundsimplexen von K hiermit definierten Abbildungen schließen (wegen des Festbleibens der Ränder) stetig aneinander und ergeben eine δ'-Abbildung φ'' des Komplexes K auf sich selbst, bei der als Bildmenge von L der ganze Komplex K auftritt. Somit ist $\varphi = \varphi''(\varphi'(K))$ eine $\delta' + h$-, also erst recht eine δ-Abbildung von K auf sich, die die gewünschten Eigenschaften besitzt. Die Sätze IV und IV′ sind hiermit bewiesen.

16. Der Überführungssatz erlaubt eine weitere metrische Eigenschaft der Dimension festzustellen.

Eine aus endlich-vielen Punkten $a_1, a_2, \cdots, a_s$ bestehende Teilmenge A der abgeschlossenen Menge F soll ein ε-Netz heißen, wenn es zu jedem Punkt von F mindestens einen, von diesem Punkt weniger als um ε entfernten Punkt von A gibt. Man betrachte jetzt einen festen Punkt x von F und diejenigen Punkte a_i von A, für welche die Entfernung $\varrho(x, a_i)$ ihren Minimalwert $r(x)$ erhält. Die Anzahl dieser Punkte soll die *Ordnung* von x in bezug auf das Netz A heißen. Sodann folgt aus dem Überführungssatz die Richtigkeit folgender Behauptung:

SATZ V. *Wenn ε hinreichend klein und A ein beliebiges ε-Netz der λ-dimensionalen abgeschlossenen Menge F ist, so enthält F Punkte, deren Ordnung in bezug auf A mindestens $\lambda + 1$ ist; durch kleine Transformationen von F, welche sämtliche Punkte des Netzes festhalten, lassen sich Punkte höherer Ordnung stets beseitigen, Punkte der Ordnung $\lambda + 1$ treten dagegen auch nach der Transformation notwendig auf.*

Es genügt den Satz unter der Voraussetzung $F \subset R^n$ zu beweisen. Man bilde nun alle $\lambda+1$-dimensionale Simplexe, deren Eckpunkte dem Netze angehören. Zu jedem dieser Simplexe gibt es eine (niemals ausartende) $n-\lambda-1$-dimensionale Hyperebene, die alle von den Eckpunkten des Simplexes gleichentfernten Punkte enthält. Die Vereinigungsmenge aller dieser Hyperebenen stellt einen $n-\lambda-1$-dimensionalen ("offenen") Komplex Q dar, welcher sämtliche Punkte des Raumes enthält, die in bezug auf das gegebene Netz eine Ordnung $>\lambda+1$ haben. Für jedes $\delta>0$ kann man aber die Menge F in einen bis auf eventuelle in A enthaltene Eckpunkte zu Q fremden λ-dimensionalen Komplex K unter Festhaltung sämtlicher Punkte des Netzes A δ-überführen;[28] da aber die Ordnung der Punkte a_i gleich 1, also unmöglich $\geqq\lambda+2$ ist, und kein sonstiger Punkt von Q zu K gehört, enthält K keinen Punkt von einer Ordnung $>\lambda+1$, womit die eine Hälfte des Satzes V bewiesen ist.

Um die andere zu beweisen, zeigen wir, daß jede abgeschlossene Menge F', die in bezug auf A keinen Punkt von einer Ordnung $>\lambda$ besitzt und in $S(A,\varepsilon)$ enthalten ist, mittels einer ε-Transformation in einen höchstens $\lambda-1$-dimensionalen Komplex übergeführt werden kann.

Es sei in der Tat eine unseren Voraussetzungen genügende F' gegeben. Man betrachte irgendeine Teilmenge

$$a_{i_1}, a_{i_2}, \cdots, a_{i_m} \tag{1}$$

der Menge A. Wenn es eine n-dimensionale Vollkugel gibt mit einem zu F' gehörenden Mittelpunkt, deren Rand die Punkte (1) enthält, deren Inneres jedoch von Punkten des Netzes A frei ist, sagen wir, daß die Punktmenge (1) *ausgezeichnet* ist. Man erhält jetzt einen Komplex $K(A)$, wenn man jede ausgezeichnete Punktmenge zum Eckpunktgerüst eines Simplexes macht. Auf Grund unserer Voraussetzung ist der Komplex $K(A)$ höchstens $\lambda-1$-dimensional. Überdies sind die Durchmesser sämtlicher Simplexe von $K(A)$ kleiner als 2ε.

[28] Man nehme in der Tat die Punkte von A als Eckpunkte in eine (hinreichend feine) Simplizialzerlegung $\mathfrak{Z}$ des Raumes R^n auf; ferner betrachte man eine so feine $(\delta, \lambda+1)$-Überdeckung $\mathfrak{B}$ von F, daß keine zwei Punkte von A zu einem Elemente der Überdeckung $\mathfrak{B}$ gehören; dann ist es leicht zu erreichen, daß die Punkte von A sämtlich unter den Eckpunkten des (geometrisch realisierten) Nerven N von $\mathfrak{B}$ vorkommen, und zwar so, daß kein Simplex von N mehr als einen zu A gehörenden Eckpunkt besitzt. Man hat dann noch Freiheit genug um (mittels eventueller kleiner Verschiebungen der übrigen Eckpunkte von N) zu erreichen, daß sämtliche Elemente von N mit eventueller Ausnahme der zu A gehörenden Eckpunkte N sich in bezug auf Q in einer allgemeinen Lage befinden, also zu Q *fremd* sind. Wenn man jetzt nach den Vorschriften des § 6 die Menge F in einen echten oder unechten Teilkomplex K von N überführt, so bleiben alle Punkte von A (die ja gleichzeitig Eckpunkte von N und der der Überführung zugrunde gelegten Unterteilung $\mathfrak{z}$ von $\mathfrak{Z}$ sind) fest, womit unsere Behauptung bewiesen ist.

Man betrachte jetzt eine Folge von unendlich fein werdenden Simplizialunterteilungen des R^n:

$$\mathfrak{d}_1, \mathfrak{d}_2, \cdots, \mathfrak{d}_k, \cdots.$$

Man nenne ein Simplex einer dieser Unterteilungen *ausgezeichnet*, wenn er im Innern oder auf dem Rande Punkte von F' enthält. Jedem ausgezeichneten Simplex t einer beliebigen Unterteilung $\mathfrak{d}_k$ lassen wir eine bestimmte Teilmenge $A_k(t)$ von A entsprechen, die aus allen Punkten a_i besteht, für die es wenigstens einen der Gleichung

$$\varrho(b, a_i) = \varrho(b, A)$$

genügenden Eckpunkt b des Simplexes t gibt. Sodann sind, wie man leicht beweist, für ein hinreichend großes k die beiden folgenden Bedingungen erfüllt:

1. Wenn t ein ausgezeichnetes Simplex aus $\mathfrak{d}_k$ ist, so ist $A_k(t)$ eine ausgezeichnete Punktmenge;
2. Alle ausgezeichneten Simplexe aus $\mathfrak{d}_k$ sind in $S(A, \varepsilon)$ enthalten.

Es sei k unter diesen Bedingungen gewählt; die Gesamtheit aller ausgezeichneter Simplexe aus $\mathfrak{d}_k$ bildet sodann eine ε-Umgebung U von F', die durch eine ε-Transformation in einen (echten oder unechten) Teilkomplex von $K(A)$ übergeht: diese Transformation erhält man, indem man jedem Eckpunkt eines ausgezeichneten Simplexes (aus $\mathfrak{d}_k$) einen unter den von ihm am wenigsten entfernten Punkte von A zuordnet: vermöge der Bedingung 1 entspricht auf diese Weise jedem Eckpunktgerüst eines ausgezeichneten Simplexes ein Eckpunktgerüst eines Simplexes des Komplexes $K(A)$, so daß eine simpliziale Abbildung von U auf einen Teilkomplex von $K(A)$ entsteht, welche vermöge der Bedingung 2 eine ε-Abbildung ist, w. z. b. w.

17. Es möge zum Schluß auf folgende zwei unentschiedene Fragen hingewiesen werden, die aufs engste mit dem Satz V zusammenhängen:

I. *Ist es möglich für jede λ-dimensionale abgeschlossene Menge F und für jedes ε ein ε-Netz zu finden, so daß kein Punkt von F eine Ordnung $> \lambda + 1$ in bezug auf dieses Netz besitzt?*

I'. *Ist es möglich vorauszusetzen, daß die kleinen Transformationen, die zur Befreiung der Menge F von Punkten von einer Ordnung $> \lambda + 1$ dienen, topologische Transformationen der Menge F sind?*

Diese beiden Fragen sind Spezialfälle einer allgemeinen Problemstellung, welche verlangt, die topologische Struktur einer beliebigen abgeschlossenen Menge auf die Struktur immer feiner werdender ε-Netze zurückzuführen. (Vgl. hierzu meine Note aus dem Bd. 94 der Math. Ann.)

Zweites Kapitel.

Abstrakte Simplizialapproximationen kompakter Räume.

1. Wir fangen an mit einigen Bemerkungen über irreduzible ε-Überdeckungen [26] und insbesondere über Folgen solcher Überdeckungen. Man hat zunächst folgende elementare Tatsache:

Eine irreduzible ε-Überdeckung einer λ-dimensionalen abgeschlossenen Menge F kann nicht von einer Ordnung $>\lambda+1$ sein; wenn ε genügend klein ist, ist somit die Ordnung jeder irreduziblen ε-Überdeckung von F genau $\lambda+1$.

Beweis. Es sei $\mathfrak{P} = (F_1, F_2, \cdots, F_s)$ eine ε-Überdeckung von F, die eine Ordnung $>\lambda+1$ besitzt, $\lambda = \dim F$. Man setze $d = d(\mathfrak{P})$ und betrachte eine $(d, \lambda+1)$-Überdeckung $\mathfrak{P}' = (F_1', F_2', \cdots, F_s')$ von F. Man bezeichne ferner durch F_1'' die Vereinigungsmenge aller F_j', die mit F_1 gemeinsame Punkte haben, und setze allgemein F_{i+1}'' gleich der Vereinigungsmenge aller F_j', die zu $F_1, F_2, \cdots, F_i$, aber *nicht* zu F_{i+1} fremd sind. Man beweist leicht, daß das System aller Mengen F_i'' von einer Ordnung $\leqq\lambda+1$ ist; da aber andererseits $\sum_{i=1}^{s} F_i'' = F$ und $F_i'' \subset S(F_i, d)$ ist, so bilden die F_i'' eine ε-Überdeckung, deren Nerv ein (wie soeben gesagt, höchstens λ-dimensionaler, also) echter Teilkomplex von $N(\mathfrak{P})$ ist; $\mathfrak{P}$ ist also keine irreduzible ε-Überdeckung, w. z. b. w.

2. Satz I. *Jede Folge von irreduziblen $\varepsilon^{(m)}$-Überdeckungen* ($\lim_{m\to\infty} \varepsilon^{(m)} = 0$) *der abgeschlossenen Menge F enthält eine Teilfolge, die einer Folge von Unterteilungen*[29] *der Menge F ähnlich*[30] *ist.*

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[29] Unter einer *Folge von Unterteilungen* von F verstehe ich eine Folge von Überdeckungen

$$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_m, \cdots$$

der Menge F, die folgenden Bedingungen genügt:

1. $\mathfrak{P}_m$ ist eine ε_m-Überdeckung, wobei $\lim_{m\to\infty} \varepsilon_m = 0$ ist;

2. Die Überdeckung $\mathfrak{P}_m$ besteht aus (endlich vielen) abgeschlossenen Mengen $F_{i_1 i_2 \cdots i_m}$, $i_k = 1, 2, \cdots, s(i_1, i_2, \cdots, i_{k-1})$, $k = 1, 2, \cdots, m$, *wobei* $F_{i_1 i_2 \cdots i_m} = \sum_{i_{m+1}=1}^{s(i_1, \cdots i_m)} F_{i_1 i_2 \cdots i_m i_{m+1}}$

(für jedes m und jedes $F_{i_1 i_2 \cdots i_m}$) ist.

Der (letzten Endes auf die Theorie der sogenannten A-Mengen zurückgehende) Begriff der Unterteilungsfolgen wurde auf dimensionstheoretische Fragen zum erstenmal von Menger und Hurewicz angewandt (Proc. Kon. Akademie Amsterdam, 29 (1926), S. 476 bzw. 1015). Der wichtige Satz, daß es zu jeder λ-dimensionalen abgeschlossenen Menge F eine Unterteilungsfolge gibt, deren Elemente Überdeckungen von der Ordnung $\lambda+1$ sind, ist zum erstenmal von Hurewicz (in der soeben zitierten Arbeit) ohne Beweis ausgesprochen worden. Unser Satz I ist eine Verschärfung des Satzes von Hurewicz.

Beweis. Man kann offenbar eine Teilfolge

$$(1) \qquad \mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_m, \cdots, \qquad N_m = N(\mathfrak{P}_m)^{31}$$

der gegebenen Überdeckungsfolge so wählen, daß folgende Bedingungen erfüllt sind:

1. $\mathfrak{P}_n$ ist eine irreduzible ε_m-Überdeckung;
2. bei jedem m ist ε_{m+1} kleiner als die beiden Zahlen $\frac{1}{2}\varepsilon_m$ und $d_m = \frac{1}{2}d(\mathfrak{P}_m)$.

Es soll jetzt bewiesen werden, daß es eine Folge von Unterteilungen der Menge F gibt, welche der Folge (1) ähnlich ist.

Es seien F_{i_1} $(i_1 = 1, 2, \cdots, s)$ die Elemente des Mengensystems $\mathfrak{P}_1$; wir nehmen an, daß die Mengen $F_{i_1 \cdots i_m}$, $i_k = 1, 2, \cdots, s$ $(i_1, i_2, \cdots, i_{k-1})$, $k = 1, 2, \cdots, m$, bereits konstruiert seien, und denken uns die Mengen $F_{i_1 \cdots i_m}$ in einer beliebigen, aber festen Reihenfolge gegeben:

$$F_{*_1}, F_{*_2}, \cdots, F_{*_r},$$

(unter $*_k$ ist also eine Indexkombination $(i_1 i_2 \cdots i_m)$ zu verstehen). Man bezeichne jetzt mit

$$F_{*_1 1}, F_{*_1 2}, \cdots, F_{*_1 s(*_1)}$$

alle diejenigen unter den (die Überdeckung $\mathfrak{P}_{m+1}$ bildenden) Mengen F_i^{m+1}, welche mit F_{*_1} gemeinsame Punkte haben; im allgemeinen bezeichne man mit

$$F_{*_k 1}, F_{*_k 2}, \cdots, F_{*_k s(*_k)}$$

alle diejenigen unter den Mengen F_i^{m+1}, die mit F_{*_k}, jedoch mit keiner F_{*_h}, $h < k$, gemeinsame Punkte haben. Auf diese Weise werden die $F_{i_1 \cdots i_{m+1}}$ und also der Reihe nach alle $F_{i_1 \cdots i_m}$ für jedes m schrittweise definiert. Dabei ist jedes $F_{i_1 \cdots i_m}$ ein bestimmtes F_i^m, und auch umgekehrt kommt jedes Element von $\mathfrak{P}_m$ (und zwar ein einziges Mal) unter den $F_{i_1 \cdots i_m}$ vor. Man setze endlich

$$F^{i_1 \cdots i_m} = \sum_{i_{m+1}=1}^{s(i_1, \cdots, i_m)} F_{i_1 \cdots i_{m+1}}.$$

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30 Zwei Folgen von Überdeckungen:

$$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_m, \cdots$$

und

$$\mathfrak{Q}_1, \mathfrak{Q}_2, \cdots, \mathfrak{Q}_m, \cdots$$

heißen *ähnlich*, wenn für jedes m der Nerv von $\mathfrak{P}_m$ mit dem Nerv von $\mathfrak{Q}_m$ identisch ist.

31 Wenn $\mathfrak{P}$ irgendein endliches Mengensystem ist, so bedeutet $N(\mathfrak{P})$ den Nerv von $\mathfrak{P}$.

Da für jedes m die Inklusion $F^{i_1 \cdots i_m} \subset S(F_{i_1 \cdots i_m}, d_m)$ stattfindet, so ist der Nerv des Systems aller nicht leeren unter den Mengen $F^{i_1 \cdots i_m}$ ein Teilkomplex von N_m, welcher (infolge der Irreduzibilität der Überdeckung $\mathfrak{B}_m$) mit N_m identisch sein muß; daraus folgt aber, daß keine der Mengen $F^{i_1 \cdots i_m}$ leer ist, mit anderen Worten: *zu jedem $F_{i_1 \cdots i_m}$ gibt es mindestens ein $F_{i_1 \cdots i_{m+1}}$.*

Man betrachte jetzt eine Mengenfolge von der Gestalt:

$$F_{i_1},\ F_{i_1 i_2},\ \cdots,\ F_{i_1 \cdots i_m},\ \cdots; \tag{2}$$

da $F_{i_1 \cdots i_{m+1}}$ mit $F_{i_1 \cdots i_m}$ gemeinsame Punkte hat und die Reihe $\sum_{m=1}^{\infty} \delta(F_{i_1 \cdots i_m})$ konvergiert, so konvergiert die ganze Mengenfolge (2) zu einem Punkt $x_{i_1 i_2 \cdots i_m \cdots}$. Ich behaupte nun, daß es zu jedem Punkt x der Menge F mindestens eine Folge natürlicher Zahlen

$$i_1, i_2, \cdots, i_m, \cdots \tag{3}$$

gibt — von der Beschaffenheit, daß $x = x_{i_1 i_2 \cdots i_m \cdots}$ ist. Es seien in der Tat die Mengen

$$F_{i_1^1},\ F_{i_1^2 i_2^2},\ \cdots,\ F_{i_1^m i_2^m \cdots i_m^m},\ \cdots \tag{4}$$

unter der Bedingung $x \subset F_{i_1^m i_2^m \cdots i_m^m}$ gewählt. Da es nur endlich viele verschiedene Werte gibt, die der erste Index in (4) annehmen kann, wird wenigstens ein Wert unendlich oft angenommen. Diesen Wert bezeichnen wir mit i_1. Man nehme jetzt an, daß eine aus m Elementen bestehende Indexkombination $(i_1 i_2 \cdots i_m)$ in (4) unendlich oft vorkommt, so daß eine Teilfolge

$$F_{i_1 \cdots i_m *_1'},\qquad F_{i_1 \cdots i_m *_2'},\ \cdots,\ F_{i_1 \cdots i_m *_k'},\ \cdots \tag{5}$$

der Folge (4) vorliegt. Da der $m+1$-er Index nur endlich vieler Werte fähig ist, tritt einer dieser Werte, etwa i_{m+1} in (5) unendlich oft vor, so daß (4) auch eine Teilfolge

$$F_{i_1 \cdots i_m i_{m+1} *_1''},\qquad F_{i_1 \cdots i_m i_{m+1} *_2''},\ \cdots,\ F_{i_1 \cdots i_m i_{m+1} *_k''},\ \cdots \tag{6}$$

enthält. Man betrachte jetzt die "Diagonalfolge"

$$F_{i_1 *_1},\ F_{i_1 i_2 *_2},\ \cdots,\ F_{i_1 i_2 \cdots i_m *_m},\ \cdots. \tag{6}$$

Da

$$x \subset F_{i_1 \cdots i_m *_m},\qquad \delta(F_{i_1 \cdots i_m \cdots i_{m+h}}) < \varepsilon_{m+h},\qquad F_{i_1 \cdots i_m \cdots i_{m+h}} \cdot F_{i_1 \cdots i_m \cdots i_{m+h+1}} \neq 0$$

ist, ist

$$\varrho(x, F_{i_1 \cdots i_m}) < \sum_{h=1}^{\infty} \varepsilon_{m+h} < \varepsilon_m,$$

so daß die Folge $F_{i_1 \cdots i_m}$ ($m = 1, 2, \cdots$ in inf.) gegen den Punkt x konvergiert, womit die Identität $x = x_{i_1 i_2 \cdots i_m \cdots}$ bewiesen ist.

Mit Hilfe desselben Verfahrens beweist sich auch folgende Tatsache (die wir sofort brauchen werden):

Wenn die Punkte

$$(7) \qquad x_{i_1 i_2 \cdots i_m h^1_{m+1} \cdots}, \quad x_{i_1 i_2 \cdots i_m h^2_{m+1} \cdots}, \cdots, x_{i_1 i_2 \cdots i_m h^k_{m+1} \cdots}, \cdots$$

gegen den Punkt x konvergieren, so ist $x = x_{i_1 i_2 \cdots i_m h_{m+1} \cdots}$. Die obige Überlegung erlaubt in der Tat aus (7) eine Teilfolge von der Gestalt

$$(8) \qquad x^1, x^2, \cdots, x^k, \cdots, \text{ mit } x^k = x_{i_1 \cdots i_m h_{m+1} \cdots h_{m+k} h^k_{m+k+1} h^k_{m+k+2} \cdots}$$

zu wählen; es ist aber

$$(9) \qquad \varrho(x^k, F_{i_1 \cdots i_m h_{m+1} \cdots h_{m+k}}) < \sum_{l=1}^{\infty} \varepsilon_{m+k+l} < \varepsilon_{m+k}$$

und da $\lim\limits_{k \to \infty} x^k = x$ ist, so folgt aus (9), daß auch $\lim\limits_{k=\infty} F_{i_1 \cdots i_m \cdots h_{m+k}} = x$, also $x = x_{i_1 i_2 \cdots i_m \cdots h_{m+k} \cdots}$ ist.

Man bezeichne jetzt mit $F'_{i_1 i_2 \cdots i_m}$ die Menge aller Punkte x, die eine Darstellung

$$x = x_{i_1 i_2 \cdots i_m h_{m+1} h_{m+2} \cdots}$$

zulassen, wobei die $i_1, i_2, \cdots, i_m$ als fest, die übrigen Indizes als variabel zu betrachten sind. Offenbar ist $F'_{i_1 i_2 \cdots i_m} = \sum\limits_{(i_{m+1})} F'_{i_1 i_2 \cdots i_m i_{m+1}}$; da ferner

$$(10) \qquad \varrho(x_{i_1 i_2 \cdots i_m h_{m+1} \cdots}, F_{i_1 \cdots i_m}) < \sum_{k=1}^{\infty} \varepsilon_{m+k} < d_m$$

ist, so ist $F'_{i_1 i_2 \cdots i_m} \subset S(F_{i_1 i_2 \cdots i_m}, d)$, also $\delta(F'_{i_1 i_2 \cdots i_m}) < \varepsilon_m$; die $F'_{i_1 i_2 \cdots i_m}$ bilden also eine Folge von Unterteilungen. Aus der Inklusion $F'_{i_1 i_2 \cdots i_m} \subset S(F_{i_1 i_2 \cdots i_m}, d)$ folgt aber weiter, daß der Nerv des Systems $\mathfrak{p}_m$ aller $F'_{i_1 i_2 \cdots i_m}$ ein (vermöge der Irreduzibilität von $\mathfrak{P}_m$) unechter Teilkomplex von $N(\mathfrak{P}_m)$ ist, so daß $\mathfrak{P}_m$ und $\mathfrak{p}_m$ ähnlich sind, w. z. b. w.

Korollar. *Jede Folge von unendlich fein werdenden Überdeckungen einer abgeschlossenen Menge F enthält eine Teilfolge*

$$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_m, \cdots$$

zu der es eine Folge von Unterteilungen

$$\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_m, \cdots,$$

gibt, so daß der $N(\mathfrak{p}_m)$ ein echter oder unechter Teilkomplex von $N(\mathfrak{P}_m)$ ist.[32]

3. Aus dem soeben bewiesenen Satze folgt, daß die Unterteilungen mit einem gewissen Recht als der Normaltyp von unendlich fein werdenden Überdeckungsfolgen betrachtet werden können. Wir wollen sie jetzt zur Konstruktion von Projektionsspektren anwenden.

Es sei

$$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_m, \cdots \tag{11}$$

eine Folge von Unterteilungen von F. Dabei sei $\mathfrak{P}_m$ eine aus den Mengen $F_{i_1 \cdots i_m}$ bestehende ε_m-Überdeckung ($\lim_{m\to\infty} \varepsilon_m = 0$), von der wir (auf Grund des Satzes I) annehmen, daß sie (falls $\dim F = \lambda < \infty$ ist) die Ordnung $\lambda + 1$ hat. Man setze ferner $N_m = N(\mathfrak{P}_m)$ und bezeichne die Eckpunkte von N_m naturgemäß mit $a_{i_1 \cdots i_m}$. Die Projektionen $\pi(N_{m+1})$ sind sodann dadurch definiert, daß man jedem Eckpunkt $a_{i_1 \cdots i_m i_{m+1}}$ von N_{m+1} den Eckpunkt $a_{i_1 \cdots i_m}$ von N_m zuordnet und die dadurch bestimmte simpliziale Abbildung von N_{m+1} auf N_m als die Abbildung $\pi(N_{m+1})$ erklärt.

Jeder Projektionsfolge

$$\mathrm{t} = (T_1, T_2, \cdots, T_m, \cdots) \tag{12}$$

wird folgendermaßen ein Punkt $x(\mathrm{t})$ von F zugeordnet. Man bezeichne mit $F(T_m)$ den Durchschnitt aller, Eckpunkten von T_m entsprechenden, Mengen $F_{i_1 \cdots i_m}$; dann gibt es, wie sofort ersichtlich, einen einzigen Punkt x, welcher zu allen (den Simplexen von (12) entsprechenden) Mengen $F(T_m)$ gehört. Dieser Punkt x soll definitionsgemäß der Punkt $x(\mathrm{t})$ heißen.

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[32] Aus dem Satze I ergibt sich leicht folgender Satz von Hurewicz (Proceed. Akad. Amsterdam, 29 (1926), S. 1015): *F ist dann und nur dann λ-dimensional, wenn man F als stetiges Bild einer abgeschlossenen nulldimensionalen Menge so darstellen kann, daß jeder Punkt von F bei dieser Abbildung höchstens $\lambda+1$ Originalpunkte hat und es keine Abbildung dieser Art gibt, bei der die Originalmenge jedes Punktes von F aus höchstens λ Punkten besteht.*

Daß eine λ-dimensionale F eine Darstellung von der erwünschten Art zuläßt, sieht man dadurch, daß man auf die Unterteilungsfolge (11), deren Elemente sämtlich von der Ordnung $\lambda+1$ sind, das in meiner Arbeit "Über stetige Abbildungen kompakter Räume" (Math. Ann., 96, SS. 555–571, insbesondere SS. 563–566) für den Fall beliebiger Folgen von Überdeckungen angegebene Abbildungsverfahren wörtlich anwendet.

Die zweite Hälfte des Satzes ergibt sich sodann wie folgt. Es sei $F = f(F')$, wobei F' eine nulldimensionale abgeschlossene Menge ist. Wenn ε beliebig gegeben ist, wähle man δ so klein, daß aus $\varrho(x', y) < \delta$ in F' folgt, daß $\varrho(x, y) < \varepsilon$ in F ist (wobei $x = f(x')$ und $y = f(y')$ gesetzt ist); wenn nun $F' = F_1' + F_2' + \cdots + F_s'$ eine Zerlegung von F' in zueinander paarweise fremde abgeschlossene Teilmengen von Durchmesser $< \delta$, $F_i = f(F_i')$ ist und jeder Punkt von F bei der Abbildung f höchstens $\lambda+1$ Originalpunkte hat, so bilden die F_i eine $(\varepsilon, \lambda+1)$-Überdeckung von F, w. z. b. w.

Jetzt läßt sich leicht zeigen, daß zwei benachbarte Projektionsfolgen stets eine umfassende Projektionsfolge besitzen und infolgedessen die Komplexe N_m durch unsere Projektionsdefinition zu einem Projektionsspektrum vereinigt werden.

Es seien in der Tat

(13′) $$\mathfrak{t}' = (T_1', T_2', \cdots, T_m', \cdots)$$

und

(13″) $$\mathfrak{t}'' = (T_1'', T_2'', \cdots, T_m'', \cdots)$$

zwei benachbarte Projektionsfolgen und t_m der Durchschnitt von T_m' und T_m''; da sowohl $F(T_m')$ als auch $F(T_m'')$ in $F(t_m)$ enthalten ist, ist

(14) $$x(\mathfrak{t}') = x(\mathfrak{t}) = x(\mathfrak{t}''),$$

wobei $\mathfrak{t} = (t_1, t_2, \cdots, t_m, \cdots)$ ist. Der Punkt (14) ist somit für jedes m in allen $F_{i_1 \cdots i_m}$ enthalten, die Eckpunkten von T_m' und T_m'' entsprechen, so daß die letztgenannten beiden Simplexe bei jedem m als Seiten eines Grundsimplexes T_m von N_m auftreten. Somit umfaßt die Projektionsfolge $\mathfrak{T} = (T_1, T_2, \cdots, T_m, \cdots)$ die beiden Folgen (13′) und (13″).

4. Die Folge

(15) $$N_1, N_2, \cdots, N_m, \cdots$$

ist also ein Projektionsspektrum. Es soll jetzt gezeigt werden, daß der durch dieses Projektionsspektrum definierte Raum $\mathfrak{F}$ mit F topologisch identisch ist. Zu diesem Zwecke soll zuerst eine eineindeutige Beziehung zwischen den Ketten

(16) $$\mathfrak{x} = (T_1, T_2, \cdots, T_m, \cdots)$$

und den Punkten x von F hergestellt werden. Wir sahen im vorigen Paragraphen, daß jeder Kette x (die ja eine Projektionsfolge ist) ein Punkt $x(\mathfrak{x})$ entspricht; jetzt zeigen wir umgekehrt, daß es zu jedem Punkt eine einzige Kette x gibt, für die $x = x(\mathfrak{x})$ ist. Um diese zu erhalten, betrachte man für jedes m die Gesamtheit aller $F_{i_1 \cdots i_m}$, die den Punkt x enthalten. Die entsprechenden Eckpunkte $a_{i_1 \cdots i_m}$ definieren sodann ein Simplex $T_m(x)$ aus N_m und man überzeugt sich leicht, daß

(17) $$T_1(x), \; T_2(x), \; \cdots, \; T_m(x), \; \cdots$$

eine Projektionsfolge ist. Diese Projektionsfolge ist aber überdies eine Kette. Es sei in der Tat

(18) $$\mathfrak{t} = (T_1, T_2, \cdots, T_m, \cdots)$$

eine umfassende Folge. Für ein bestimmtes m ist dann $T_m(x)$ eine Seite von T_m und somit ist für jedes $h > m$ $T_h(x)$ eine Seite von T_h. Da aber $F(T_h)$ unter diesen Umständen in $F(T_h(x))$ enthalten ist, so ist $x(t) = x$ und folglich $x \subset F(T_h)$, was der Definition von $T_h(x)$ widerspricht.

Die Projektionsfolge (17) ist also tatsächlich eine Kette x, und es ist $x(\mathfrak{x}) = x$. Daß auf die soeben geschilderte Weise einem Punkte x von F nur eine Kette entsprechen kann, folgt aus der leicht beweisbaren Tatsache, daß zwei verschiedene Projektionsfolgen t' und t'' nur dann denselben Punkt $x = x(t') = x(t'')$ bestimmen können, wenn sie von einer dritten Folge umfaßt werden.

5. Endlich soll gezeigt werden, daß die soeben definierte eineindeutige Zuordnung f zwischen den Punkten x von F und den Punkten $\mathfrak{x}$ des durch das Spektrum definierten Raumes $\mathfrak{F}$ eine beiderseits stetige ist. Dazu beweisen wir, daß vermöge f

a) F auf $\mathfrak{F}$ bzw. b) $\mathfrak{F}$ auf F

stetig abgebildet werden.

Ad a) Es sei für jeden Punkt $x \subset F$ ε_x^m die Entfernung von x bis zur Vereinigungsmenge ψ_x^m aller den Punkt x nicht enthaltenden Mengen $F_{i_1 \cdots i_m}$. Wenn ϱ_x^m die kleinste unter den positiven Zahlen $\varepsilon_x^1, \varepsilon_x^2, \cdots, \varepsilon_x^m$ bedeutet und $\mathfrak{U}_m(\mathfrak{x})$ die m-te Umgebung des Punktes $\mathfrak{x}_0 = f(x_0)$ in $\mathfrak{F}$ ist, so folgt, wie leicht ersichtlich, aus $\varrho(x_0, x) < \varrho_{x_0}^m$ die Inklusion $\mathfrak{x} \subset \mathfrak{U}_m(\mathfrak{x}_0)$ (in $\mathfrak{F}$); da dies für jedes m (also für jede Umgebung von $\mathfrak{x}_0$) zutrifft, ist die Behauptung a) bewiesen.

Ad b) Es seien $\varepsilon > 0$ und $\mathfrak{x}_0 \subset \mathfrak{F}$ beliebig gegeben. Wenn m so groß ist, daß $\varepsilon_m < \varepsilon$ ausfällt und $\mathfrak{x} \subset \mathfrak{U}_m(\mathfrak{x}_0)$ beliebig gewählt ist, so ist (da x unter diesen Bedingungen nur zu solchen $F_{i_1 \cdots i_m}$ gehören kann, die x_0 enthalten) $\varrho(x_0, x) < \varepsilon$, womit b) bewiesen ist.

Hiermit ist bewiesen:

SATZ II. *Jeder kompakte metrisierbare Raum läßt sich durch ein Projektionsspektrum definieren; dabei lassen sich Räume, deren Dimension $\leqq \lambda$ ist, immer durch höchstens n-dimensionale Projektionsspektra definieren.*

6. Es gilt auch die Umkehrung dieses Satzes, nämlich der

SATZ III. *Jeder durch ein Projektionsspektrum definierte Raum ist ein kompakter metrisierbarer topologischer Raum; wenn sämtliche Komplexe des Projektionsspektrums von einer Dimension $\leqq \lambda$ sind, so ist auch der durch das Spektrum definierte Raum höchstens λ-dimensional.*

Der Beweis unterscheidet sich nur wenig von dem Beweise der entsprechenden Tatsache in den "Simpl. Appr." (§§ 10–12): daß der Raum topologisch ist und dem II. Abzählbarkeitsaxiom genügt, folgt sofort aus unserer Umgebungsdefinition. Die Kompaktheit folgt leicht aus der Tatsache, daß jeder Komplex aus endlich vielen Simplexen besteht und infolge-

dessen die m-te"Koordinate" (m-beliebig) eines (variablen) Punktes nur endlich vieler verschiedener Werte fähig ist. Daraus folgt bereits die Metrisierbarkeit unseres Raumes. Man bezeichne ferner durch $\Phi_{i_1 i_2 \cdots i_m}$ die Gesamtheit aller Punkte, deren erste m Koordinaten der Reihe nach die Eckpunkte

$$a_{i_1},\ a_{i_1 i_2},\ \cdots,\ a_{i_1 i_2 \cdots i_m},$$

enthalten. Der Durchschnitt einer Mengenfolge von der Gestalt

$$\Phi_{i_1},\ \Phi_{i_1 i_2},\ \cdots,\ \Phi_{i_1 i_2 \cdots i_m}$$

besteht aber nur aus einem Punkte, woraus wiederum folgt, daß in jeder Metrik, die man dem durch unser Projektionsspektrum definierten Raum vorschreiben kann, die Durchmesser der Mengen $\Phi_{i_1 i_2 \cdots i_m}$ mit $\frac{1}{m}$ beliebig klein werden.[33] Endlich ist in der Definition der Mengen $\Phi_{i_1 i_2 \cdots i_m}$ selbst die Tatsache enthalten, daß der Nerv des Systems aller $\Phi_{i_1 i_2 \cdots i_m}$ (m beliebig aber fest) mit dem Komplex K_m identisch ist. Das bedeutet aber nichts anderes, als daß man unseren Raum beliebig fein so überdecken kann, daß der Nerv dieser Überdeckung ein Komplex des (den Raum definierenden) Projektionsspektrums ist; wenn dieses Spektrum kein Komplex von einer Dimension $> \lambda$ enthält, so ist also auch der durch dasselbe definierte Raum höchstens λ-dimensional, w. z. b. w.

7. Es seien jetzt zwei Projektionsspektra

(19) $$K_1, K_2, \cdots, K_m, \cdots$$

und

(20) $$Q_1, Q_2, \cdots, Q_m, \cdots$$

gegeben; so entsteht die natürliche Frage:

Unter welchen Bedingungen definieren die beiden Spektra (19) *und* (20) *denselben topologischen Raum?*

Diese Frage soll jetzt beantwortet werden.

Vorbemerkung I. Es sei eine wachsende Folge natürlicher Zahlen gegeben:

(21) $$i_1, i_2, \cdots, i_m, \cdots.$$

Einfachheitshalber schreibe man

(22) $$K_{i_m} = A_m \quad \text{und} \quad Q_{i_m} = B_m.$$

Da die A_m bzw. die B_m je zu einem Projektionsspektrum gehören, ist für jedes Paar natürlicher Zahlen $r \leqq s$ die *Projektion* von A_s auf A_r sowie

[33] Es gilt ja für jedes m $\Phi_{i_1 i_2 \cdots i_m} \supset \Phi_{i_1 i_2 \cdots i_m i_{m+1}}$!

die *Projektion* von B_s auf B_r definiert; diese beiden Abbildungen werden durch $\pi_r^s(A_s)$ bzw. durch $\pi_r^s(B_s)$ bezeichnet, so daß

$$A_r = \pi_r^s(A_s) \quad \text{und} \quad B_r = \pi_r^s(B_s) \tag{23}$$

ist.

Vorbemerkung II. Wenn a_i und a_j zwei benachbarte Eckpunkte des Komplexes K sind,[34] so schreiben wir $a_i \cdot\!\!-\!\!\cdot\, a_j$.

8. Es gilt nun folgender

HOMÖOMORPHIESATZ. *Zwei Projektionsspektra* (19) *und* (20) *definieren dann und nur dann denselben topologischen Raum, wenn eine Teilfolge*

$$A_1, A_3, \cdots, A_{2m+1}, \cdots, \qquad (A_{2m+1} = K_{h_{2m+1}}), \tag{24}$$

eine Teilfolge

$$B_2, B_4, \cdots, B_{2m}, \cdots, \qquad (B_{2m} = Q_{k_{2m}}), \tag{25}$$

und für jedes m eine simpliziale Abbildung $f_{2m}(A_{2m+1})$ von A_{2m+1} auf einen (echten oder unechten) Teilkomplex von B_{2m} und eine simpliziale Abbildung $f_{2m-1}(B_{2m})$ von B_{2m} auf einen (echten oder unechten) Teilkomplex von A_{2m-1} so gefunden werden können, daß für beliebige m und r, $m \leqq r$, und einen beliebigen Eckpunkt a^{2r+1} von A_{2r+1} bzw. b^{2r} von B_{2r} die Relationen

$$\pi_{2m+1}^{2r-1} f_{2r-1} f_{2r}(a^{2r+1}) \cdot\!\!-\!\!\cdot\, \pi_{2m+1}^{2r+1}(a^{2r+1});$$
$$\pi_{2m}^{2r} f_{2r}(a^{2r+1}) \cdot\!\!-\!\!\cdot\, f_{2m} \pi_{2m+1}^{2r+1}(a^{2r+1})$$

bzw.

$$\pi_{2m}^{2r-2} f_{2r-2} f_{2r-1}(b^{2r}) \cdot\!\!-\!\!\cdot\, \pi_{2m}^{2r}(b^{2r}); \quad \pi_{2m-1}^{2r-1} f_{2r-1}(b^{2r}) \cdot\!\!-\!\!\cdot\, f_{2m-1} \pi_{2m}^{2r}(b^{2r})$$

in A_{2m+1}; B_{2m} bzw. in B_{2m}; A_{2m-1} stattfinden.

Wenn man (im Anschluß an einen von Alexander herrührenden Sprachgebrauch) eine simpliziale Abbildung eines Komplexes K in sich *pseudoidentisch* nennt, wenn sie jedem Eckpunkt von K einen benachbarten Eckpunkt entsprechen läßt, kann man die soeben formulierte Bedingung folgendermaßen aussprechen:

Bei beliebigen m und r, $m \leqq r$, können durch passend gewählte pseudoidentische Abbildungen von A_{2m+1}; B_{2m} bzw. B_{2m}; A_{2m-1} in sich

der Komplex $\pi_{2m+1}^{2r-1} f_{2r-1} f_{2r}(A_{2r+1})$ in den (ganzen) Komplex A_{2m+1};

der Komplex $\pi_{2m}^{2r} f_{2r}(A_{2r+1})$ in den Komplex $f_{2m} \pi_{2m+1}^{2r+1}(A_{2r+1})$

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[34] Zwei Eckpunkte a und b des Komplexes K heißen *benachbart* (in K), wenn es in K ein eindimensionales Simplex (= eine Kante) gibt, welches a und b als seine beiden Endpunkte besitzt. Wenn die Punkte a und b zusammenfallen oder benachbart sind, schreiben wir $a \cdot\!\!-\!\!\cdot\, b$, wenn man a und b mit einer aus höchstens zwei Kanten bestehenden Kette verbinden kann, schreiben wir $a \cdot\!\!-\!\!\cdot\!\!-\!\!\cdot\, b$ usw.

sowie

der Komplex $\pi_{2m}^{2r-2} f_{2r-2} f_{2r-1}(B_{2r})$ *in* B_{2m}

und

der Komplex $\pi_{2m-1}^{2r-1} f_{2r-1}(B_{2r})$ *in* $f_{2m-1}\pi_{2m}^{2r}(B_{2r})$

übergeführt werden.

9. Der geometrische Inhalt dieses vielleicht kompliziert klingenden Satzes ist in Wirklichkeit sehr einfach. In der Tat, wenn man zwei durch Abbildungen $f_{2m}(A_{2m+1})$ und $f_{2m-1}(B_{2m})$ miteinander verbundene Projektionsspektra hat, so sind die ersten Fragen, die man sich stellen kann, die folgenden:

I. Wie kommt man am kürzesten (d. h. mit möglichst wenig Schritten) von einem hohen Range des einen Spektrums auf einen niedrigeren Rang desselben Spektrums — also z. B. vom Range $2r+1$ des A-Spektrums auf den Rang $2m+1$ desselben Spektrums?

II. Wie kommt man am kürzesten von einem hohen Range des einen Spektrums auf einen niedrigeren Rang des anderen — also z. B. vom Range $2r+1$ des A-Spektrums auf den Rang $2m$ des B-Spektrums?

Die Antwort auf die erste Frage lautet selbstverständlich:

"Durch Projektion $\pi_{2m+1}^{2r+1}(a^{2r+1}) = \pi_{2m+1}^{2m+3} \cdots \pi_{2r-3}^{2r-1} \pi_{2r-1}^{2r+1}(a^{2r+1})$."

Nun besteht der eine Teil unserer Bedingung gerade in der Behauptung, daß man diesen "kürzesten Weg" nur wenig (nur um eine "Pseudo-Identität") verlängert, wenn man seinen Anfang durch einen Sprung (*dann aber notwendig hin und zurück*) auf das zweite Spektrum ersetzt, also statt $\pi_{2r-3}^{2r-1} \pi_{2r-1}^{2r+1}(a)$ die Abbildung $f_{2r-1} f_{2r}(a^{2r+1})$ macht.

Als Antwort auf die zweite Frage kann man zwei a priori gleichberechtigte Vorschläge machen, und zwar hat man den jedenfalls notwendigen Übergang ins zweite Spektrum entweder am Anfang oder am Ende zu machen. Im ersten Falle beginnt man mit dem Übergang $f_{2r}(a^{2r+1})$ von A_{2r+1} auf den nächstliegenden Rang B_{2r}, und läßt sich dann nach B_{2m} hinunterprojizieren, so daß man insgesamt den Weg $\pi_{2m}^{2r} f_{2r}(a^{2r+1})$ zurücklegt; im zweiten Falle macht man die Projektion (dann aber gleich möglichst weit) im A-Spektrum, was $\pi_{2m+1}^{2r+1}(a^{2r+1})$ ergibt, und hat dann noch den Übergang f_{2m} zu machen, wodurch man nach $f_{2m}\pi_{2m+1}^{2r+1}(a^{2r+1})$ gelangt. Nun besteht der zweite Teil unserer Bedingung gerade in der Behauptung, daß diese beiden — a priori als gleichberechtigt erscheinenden — Wege es auch tatsächlich (wieder bis auf eine "Pseudo-Identität") sind.

Beweis des Homöomorphiesatzes.

10. Wir nehmen zuerst an, daß die beiden Spektra (19) und (20) denselben Raum R definieren, den wir uns als eine kompakte abgeschlossene

Teilmenge F des Hilbertschen Raumes denken. In F (und also auch in R) ist somit eine feste Metrik endgültig gewählt.

Die beiden Spektra bestimmen in bekannter Weise zwei Folgen von Unterteilungen

(26) $$\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_k, \cdots$$

und

(27) $$\mathfrak{V}_1, \mathfrak{V}_2, \cdots, \mathfrak{V}_k, \cdots.$$

Die Mengensysteme $\mathfrak{U}$ bzw. $\mathfrak{V}$ bilden dabei (für passend gewählte positive Zahlen η_k bzw. ϑ_k) η_k- bzw. ϑ_k-Überdeckungen von F, und es ist $\lim\limits_{k\to\infty} \eta_k = \lim\limits_{k\to\infty} \vartheta_k = 0$.

Man setze jetzt $i_1 = 1$ und $A_1 = K_1$; man wähle ferner i_2 so groß, daß

$$\vartheta_2 < d(\mathfrak{U}_1)$$

ist und setze $B_2 = Q_{i_2}$. Allgemein wähle man i_{2m+1} so groß, daß

$$\eta_{i_{2m+1}} < d(\mathfrak{V}_{i_{2m}})$$

ist, setze $A_{2m+1} = K_{i_{2m+1}}$, wähle dann i_{2m+2} unter der Bedingung, daß

$$\vartheta_{i_{2m+2}} < d(\mathfrak{U}_{i_{2m+1}})$$

ist und setze $B_{2m+2} = Q_{i_{2m+2}}$.

Auf diese Weise erhält man die beiden Teilfolgen (24) bzw. (25) von (19) bzw. (20).

(24) und (25) sind Projektionsspektra und A_{2m+1} bzw. B_{2m} ist für jedes m der Nerv des Mengensystems $\mathfrak{U}_{i_{2m+1}}$ bzw. $\mathfrak{V}_{i_{2m}}$. Nun können wir die Elemente dieser Mengensysteme durch $F^a_{h_1 h_3 \cdots h_{2m+1}}$ bzw. $F^b_{k_2 k_4 \cdots k_{2m}}$ bezeichnen (wobei eine Umnumerierung der Mengen aus $\mathfrak{U}_{i_{2m+1}}$ bzw. $\mathfrak{V}_{i_{2m}}$ stattgefunden hat, bei der nur die an der $i_1, i_3, \cdots, i_{2m+1}$ bzw. $i_2, i_4, \cdots, i_{2m}$ Stellen gestandenen Indizes berücksichtigt wurden). Wir setzen endlich

$$\mathfrak{A}_{2m+1} = \mathfrak{U}_{i_{2m+1}}, \qquad \mathfrak{B}_{2m} = \mathfrak{V}_{i_{2m}},$$
$$\alpha_{2m+1} = \eta_{i_{2m+1}}, \qquad \beta_{2m} = \vartheta_{i_{2m}}.$$

Die Komplexe A_{2m+1} bzw. B_{2m} können (als Nerven von $\mathfrak{A}_{2m+1}$ bzw. $\mathfrak{B}_{2m}$) dadurch geometrisch realisiert werden, daß man sich den, einer Menge $F^a_{i_1 i_3 \cdots i_{2m+1}}$ bzw. $F^b_{j_2 j_4 \cdots j_{2m}}$ entsprechenden, Eckpunkt von A_{2m+1} bzw. B_{2m} als einen zu dieser (evtl. auch zu anderen Mengen desselben Ranges) gehörenden Punkt $a_{i_1 i_3 \cdots i_{2m+1}}$ bzw. $b_{j_2 j_4 \cdots j_{2m}}$ denkt, wobei zu beachten ist, daß

infolge dieser Wahl die ersten $m+1$ bzw. m Koordinaten des Punktes[35] $\mathfrak{a}_{i_1 i_3 \cdots i_{2m+1}} \sim a_{i_1 i_3 \cdots i_{2m+1}}$ bzw. $\mathfrak{b}_{j_2 j_4 \cdots j_{2m}} \sim b_{j_2 j_4 \cdots j_{2m}}$ der Reihe nach die den Elementen $F_{i_1}, F_{i_1 i_3}, \cdots, F_{i_1 i_3 \cdots i_{2m+1}}$ von $\mathfrak{A}_1, \mathfrak{A}_3, \cdots, \mathfrak{A}_{2m+1}$ (bzw. $F_{j_2}, F_{j_2 j_4}, \cdots, F_{j_2 j_4 \cdots j_{2m}}$ von $\mathfrak{B}_2, \mathfrak{B}_4, \cdots, \mathfrak{B}_{2m}$) entsprechenden Eckpunkte $a_i, a_{i_1 i_3}, \cdots, a_{i_1 i_3 \cdots i_{2m+1}}$ von $A_1, A_3, \cdots, A_{2m+1}$ bzw. $B_2, B_4, \cdots, B_{2m}$ sind, so daß man

$$\begin{aligned} \mathfrak{a}_{i_1 i_3 \cdots i_{2m+1}} &= (a_{i_1}, a_{i_1 i_3}, \cdots, a_{i_1 i_3 \cdots i_{2m+1}}, T_{2m+3}, \cdots), \\ \mathfrak{b}_{i_2 i_4 \cdots i_{2m}} &= (b_{i_2}, b_{i_2 i_4}, \cdots, b_{i_2 i_4 \cdots i_{2m}}, T_{2m+2}, \cdots) \end{aligned} \tag{28}$$

schreiben kann. Man wähle jetzt zu jeder Indexkombination $(i_1 i_3 \cdots i_{2m+1})$ (m beliebig) eine feste Indexkombination $(j_2 j_4 \cdots j_{2m})$ unter der einzigen Bedingung, daß $F_{j_2 j_4 \cdots j_{2m}}$ den (geometrisch realisierten) Punkt $a_{i_1 i_3 \cdots i_{2m+1}}$ enthält. Diese Wahl denken wir uns ein für allemal getroffen und setzen $f_{2m}(a_{i_1 i_3 \cdots i_{2m+1}}) = b_{j_2 j_4 \cdots j_{2m}}$. Aus der Ungleichung $\alpha_{2m+1} < d(\mathfrak{B}_{2m})$ folgt sodann, daß, wenn die Punkte $a_{i_1' \cdots i_{2m+1}'}, \cdots, a_{i_1^{(\nu)} \cdots i_{2m+1}^{(\nu)}}$ Eckpunkte eines Simplexes von A_{2m+1} sind, die Punkte $f_{2m}(a_{i_1' \cdots i_{2m+1}'}), \cdots, f_{2m}(a_{i_1^{(\nu)} \cdots i_{2m+1}^{(\nu)}})$ das Eckpunktsystem eines Simplexes von B_{2m} bilden, so daß die soeben festgelegte Eckpunktzuordnung eine simpliziale Abbildung von A_{2m+1} auf B_{2m} definiert. Diese Abbildung soll $f_{2m}(A_{2m+1})$ heißen.

In analoger Weise wählen wir für jede Indexkombination $(k_2 k_4 \cdots k_{2m})$ eine Kombination $(h_1 h_3 \cdots h_{2m-1})$ unter der Bedingung, daß $b_{k_2 \cdots k_{2m}} \subset F^a_{h_1 \cdots h_{2m-1}}$ ist, setzen sodann $f_{2m-1}(b_{k_2 \cdots k_{2m}}) = a_{h_1 \cdots h_{2m-1}}$, wodurch die simplizialen Abbildungen $f_{2m-1}(B_{2m})$ definiert sind.

11. Jetzt soll bewiesen werden, daß bei der soeben gemachten Definition von f_{2m} bzw. f_{2m-1} die Bedingungen des Homöomorphiesatzes erfüllt sind.

I. Es sei

$$f_{2r}(a_{i_1 \cdots i_{2r+1}}) = b_{j_1 \cdots j_{2r}}, \quad \text{also } a_{i_1 \cdots i_{2r+1}} \subset F_{j_1 \cdots j_{2r}},$$

ferner

$$f_{2r-1}(b_{j_1 \cdots j_{2r}}) = a_{i_1' \cdots i_{2r-1}'}, \quad \text{also } b_{j_1 \cdots j_{2r}} \subset F_{i_1' \cdots i_{2r-1}'}.$$

Da außerdem

$$a_{i_1 \cdots i_{2r+1}} \subset F_{i_1 \cdots i_{2r+1}} \subset F_{i_1 \cdots i_{2r-1}} \quad \text{und} \quad b_{j_1 \cdots j_{2r}} \subset F_{j_1 \cdots j_{2r}}$$

ist, so ist

$$F_{j_1 \cdots j_{2r}} \cdot F_{i_1 \cdots i_{2r-1}} \neq 0 \neq F_{j_1 \cdots j_{2r}} \cdot F_{i_1' \cdots i_{2r-1}'},$$

woraus (vermöge der Ungleichung $\beta_{2r} < d(\mathfrak{A}_{2r-1})$) folgt, daß

$$F_{i_1 \cdots i_{2r-1}} F_{i_1' \cdots i_{2r-1}'} \neq 0, \quad \text{also umsomehr} \quad F_{i_1 \cdots i_{2m+1}} \cdot F_{i_1' \cdots i_{2m+1}'} \neq 0,$$

d. h. $a_{i_1 \cdots i_{2m+1}} \cdot - \cdot a_{i_1' \cdots i_{2m+1}'}$ ist.

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[35] $\mathfrak{a}_{i_1 i_3 \cdots i_{2m+1}}$ ist der Punkt $a_{i_1 i_3 \cdots i_{2m+1}}$ der Menge F, betrachtet als Kette des Projektionsspektrums (24); analoges gilt für $\mathfrak{b}_{j_2 j_4 \cdots j_{2m}}$.

II. Es sei

$$f_{2r}(a_{i_1}\cdots i_{2r+1}) = b_{j'_1}\cdots j'_{2r}, \qquad f_{2m}(a_{i_1}\cdots i_{2m+1}) = b_{j_1}\cdots j_{2m};$$

zu beweisen ist, daß $b_{j'_1}\cdots j'_{2m} \cdot - \cdot b_{j_1}\cdots j_{2m}$ ist.

Es ist dann

$$a_{i_1}\cdots i_{2r+1} \subset F_{j'_1}\cdots j'_{2r} \subset F_{j'_1}\cdots j'_{2m} \text{ und } a_{i_1}\cdots i_{2r+1} \subset F_{i_1}\cdots i_{2r+1} \subset F_{i_1}\cdots i_{2m+1},$$

also

$$F_{j'_1}\cdots j'_{2m} \cdot F_{i_1}\cdots i_{2m+1} \neq 0; \tag{29}$$

gleichzeitig damit ist

$$a_{i_1}\cdots i_{2m+1} \subset F_{j_1}\cdots j_{2m} \text{ und } a_{i_1}\cdots i_{2m+1} \subset F_{i_1}\cdots i_{2m+1},$$

also

$$F_{j_1}\cdots j_{2m} \cdot F_{i_1}\cdots i_{2m+1} \neq 0; \tag{30}$$

aus (29) und (30) folgt (vermöge der Ungleichung $\alpha_{2m+1} < d(\mathfrak{B}_{2m})$), daß auch $F_{j_1}\cdots j_{2m} \cdot F_{j'_1}\cdots j'_{2m} \neq 0$ ist, also $b_{j_1}\cdots j_{2m} \cdot - \cdot b_{j'_1}\cdots j'_{2m}$ ist.

Es ist hiermit bewiesen, daß

$$\pi^{2r-1}_{2m+1} f_{2r-1} f_{2r}(a^{2r+1}) \cdot - \cdot \pi^{2r+1}_{2m+1}(a^{2r+1}) \tag{31}$$

und

$$\pi^{2r}_{2m} f_{2r}(a^{2r+1}) \cdot - \cdot f_{2m}\, \pi^{2r+1}_{2m+1}(a^{2r+1}) \tag{32}$$

ist.

Genau auf dieselbe Weise beweist man, daß auch

$$\pi^{2r-2}_{2m} f_{2r-2} f_{2r-1}(b^{2r}) \cdot - \cdot \pi^{2r}_{2m}(b^{2r}) \tag{33}$$

und

$$\pi^{2r-2}_{2m-1} f_{2r-1}(b^{2r}) \cdot - \cdot f_{2m-1}\, \pi^{2r}_{2m}(b^{2r}) \tag{34}$$

ist, woraus der erste Teil des Homöomorphiesatzes (die Notwendigkeit der daselbst ausgesprochenen Bedingung) folgt.

12. Es seien jetzt die Projektionsspektra (19) und (20), die Teilfolgen (24) und (25), die Abbildungen $f_{2r}(A_{2r+1})$ und $f_{2r-1}(B_{2r})$ a priori gegeben, und zwar so, daß die Bedingungen (31), (32), (33), (34) erfüllt sind. Es seien A und B die durch (19) bzw. (20) definierten Räume, die wir uns sogleich in einer festen Metrik denken. Die Spektra (19) und (20) definieren dann in üblicher Weise die Unterteilungen $\mathfrak{A}_{2r+1}$ bzw. $\mathfrak{B}_{2r}$ von A bzw. B, deren Elemente $F_{i_1}\cdots i_{2r+1}$ bzw. $F_{i_2}\cdots i_{2r}$ sind und die α_{2r+1}- bzw. β_{2r}-Überdeckungen von A bzw. B darstellen, wobei $\lim_{r\to\infty} \alpha_{2r+1} = \lim_{r\to\infty} \beta_{2r} = 0$ ist.

Es ist zu zeigen, daß unter diesen Bedingungen die Räume A und B homöomorph sind.

Hilfssatz I. Wenn

$$b_{j_2 \cdots j_{2r}} = f_{2r}(a_{i_1 \cdots i_{2r+1}}) \quad \text{und} \quad a_{i_1 \cdots i_{2r+1}} = f_{2r+1}(b_{h_2 \cdots h_{2r+2}})$$

ist (wobei $a_{i_1 \cdots i_{2r+1}}$ bzw. $b_{j_2 \cdots j_{2r}}$ zugleich die geometrische Realisation der Punkte $a_{i_1 \cdots i_{2r+1}}$ bzw. $b_{j_2 \cdots j_{2r}}$ bedeutet), so ist (in der Metrik von B)

$$\varrho(b_{j_2 \cdots j_{2r}},\ b_{h_2 \cdots h_{2r+2}}) < 2\beta_{2r}.$$

Es ist in der Tat

$$b_{j_2 \cdots j_{2r}} = f_{2r} f_{2r+1}(b_{h_2 \cdots h_{2r+2}}); \qquad b_{h_2 \cdots h_{2r}} = \pi(b_{h_2 \cdots h_{2r+2}}),$$

also vermöge (33)

$$b_{h_2 \cdots h_{2r}} \cdot - \cdot b_{j_2 \cdots j_{2r}},$$

woraus (da $b_{h_2 \cdots h_{2r+2}} \subset F_{h_2 \cdots h_{2r}}$ ist) $\varrho(b_{h_2 \cdots h_{2r+2}}, b_{j_2 \cdots j_{2r}}) < 2\beta_{2r}$ folgt.

Hilfssatz II. Es sei x ein durch seine Koordinaten (im Spektrum (24)) bestimmter Punkt des Raumes A:

$$x = (T_1,\ T_3,\ \cdots,\ T_{2r+1},\ \cdots) \tag{35}$$

Man bezeichne durch $B(T_{2r+1})$ die folgendermaßen definierte endliche Teilmenge des Raumes B: $B(T_{2r+1})$ besteht aus allen denjenigen Eckpunkten von B_{2r} und B_{2r+2}, die bei der Abbildung $f_{2r}(A_{2r+1})$ Bildpunkte bzw. bei der Abbildung $f_{2r+1}(B_{2r+2})$ Originalpunkte irgendeines Eckpunktes des Simplexes T_{2r+1} sind. Somit behauptet Hilfssatz II:

Die Mengen $B(T_{2r+1})$ konvergieren bei unendlich wachsendem r gegen einen Punkt y von B, den wir durch $\varphi(x)$ bezeichnen werden.

Es ist zunächst $B(T_{2r+1}) = B_*(T_{2r+1}) + B^*(T_{2r+1})$, wobei $B_*(T_{2r+1})$ die Gesamtheit aller zu B_{2r} und $B^*(T_{2r+1})$ die Gesamtheit aller zu B_{2r+2} gehörenden Punkte von $B(T_{2r+1})$ bezeichnet. Da es aber vermöge des Hilfssatzes I zu jedem Punkt von $B^*(T_{2r+1})$ einen von demselben weniger als um $2\beta_{2r}$ entfernten Punkt von $B_*(T_{2r+1})$ gibt, genügt es zu zeigen, daß die Mengen $B_*(T_{2r+1})$ gegen einen Punkt konvergieren. Zum letzteren Zweck genügt es wiederum zu beweisen, daß man zu jedem $\varepsilon > 0$ eine natürliche Zahl m so finden kann, daß aus

$$s \geqq r > m; \qquad a_{i_1 \cdots i_{2r+1}} \subset T_{2r+1}; \qquad a_{h_1 \cdots h_{2s+1}} \subset T_{2s+1},$$
$$b_{j_2 \cdots j_{2r}} = f_{2r}(a_{i_1 \cdots i_{2r+1}}); \qquad b_{k_2 \cdots k_{2s}} = f_{2s}(a_{h_1 \cdots h_{2s+1}})$$

folgt, daß $\varrho(b_{j_2 \cdots j_{2r}}, b_{k_2 \cdots k_{2s}}) < \varepsilon$ ist.

Es sei nun ε gegeben und m so groß, daß $4\beta_{2m} < \varepsilon$ ist. Dann ist (vermöge (32)):

$$(36)\qquad \begin{aligned} b_{j_1\cdots j_{2m}} &= \pi^{2r}_{2m} f_{2r}(a_{i_1\cdots i_{2r+1}}) \cdot — \cdot f_{2m}(a_{i_1\cdots i_{2m+1}}),\\ b_{k_1\cdots k_{2m}} &= \pi^{2s}_{2m} f_{2s}(a_{h_1\cdots h_{2s+1}}) \cdot — \cdot f_{2m}(a_{h_1\cdots h_{2m+1}}). \end{aligned}$$

Da $\pi^{2s+1}_{2r+1}(a_{h_1\cdots h_{2s+1}}) \subset T_{2r+1}$ ist, so ist

$$a_{h_1\cdots h_{2r+1}} \cdot — \cdot a_{i_1\cdots i_{2r+1}}$$

und somit auch

$$a_{h_1\cdots h_{2m+1}} \cdot — \cdot a_{i_1\cdots i_{2m+1}}$$

und

$$f_{2m}(a_{h_1\cdots h_{2m+1}}) \cdot — \cdot f_{2m}(a_{i_1\cdots i_{2m+1}}),$$

also (vermöge (36))

$$b_{j_1\cdots j_{2m}} \cdot — \cdot \cdot — \cdot \cdot — \cdot b_{k_1\cdots k_{2m}};$$

da aber $b_{j_1\cdots j_{2r}} \subset F_{j_1\cdots j_{2m}}$ und $b_{k_1\cdots k_{2s}} \subset F_{k_1\cdots k_{2m}}$, so ist $\varrho(b_{j_1\cdots j_{2r}}, b_{k_1\cdots k_{2s}}) < 4\beta_{2m} < \varepsilon$, womit Hilfssatz II bewiesen ist.

13. Aus dem soeben bewiesenen Hilfssatz ergibt sich folgende für unsere Zwecke wichtige Bemerkung.

Es seien

$$(37)\qquad T^y_2, T^y_4, \cdots, T^y_{2r}, \cdots$$

die Koordinaten des Punktes $y = \varphi(x)$; der Symmetrie wegen werden wir für den Augenblick die Koordinaten von x durch T^x_{2r+1} bezeichnen. Da die $B(T^x_{2r+1})$ gegen y konvergieren, so gibt es zu jedem m ein so großes r_m, daß alle $B(T^x_{2r+1})$, $r > r_m$ in der $2m$-ten Umgebung des Punktes y enthalten sind; auf Grund der durch das Projektionsspektrum gegebenen Umgebungsdefinition bedeutet jedoch letztere Tatsache nichts anderes, als *daß für alle* $r > r_m$

$$(38)\qquad \pi^{2r}_{2m} f_{2r}(T^x_{2r+1}) + \pi^{2r+2}_{2m} f^{-1}_{2r+2}(T^x_{2r+1}) \subset T^y_{2m}$$

ist.

14. *Hilfssatz* III. Wenn

$$(39)\qquad x' = (T'_1, T'_3, \cdots, T'_{2r+1}, \cdots)$$

und

$$(40)\qquad x'' = (T''_1, T''_3, \cdots, T''_{2r+1}, \cdots)$$

zwei verschiedene Punkte von A sind, so sind die Punkte $y' = \varphi(x')$ und $y'' = \varphi(x'')$ auch verschieden.

Um diesen Hilfssatz zu beweisen, wähle man eine Zahl m so groß, daß $6\alpha_{2m-1} < \varrho(x', x'')$, woraus folgt, daß man einen Eckpunkt von T'_{2m-1} mit einem Eckpunkt von T''_{2m-1} in A_{2m-1} unmöglich mit einem höchstens aus 5 Strecken bestehenden Streckenzuge verbinden kann.

Wir nehmen an, der Hilfssatz III sei falsch, und es sei

$$\varphi(x') = \varphi(x'') = y. \tag{41}$$

Man wähle r so groß, daß

$$\pi^{2r}_{2m} f_{2r}(T'_{2r+1}) + \pi^{2r}_{2m} f_{2r}(T''_{2r+1}) \subset T^{y}_{2m} \tag{42}$$

sei und bezeichne mit a' bzw. a'' irgendeinen Eckpunkt von T'_{2r+1} bzw. T''_{2r+1}. Dann folgt aus (42), daß

$$\pi^{2r}_{2m} f_{2r}(a') \cdot - \cdot \pi^{2r}_{2m} f_{2r}(a'') \tag{43}$$

ist. Da aber

$$f_{2m}\, \pi^{2r+1}_{2m+1}(a') \cdot - \cdot \pi^{2r}_{2m} f_{2r}(a') \quad \text{und} \quad f_{2m}\, \pi^{2r+1}_{2m+1}(a'') \cdot - \cdot \pi^{2r}_{2m} f_{2r}(a'')$$

ist, so ist

$$f_{2m}\, \pi^{2r+1}_{2m+1}(a') \cdot - \cdot \cdot - \cdot \cdot - \cdot f_{2m}\, \pi^{2r+1}_{2m+1}(a'')$$

und

$$f_{2m-1} f_{2m}\, \pi^{2r+1}_{2m+1}(a') \quad - \cdot \cdot - \cdot \cdot - \cdot f_{2m-1} f_{2m}\, \pi^{2r+1}_{2m+1}(a''). \tag{44}$$

Nun ist aber

$$\pi^{2m+1}_{2m-1}(\pi^{2r+1}_{2m+1}(a')) \cdot - \cdot f_{2m-1} f_{2m}(\pi^{2r+1}_{2m+1}(a')),$$
$$\pi^{2m+1}_{2m-1}(\pi^{2r+1}_{2m+1}(a'')) \cdot - \cdot f_{2m-1} f_{2m}(\pi^{2r+1}_{2m+1}(a'')),$$

was zusammen mit (44)

$$\pi^{2r+1}_{2m-1}(a') \cdot - \cdot \cdot - \cdot \cdot - \cdot \cdot - \cdot \cdot - \cdot \pi^{2r+1}_{2m-1}(a'')$$

ergibt, was im Widerspruch mit der Definition der Zahl m steht.

15. Analog zu der bei der Formulierung des Hilfssatzes II gemachten Definition der Funktion $\varphi(x)$ läßt sich auch die Definition einer Funktion $\psi(y)$ machen, die jedem Punkte

$$y = (T_2,\ T_4,\ \cdots,\ T_{2r},\ \cdots) \tag{45}$$

von B eindeutig einen Punkt

$$\psi(y) = (T_1,\ T_3,\ \cdots,\ T_{2r+1},\ \cdots) \tag{46}$$

von A entsprechen läßt, und zwar ist $\psi(y)$ der Limes der stets gegen einen Punkt konvergierenden Mengenfolge $A(T_{2r})$, wobei definitionsgemäß $A(T_{2r})$ aus allen denjenigen Eckpunkten von A_{2r-1} und A_{2r+1} besteht, die bei der Abbildung $f_{2r-1}(B_{2r})$ Bildpunkte bzw. bei der Abbildung $f_{2r}(A_{2r+1})$ Originalpunkte irgendeines Eckpunktes des Simplexes T_{2r} sind.

Dabei gelten (mit auf der Hand liegenden Modifikationen) die den Hilfssätzen I—III analogen Sätze. Insbesondere gilt

(38 bis) $$\pi_{2m-1}^{2r-1} f_{2r-1}(T_{2r}) + \pi_{2m-1}^{2r+1} f_{2r+1}^{-1}(T_{2r}) \subset T_{2m-1},$$

woraus (genau wie im § 14) folgt, daß zwei verschiedenen Punkten y von B auch immer verschiedene Punkte $\psi(y)$ von A entsprechen; dagegen haben wir noch nicht bewiesen, daß es zu jedem Punkt y von B einen Punkt x von A gibt, für welchen $\varphi(x) = y$ ist, und können deshalb auch nicht schließen, daß es für jedes x aus A einen Punkt y von B gibt, für den $\psi(y) = x$. Diese beiden Behauptungen folgen aus dem

Hilfssatz IV. Wenn x bzw. y ein beliebiger Punkt von A bzw. B ist, so ist

(47) $$\psi\varphi(x) = x \quad \text{und} \quad \varphi\psi(y) = y.$$

Die beiden Gleichungen (47) lassen sich auf dieselbe Weise beweisen, es genügt also, den Beweis für eine, z. B. für $\varphi\psi(y) = y$, zu führen.

16. Es seien

(48) $$y = (T_2, T_4, \cdots, T_{2r}, \cdots),$$

(49) $$\psi(y) = (T_1, T_3, \cdots, T_{2r+1}, \cdots);$$

es sei ferner $a_{i_1 \cdots i_{2r+1}}$ ein beliebiger Eckpunkt von T_{2r+1} und $b_{j_2 \cdots j_{2r}} = f_{2r}(a_{i_1 \cdots i_{2r+1}})$. Dann ist $\lim\limits_{r\to\infty} b_{j_2 \cdots j_{2r}} = \varphi\psi(y)$; um zu beweisen, daß letzterer Punkt mit y identisch ist, genügt es, zu zeigen, daß für ein hinreichend großes r die Entfernung zwischen $b_{j_2 \cdots j_{2r}}$ und der Menge aller Eckpunkte von T_{2r} unter jede Grenze sinkt.

Es sei $\varepsilon > 0$ beliebig gegeben, m so groß, daß $5\beta_{2m-2} < \varepsilon$ ist, r so groß, daß

(50) $$\pi_{2m-1}^{2r-1} f_{2r-1}(T_{2r}) \subset T_{2m-1}$$

ist; es sei ferner (für jedes s) $b_{h_2 \cdots h_{2s}}$ ein Eckpunkt von T_{2s}. Dann folgt aus (50), daß

(51) $$a_{i_1 \cdots i_{2m-1}} \cdot - \cdot \pi_{2m-1}^{2r-1} f_{2r-1}'(b_{h_2 \cdots h_{2r}}).$$

Andererseits ist

$$\pi_{2m-2}^{2r}(b_{j_2 \cdots j_{2r}}) = \pi_{2m-2}^{2r} f_{2r}(a_{i_1 \cdots i_{2r+1}}) \cdot - \cdot f_{2m-2} \pi_{2m-1}^{2r+1}(a_{i_1 \cdots i_{2r+1}}),$$

also

(52) $$b_{j_2 \cdots j_{2m-2}} \cdot - \cdot f_{2m-2}(a_{i_1 \cdots i_{2m-1}}).$$

Aus (51) folgt ferner

(53) $$f_{2m-2}(a_{i_1 \cdots i_{2m-1}}) \cdot - \cdot f_{2m-2} \pi_{2m-1}^{2r-1} f_{2r-1}(b_{h_2 \cdots h_{2r}}),$$

und da

$$\pi_{2m-1}^{2r-1} f_{2r-1}(b_{h_2 \cdots h_{2r}}) \cdot - \quad f_{2m-1} \pi_{2m}^{2r}(b_{h_2 \cdots h_{2r}})$$

ist, so ist

$$f_{2m-2}\,\pi_{2m-1}^{2r-1} f_{2r-1}\,(b_{h_2\cdots h_{2r}}) \cdot - \cdot f_{2m-2} f_{2m-1}\,\pi_{2m}^{2r}\,(b_{h_2\cdots h_{2r}})$$
$$= f_{2m-2} f_{2m-1}\,(\pi_{2m}^{2r}\,(b_{h_2\cdots h_{2r}})) \cdot - \cdot \pi_{2m-2}^{2m}\,(\pi_{2m}^{2r}\,(b_{h_2\cdots h_{2r}})) = b_{h_2\cdots h_{2m-2}},$$

was zusammen mit (52) und (53)

$$(54) \qquad b_{j_2\cdots j_{2m-2}} \cdot - \cdot \cdot - \cdot \cdot - \cdot \cdot - \cdot b_{h_2\cdots h_{2m-2}}$$

ergibt; da aber $b_{j_2\cdots j_{2r}} \subset F_{j_2\cdots j_{2m-2}}$ und $b_{h_2\cdots h_{2r}} \subset F_{h_2\cdots h_{2m-2}}$ ist, so folgt aus (54) ohne weiteres, daß $\varrho(b_{j_2\cdots j_{2r}}, b_{h_2\cdots h_{2r}}) < 5\beta_{2m-2} < \varepsilon$ ist.

17. Der Hilfssatz IV ist hiermit bewiesen. Sein Ergebnis ist aber nichts anderes, als *daß die Abbildung $\varphi(A)$ eine eineindeutige Abbildung des ganzen Raumes A auf den ganzen Raum B und daß ψ die Umkehrung* dieser eineindeutigen Abbildung ist. Man kann also schreiben

$$(55) \qquad B = \varphi(A).$$

Um jetzt den Beweis des Homöomorphiesatzes zu Ende zu bringen, genügt es zu zeigen, daß die Abbildung (55) auch stetig ist: da die beiden Räume A und B kompakt sind und (55) eineindeutig ist, wird daraus folgen, daß auch die Umkehrung von (55) stetig ist, d. h., daß die Räume A und B homöomorph sind.

Es sei $\varepsilon > 0$ beliebig, m so groß, daß $4\beta_{2m} < \varepsilon$ ist und $\delta < d(\mathfrak{A}_{2m+1})$; es seien sodann

$$x' = (T_1', T_3', \cdots, T_{2r+1}', \cdots)$$

und

$$x'' = (T_1''; T_3'', \cdots, T_{2r+1}'', \cdots)$$

zwei beliebige Punkte von A, deren Entfernung kleiner als δ ist; es soll gezeigt werden (und damit wird die Stetigkeit von φ und folglich der ganze Satz bewiesen), daß $\varrho(\varphi(x'), \varphi(x'')) \leqq \varepsilon$ ist.

Wir bezeichnen wie immer durch $a_{i_1'\cdots i_{2r+1}'}$ irgendeinen (fest gewählten) Eckpunkt von T_{2r+1}', durch $a_{i_1''\cdots i_{2r+1}''}$ irgendeinen Eckpunkt von T_{2r+1}'' und setzen $b_{j_2'\cdots j_{2r}'} = f_{2r}(a_{i_1'\cdots i_{2r+1}'})$ und $b_{j_2''\cdots j_{2r}''} = f_{2r}(a_{i_1''\cdots i_{2r+1}''})$; dann ist $\varphi(x') = \lim\limits_{r\to\infty} b_{j_2'\cdots j_{2r}'}$, $\varphi(x'') = \lim\limits_{r\to\infty} b_{j_2''\cdots j_{2r}''}$, und es genügt zu zeigen, daß für alle $r > m$

$$\varrho(b_{j_2'\cdots j_{2r}'}, b_{j_2''\cdots j_{2r}''}) < \varepsilon$$

ist.

Dies ergibt sich wie folgt. Zunächst ist:

$$(56) \quad \begin{cases} b_{j_2'\cdots j_{2m}'} = \pi_{2m}^{2r}(b_{j_2'\cdots j_{2r}'}) = \pi_{2m}^{2r} f_{2r}(a_{i_1'\cdots i_{2r+1}'}) \cdot - \cdot f_{2m}(a_{i_1'\cdots i_{2m+1}'}), \\ b_{j_2''\cdots j_{2m}''} = \pi_{2m}^{2r}(b_{j_2''\cdots j_{2r}''}) = \pi_{2m}^{2r} f_{2r}(a_{i_1''\cdots i_{2r+1}''}) \cdot - \cdot f_{2m}(a_{i_1''\cdots i_{2m+1}''}). \end{cases}$$

Nun ist aber die Entfernung $\varrho(x', x'')$ so gewählt, daß

$$a_{i_1' \cdots i_{2m+1}'} \cdot — \cdot a_{i_1'' \cdots i_{2m+1}''},$$

also auch

$$f_{2m}(a_{i_1' \cdots i_{2m+1}'}) \cdot — \cdot f_{2m}(a_{i_1'' \cdots i_{2m+1}''}),$$

was zusammen mit (56)

$$b_{j_2' \cdots j_{2m}'} \cdot — \cdot \cdot — \cdot \cdot — \cdot b_{j_2'' \cdots j_{2m}''}$$

ergibt; aus dieser Relation folgt aber (da $b_{j_2' \cdots j_{2r}'} \subset F_{j_2' \cdots j_{2m}'}$ und $b_{j_2'' \cdots j_{2r}''} \subset F_{j_2'' \cdots j_{2m}''}$ ist), daß $\varrho(b_{j_2' \cdots j_{2r}'}, b_{j_2'' \cdots j_{2r}'}) < 4\beta_{2m} < \varepsilon$ ist, w. z. b. w.

18. Zum Schluß möchte ich noch einmal einen neuen Beweis des Urysohnschen Satzes über die Zerlegbarkeit jeder λ-dimensionalen abgeschlossenen Menge in $\lambda+1$ nulldimensionale Mengen geben, der sich an die Betrachtungen dieses Kapitel anschließt und die diesem Satze zugrunde liegenden Tatsachen, wie mir scheint, besonders klar hervorhebt.

Wenn die λ-dimensionale Menge F durch ein Projektionsspektrum

$$K_1, K_2, \cdots, K_m, \cdots \tag{57}$$

definiert ist, so entspricht jedem Punkt

$$x = (T_1, T_2, \cdots, T_m, \cdots) \tag{58}$$

von F eine bestimmte nicht negative ganze Zahl $k(x)$ — die Höchstdimension der in der Kette (58) auftretenden Simplexe. Man bezeichne jetzt durch P_n die Menge aller Punkte x von F, für die $k(x) = n$ ist. Da offenbar

$$F = \sum_{n=0}^{\lambda'} P_n$$

ist, wobei λ' die Dimension des Spektrums bedeutet, und das Spektrum stets so gewählt werden kann, daß $\lambda' = \lambda$ ist, so genügt es zu zeigen, daß *jede der Mengen P_n nulldimensional ist.*

Die Dimension der in (58) vorkommenden Simplexe T_m kann bei zunehmendem m nur wachsen, also bleibt sie von einem bestimmtem m an konstant. Jede der Mengen P_n kann also als Vereinigungsmenge von höchstens abzählbar vielen Mengen P_n^k dargestellt werden, wobei P_n^k die Menge aller Punkte (58) ist, die für alle $h \geqq k$ der Bedingung

$$\dim T_h = n$$

genügen. Da die Summe von abzählbar vielen nulldimensionalen Mengen, die in ihrer Vereinigungsmenge abgeschlossen sind, selbst nulldimensional

ist,[36] genügt es zu zeigen, daß stets P_n^k erstens nulldimensional und zweitens abgeschlossen (rel P_n) ist. Um die erste Behauptung zu beweisen, genügt es zu zeigen, daß jede Relativumgebung $P_n^k \cdot U_{k+h}(x)$, $x \subset P_n^k$, eine verschwindende Begrenzung (rel P_n^k) hat, d. h. in P_n^k abgeschlossen ist, oder, mit anderen Worten, daß jeder Punkt

$$\xi = (\tau_1, \tau_2, \cdots, \tau_m, \cdots)$$

von P_n^k, welcher Häufungspunkt von $P_n^k \cdot U_{k+h}(x)$ ist, selbst zu $U_{k+h}(x)$ gehört. Nun folgt aber aus $\xi \subset \overline{P_n^k \cdot U_{k+h}(x)}$, daß für jedes i, $k \leqq i \leqq k+h$, $\tau_i \supset T_i$ ist; da aber $\xi \subset P_n^k$ ist, so sind die Simplexe τ_k, τ_{k+1}, $\cdots$, τ_{k+h} n-dimensional, also der Reihe nach mit T_k, T_{k+1}, $\cdots$, T_{k+h} identisch; somit gilt $\tau_i = T_i$ auch für $i < k$, woraus $\xi \subset U_{k+h}(x)$ folgt.

Analog beweist man auch die zweite Behauptung: aus $\xi \subset \bar{P}_n^k$ folgt nämlich, daß $U_k(\xi)$ Punkte x von P_n^k enthält, also $\tau_k \supset T_k(x)$ ist; da aber $\xi \subset P_n$ ist, so ist $\dim \tau_k \leqq n$, also $\tau_k = T_k(x)$ und $\dim \tau_k = n$; mithin ist $\xi \subset P_n^k$, w. z. b. w.

Aus den obigen Überlegungen folgt ohne weiteres: *Wenn ein Projektionsspektrum endlicher Dimension eine λ-dimensionale Menge F definiert, so nimmt die in bezug auf dieses Spektrum erklärte Funktion $k(x)$ mindestens $\lambda+1$ verschiedene Werte an.*

Drittes Kapitel.

Der Zerlegungssatz.

1. Definition. *Es sei J eine irgendwie definierte topologische Invariante, die für einen beliebigen Komplex einen Sinn hat, und zwar als eine nicht negative ganze Zahl definiert ist. Wenn F eine abgeschlossene Menge ist, so soll durch $J(F)$ die kleinste Zahl k bezeichnet werden, die die Eigenschaft hat, daß es für jedes ε eine ε-Überdeckung von F gibt, deren Nerv N der Bedingung $J(N) = k$ genügt. Wenn für kein endliches k diese Bedingung erfüllt ist, soll definitionsgemäß $J(F) = \infty$ gesetzt werden.*

Die am nächsten liegende Invariante J ist natürlich die Dimension eines Komplexes (im elementaren Sinne). Aus den Resultaten des ersten Kapitels

[36] Der Satz, daß die Summe von abzählbar vielen *nulldimensionalen* Teilmengen eines gegebenen separablen metrischen Raumes nulldimensional ist, sobald diese Teilmengen (in bezug auf den Raum) abgeschlossen sind, ist von Hurewicz und Tumarkin (Math. Ann., 96 (1926), S. 745, bzw. Proceed. Kon. Akademie Amsterdam, 28 (1925), S. 995, und Math. Ann., 98 (1928), S. 641) sehr einfach bewiesen worden; dieser Satz kann (im Unterschied zu seinem von denselben Autoren a. a. O. bewiesenen n-dimensionalen Analogon) als ein *elementarer* dimensionstheoretischer Satz gelten.

folgt sodann, daß die durch Anwendung der obigen Definition gewonnene Übertragung dieser Invariante auf beliebige abgeschlossene Mengen nichts anderes als die Brouwer-Urysohn-Mengersche Dimension liefert.

2. In der soeben aufgestellten Definition sind *alle* ε-Überdeckungen zur Konkurrenz zugelassen; diese Definition würde automatische Änderungen erleiden, wenn man für jedes ε nicht alle, sondern nur gewissen Nebenbedingungen genügende ε-Überdeckungen zuließe.

Die natürlichste Einschränkung dieser Art besteht darin, daß man für λ-dimensionale Mengen nur Überdeckungen von der Ordnung $\lambda+1$ (also Überdeckungen mit einem λ-dimensionalen Nerv) betrachtet; die dahin modifizierte Definition des vorigen Paragraphen ergibt dann für F einen eventuell anderen Wert der dort betrachteten Invariante, den wir mit $J'(F)$ bezeichnen.

Eine weitere Einschränkung der konkurrenzfähigen ε-Überdeckungen erhält man, wenn man nur irreduzible ε-Überdeckungen betrachtet; die entsprechende Invariante der Menge F soll $J''(F)$ heißen und es ist offenbar *allgemein*:

$$J''(F) \geqq J'(F) \geqq J(F), \tag{1}$$

wobei für unendlich-dimensionale F trivialerweise

$$J'(F) = J(F). \tag{1∞}$$

ausfällt.

Außerdem folgt aus dem Satz I des Kap. II, daß für jede F ein Projektionsspektrum (von derselben Dimension wie F selbst) existiert, so daß für sämtliche Komplexe K dieses Spektrums

$$J(K) \leqq J''(F)$$

ist.

3. Man könnte sich aber von vornherein auf einen modifizierten Standpunkt stellen, indem man in der Definition des § 1 Komplexe betrachtet, in die F durch k. T. (oder, was praktisch dasselbe ist, k. A.) übergeht. Die auf diese Weise erhaltene Invariante der Menge F soll etwa mit $\overline{J}(F)$ bezeichnet werden; für λ-dimensionale F liefert die durch die Bedingung $\dim K = \lambda$ gegebene Einschränkung der zur Konkurrenz zugelassenen Komplexe (wie im § 2) die Invariante $\overline{J}'(F)$ mit

$$\overline{J}'(F) \geqq \overline{J}(F); \tag{2}$$

aus dem Satz III des Kap. I folgt ferner, daß

$$\overline{J}(F) \geqq J(F) \text{ und } \overline{J}'(F) \geqq J'(F), \tag{3}$$

während aus dem Satz IV folgt, daß

$$J''(F) \geq J'(F) \tag{4}$$

ist. Endlich folgt aus Satz IV′, Kap. I, daß, *wenn F zusammenhängend ist,*

$$\bar{J}(F) = J(F) \text{ und } \bar{J}'(F) = J'(F) \tag{5}$$

ist. Aus (1)–(4) folgt überdies, das $J(F)$ die kleinste und $J''(F)$ die größte unter allen Zahlen $J(F)$, $J'(F)$, $J''(F)$, $\bar{J}(F)$, $\bar{J}'(F)$ ist.

4. Es wäre vielleicht interessant, die Beziehungen zwischen allen diesen Übertragungsweisen einer topologischen Invariante J auf abgeschlossene Mengen genau zu kennen, höchst wahrscheinlich werden alle diese verschiedenen Weisen, wenigstens für wichtigere Invarianten (wie z. B. die Bettischen Zahlen) dasselbe Resultat liefern.

5. Nach diesen Vorbemerkungen wenden wir uns zur Untersuchung der Eigenschaften einer Menge F, von welchen die Anzahl der durch F im R^n bestimmten Gebiete abhängt.

Eine höchstens $n-2$-dimensionale Menge kann bekanntlich den R^n überhaupt nicht zerlegen, so daß für unsere Fragestellung von vornherein nur n- und $n-1$-dimensionale Mengen in Betracht kommen. Die wirklich interessanten sind aber dabei die $n-1$-dimensionalen Mengen: in der Tat, wenn $F \subset R^n$ und dim $F = n$ ist, so ist der Rand Φ von F in bezug auf R^n $n-1$-dimensional; die Menge F ensteht aus Φ einfach dadurch, daß man die Vereinigungsmenge von Φ und einigen durch letztere Menge im R^n bestimmten Gebieten nimmt. Wir wenden uns also zunächst den $n-1$-dimensionalen Mengen zu. Indem wir mit

$$p_r(F),\ p_r'(F),\ p_r''(F),\ \bar{p}_r(F),\ \bar{p}_r'(F)$$

die gemäß den verschiedenen Erklärungsweisen der §§ 1–3 festgestellten Werte der r-ten Bettischen Zahl[37] von F bezeichnen, können wir folgenden Satz aussprechen:

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[37] Alle der kombinatorischen Topologie entnommenen Begriffe sind im Sinne der Definitionen "modulo 2" verstanden (vgl. Fußnote [1]). Berandungsrelationen werden mit einem Pfeil angegeben (wie bei Alexander, CAS); *homogene* n-dimensionale Komplexe (d.h. solche, deren Grundsimplexe *sämtlich* n-dimensional sind) werden mit K^n, Q^n u. dgl. bezeichnet; n-dimensionale geschlossene Komplexe (Zyklen) werden mit Γ^n, γ^n bezeichnet; $K^{n+1} \to \gamma^n$ heißt, daß γ^n der Rand von K^{n+1} (mod. 2) ist.

Die Bettischen Zahlen werden mit p bezeichnet, wobei der Dimensionsindex oben oder unten angebracht ist.

R^n bedeutet den durch Hinzufügung des unendlich-fernen Punktes ergänzten n-dimensionalen Euklidischen Raum.

SATZ I. *Es sei F eine $n-1$-dimensionale beschränkte abgeschlossene Teilmenge des R^n. Dann ist:*

(p) $$p_{n-1}(F) = p'_{n-1}(F) = p''_{n-1}(F) = \bar{p}_{n-1}(F) = \bar{p}'_{n-1}(F).$$

Diese Zahl $p = p(F)$ ist um 1 kleiner als die Komponentenzahl von $R^n - F$.

6. Die Beweisanordnung ist die folgende. Man beginnt mit dem

SATZ II. *Es seien $F \subset R^n$ und $r < n$ beliebig gegeben. Dann ist*

(6) $$p_{n-r-1}(R^n - F) \leqq p_r(F),$$

wobei $p_{n-r-1}(R^n - F)$ die Bettische Zahl von $R^n - F$ bedeutet.[38]

Nachdem das bewiesen ist, beweist man für eine *beliebige* $F \subset R^n$ die Ungleichung

(7) $$p_0(R^n - F) \geqq \bar{p}'_{n-1}(F).$$

Die Ungleichungen (6), (7) und (1), (2), (3) ergeben dann für *jedes* $F \subset R^n$

(8) $$p_0(R^n - F) = p_{n-1}(F) = p'_{n-1}(F) = \bar{p}_{n-1}(F) = \bar{p}'_{n-1}(F),$$

womit insbesondere die beiden im § 15 *der Einleitung ausgesprochenen Formen des Zerlegungssatzes bewiesen sind.*

Im Falle $\dim F = n-1$ folgt aus (8) vermöge (1) die Identität **(p)** und somit der ganze Satz I.

Auf Grund der Schlußbemerkung des § 2 folgt überdies aus dem Satz I, daß man F ($\dim F = n-1$) mittels eines $n-1$-dimensionalen Projektionsspektrums definieren kann, für dessen sämtliche Komplexe K_m die *höchste* (also in unserem Falle $n-1$-te) Bettische Zahl entweder endlich, und zwar gleich p ist, oder mit m über alle Grenzen wächst. Wenn wir im allgemeinen ein Projektionsspektrum im ersten Fall *endlich, und zwar $p+1$-fach zusammenhängend,* im zweiten *unendlich*-hoch-zusammenhängend nennen, können wir den wesentlichen Inhalt der hier ausgesprochenen Zerlegungssätze einfach so formulieren:

Die endliche oder unendliche Anzahl von Gebieten, die eine $n-1$-dimensionale abgeschlossene Menge im R^n bestimmt, ist gleich der kleinsten Zahl, die als Zusammenhangszahl eines diese Menge definierenden $n-1$-dimensionalen Projektionsspektrums auftritt.

7. Beweis des Satzes II. Es sei k eine Zahl, die $= p_{n-r-1}(R^n - F)$, falls letztere Zahl endlich, und beliebig groß, falls $p_{n-r-1}(R^n - F) = \infty$, ist. Es seien

[38] Dieser Satz wird sich auch aus den Resultaten des nächsten Kapitels (Formel (1) und Definition der Zahl $B^r(F)$ im Kap. VI) ergeben.

$$\gamma_1^{n-r-1},\ \gamma_2^{n-r-1},\ \cdots,\ \gamma_k^{n-r-1} \tag{9}$$

in $R^n - F$ liegende und daselbst unabhängige Zyklen. Aus einem Satze von Pontrjagin[39] folgt, daß, wenn man F stetig in einer Menge F' deformiert, ohne daß während dieser Deformation die Komplexe (9) von der sich deformierenden Menge berührt werden, die γ_i^{n-r-1} auch in $R^n - F'$ unabhängig bleiben. Da die γ_i^{n-r-1} von F eine positive Minimalentfernung haben, kann man stets annehmen, daß die k. T. durch solche Deformationen erzeugt werden, bei denen die Menge immer zu den (9) fremd bleibt, so daß diese auch außerhalb der Komplexe, in die F durch k. T. übergeht, unabhängig sind. Dann ist aber (vermöge des Alexanderschen Dualitätssatzes) die r-te Bettische Zahl der letztgenannten Komplexe gewiß $\geqq k$. Damit ist bewiesen, daß $\bar{p}_r(F) > p_{n-r-1}(R^n - F)$ ist, woraus der Satz II für zusammenhängende Mengen F folgt.

Der allgemeine Fall ist ein wenig komplizierter, weil man folgenden Hilfssatz braucht:

Es sei $F \subset R^m \subset R^{m+h}$ *und die Zyklen*

$$\gamma_1^s,\ \gamma_2^s,\ \cdots,\ \gamma_k^s \tag{9'}$$

unabhängig in $R^m - F$; *dann existieren im* $R^{m+h} - F$ *unabhängige Zyklen*

$$\gamma_1^{s+h},\ \gamma_2^{s+h},\ \cdots,\ \gamma_k^{s+h}, \tag{10}$$

die der Bedingung $\gamma_i^{s+h} \cdot R^m = \gamma^s$ *genügen.*

Beweis: Der Hilfssatz ist trivial für $h = 0$; wir nehmen ihn für alle $h' < h$ als bewiesen an und zeigen seine Richtigkeit für h. Es werde nun der R^{m+h-1} (für den der Satz schon bewiesen ist) in einen R^{m+h} eingebettet, jedes γ_i^{s+h-1} in eine zu F fremde Umgebung U_i (in bezug auf R^{m+h}) eingeschlossen, und hierin jeweils zwei Punkte p_i und q_i ausgewählt, die durch R^{m+h-1} getrennt werden. Verbindet man p_i bzw. q_i mit allen Punkten von γ_i^{s+h-1} mit geradlinigen Strecken, so entstehen zwei Komplexe $K_{p_i}^{s+h}$ und $K_{q_i}^{s+h}$ (beide in $R^{m+h} - F$), deren Summe ein Zyklus $\gamma_i^{s+h} \subset R^{m+h} - F$. Offenbar ist

[39] Göttinger Nachrichten, 25. Nov. 1927. Man könnte übrigens den Beweis auch ohne Zuhilfenahme des Pontrjaginschen Satzes führen, indem man den R^n in einen $2n+3$-dimensionalen Raum einbettet: dann kann man voraussetzen, daß der durch die Deformation einer Umgebung U von F (in bezug auf R^n) erzeugte $n+1$-dimensionale Komplex singularitätenfrei im R^{2n+3} liegt, woraus folgt, daß die Bettischen Zahlen dieser Umgebung sich während der Deformation nicht ändern, dies erlaubt ohne weiteres auf die Richtigkeit der Ungleichung (6) zu schließen.

$$\gamma_i^{s+h} \cdot R^m = (\gamma_i^{s+h} \cdot R^{m+h-1}) \cdot R^m = \gamma_i^{s+h-1} \cdot R^m = \gamma_i^s.$$

Überdies sind die γ_i^{s+h} in $R^m - F$ unabhängig, weil aus

$$Q^{s+h+1} \to \lambda^i \gamma_i^{s+h} \quad (\text{in } R^{m+h} - F)$$

(in der Annahme, daß Q^{s+h+1} sich zu R_m in einer allgemeinen Lage befindet und infolgedessen $Q^{s+h+1} \cdot R^m = Q^{s+1}$ ist) die Relation

$$Q^{s+1} \to \lambda^i \gamma_i^s \quad (\text{in } R^m)$$

gegen die Voraussetzung der linearen Unabhängigkeit der γ_i^s in $R^m - F$ folgen würde.

8. Wir kehren jetzt zum Beweise der Ungleichung $p_r(F) \geqq k$ zurück. Es sei $\dim F = h$. Bettet man R^n in einen R^{n+2h+1} ein, so gibt es nach dem soeben bewiesenen Hilfssatze in $R^{n+2h+1} - F$ linear-unabhängige Zyklen

$$\gamma_1^{n-r-1+2h+1},\ \gamma_2^{n-r-1+2h+1},\ \ldots,\ \gamma_k^{n-r-1+2h+1}. \tag{11}$$

Es sei ε hinreichend klein und $\mathfrak{P}$ eine $(\varepsilon, h+1)$-Überdeckung von F; vermöge des Satzes IV″ (Kap. I) kann man eine zu den (11) fremde Umgebung U von F, ohne die Komplexe (11) zu berühren, in den geometrisch realisierten Nerv N von $\mathfrak{P}$ stetig überführen; da N ein h-dimensionaler Komplex ist, kann man reichlich annehmen, daß N in R^{n+2h+1} ohne Singularitäten liegt; aus dem erwähnten Pontrjaginschen Satze folgt nunmehr, daß die Zyklen (11) auch in $R^{n+2h+1} - F$ unabhängig sind, was vermöge des Alexanderschen Dualitätssatzes $p^r(N) \geqq k$ ergibt, w. z. b. w.

9. Beweis der Ungleichung (7). Falls die Komponentenanzahl von $R^n - F$ (und also $p_0(R^n - F)$) unendlich ist, so ist die Ungleichung (7) gewiß richtig. Man kann sich also auf den Fall beschränken, in dem $R^n - F$ aus endlich-vielen — wir nehmen an, aus $k+1$ — Komponenten besteht, und also $p_0(R^n - F) = k$ ist. Es soll gezeigt werden, daß F durch k. T. in Komplexe K übergeht, für die $p_{n-1}(K) \leqq k$ ist.

Es sei $\varepsilon > 0$ beliebig gegeben. Man betrachte eine Simplizialzerlegung $\mathfrak{z}'$ des R^n, deren Elemente sämtlich kleiner als ε sind. Die Gesamtheit aller zu F nicht fremden Simplexe dieser Zerlegung bildet einen Komplex Q'. Es seien

$$G_0,\ G_1,\ G_2,\ \ldots,\ G_k \tag{12}$$

sämtliche Komponenten von $R^n - F$,

$$H_0',\ H_1',\ H_2',\ \ldots,\ H_{q'}' \tag{13}$$

sämtliche Komponenten von $R^n - Q'$; für jedes H_j' gibt es ein eindeutig bestimmtes G_i, in welchem es enthalten ist, mehrere H_j' können aber in demselben G_i enthalten sein.

Die Grundsimplexe von Q' seien mit T' bezeichnet; falls das gegebene T' nicht in F enthalten ist, wähle man im Innern von T' ein zu F fremdes Simplex t' und entferne aus T' das Innere von t'; indem man das für jedes T' tut, erhält man einen Komplex Q'', wobei offenbar

$$F \subset Q'' \subset Q'$$

ist. Man kann voraussetzen, daß Q'' aus Simplexen einer Unterteilung $\mathfrak{z}''$ von $\mathfrak{z}'$ besteht. Jede Komponente H_j'' von $R^n - Q''$ ist in einer bestimmten Komponente G_i von $R^n - F$ enthalten; es kann aber vorkommen, daß zwei Komponenten, z. B. H_j'' und H_h'' von $R^n - Q''$ in einem und demselben Gebiete G_i enthalten sind. Dann verbinde man H_j'' und H_h'' innerhalb von G_i mit einem Streckenzuge L_{jh} und entferne aus Q'' eine aus Simplexen einer genügend feinen Unterteilung von $\mathfrak{z}''$ gebildete, einem gewöhnlichen n-dimensionalen Zylinderkörper homöomorphe Umgebung von L_{jh}. Wenn man dies für alle in Frage kommenden Komponentenpaare H_j'', H_h'' macht, erhält man einen Komplex Q und es ist

$$F \subset Q \subset Q'' \subset Q'. \tag{14}$$

Man kann voraussetzen, daß Q aus Simplexen T einer Unterteilung $\mathfrak{z}$ von $\mathfrak{z}''$ (die also erst recht eine Unterteilung von $\mathfrak{z}'$ ist) gebildet ist.

Es ist leicht einzusehen, daß keine zwei Komponenten H_j und H_h von $R^n - Q$ in einer Komponente G_i von $R^n - F$ enthalten sein können, so daß die Anzahl $q+1$ dieser Komponenten höchstens gleich $k+1$ ist. Somit gilt aber (vermöge des Alexanderschen Dualitätssatzes):

$$p_{n-1}(Q) = p_0(R^n - Q) \leqq k. \tag{15}$$

10. Man betrachte jetzt ein beliebiges T'. Wenn $T' \subset F$ ist, so ist um so mehr $T' \subset Q$; wenn aber T' nicht in F enthalten ist, so besteht der zu T' gehörende Teil von Q aus einer endlichen Gesamtheit S von Simplexen von $\mathfrak{z}$, *die zu t' fremd sind*; deshalb gibt es in S notwendig Simplexe T mit zum Rande von Q gehörenden (wir sagen kurz: *freien*) $n-1$-dimensionalen Seiten. Solche Simplexe T wollen wir *Grenzsimplexe* nennen. Man definiere jetzt eine stetige Deformation eines Grenzsimplexes, indem man gleichförmig und geradlinig den Schwerpunkt seiner freien Seite in den gegenüberliegenden Eckpunkt bewegt, wodurch das ganze T in die Gesamtheit seiner von der gewählten freien Seite verschiedenen $n-1$-dimensionalen

Seiten übergeht. Dieser Prozess wird zunächst für jedes S so lange fortgesetzt, bis keine (n-dimensionale) Grenzsimplexe mehr vorkommen. Wenn man dies für jedes S macht, entsteht eine offenbar stetige Deformation Δ des Komplexes Q in sich, deren Endresultat ein aus allen in F enthaltenen n-dimensionalen Simplexen T' (wenn es solche gibt) und aus gewissen $n-1$-dimensionalen Simplexen zusammengesetzter Komplex K ist. Diese Deformation ist eine ε-Deformation, weil sie jeden Punkt von Q in *dem* T', zu dem er ursprünglich gehörte, bleiben läßt; *überdies hat unsere Deformation die Eigenschaft, sämtliche Punkte von K festzuhalten.* Es ist nun leicht zu zeigen, daß während dieser Deformation die Zahl $p_{n-1}(Q)$ unverändert bleibt. Es sei in der Tat

$$\Gamma_1^{n-1}, \Gamma_2^{n-1}, \cdots, \Gamma_q^{n-1} \tag{16}$$

die Homologiebasis von Q. Wenn $\gamma_i^{n-1} \subset K$ das Bild von Γ_i^{n-1} bedeutet, so ist $\gamma_i^{n-1} \sim \Gamma_i^{n-1}$ in Q, woraus folgt, daß die γ_i^{n-1} in Q, also erst recht in K unabhängig sind; es sei andererseits γ^{n-1} irgendein Zyklus in K; als Zyklus in Q betrachtet ist $\gamma^{n-1} \sim \lambda^i \Gamma_i^{n-1}$; somit existiert ein $Q^n \subset Q$, so daß

$$Q^n \to \gamma^{n-1} + \lambda^i \Gamma_i^{n-1} \text{ (in } Q)$$

und folglich

$$\overset{*}{Q}{}^n \to \gamma^{n-1} + \lambda^i \gamma_i^{n-1}, \text{ also } \gamma^{n-1} \sim \lambda^i \gamma_i^{n-1} \text{ (in } K)$$

ist. Die γ_i^{n-1} bilden also eine Homologiebasis von K, so daß

$$p_{n-1}(K) = p_{n-1}(Q) \leqq k \tag{17}$$

ist.

11. Durch die Deformation Δ entsteht eine stetige ε-Abbildung von F auf eine echte oder unechte Teilmenge F' von K, *wobei F' jedenfalls alle n-dimensionalen Simplexe von K enthält.* Durch das Ausfegeverfahren von Kap. I, § 2 geht (wenn nicht von vornherein $F' = K$ ist) F' in einen Teilkomplex K' von K über, wobei K' sich von K nur dadurch unterscheidet, daß einige höchstens $n-1$-dimensionale Simplexe von K in K' nicht auftreten; daraus folgt aber, daß die Elemente der $n-1$-dimensionalen Basis von K', insofern sie durch den Übergang von K zu K' nicht zerstört worden sind, gewiß genügen, um eine Basis von K' zu bilden; somit ist $p_{n-1}(K') \leqq p_{n-1}(K) \leqq k$, und da der ganze Übergang von F zu K' eine ε-Transformation und ε beliebig war, ist die Formel (7) und folglich der Zerlegungssatz in allen seinen Teilen bewiesen.

12. Ich will jetzt einige Folgerungen aus den soeben bewiesenen Resultaten, namentlich aus dem Satz II, ziehen.

SATZ III. *Es sei K ein Komplex. Wenn ε genügend klein ist, gilt für den Nerv $N = N(P)$ einer beliebigen ε-Überdeckung P von K die Ungleichung* $p^r(N) \geqq p_r(K)$.

Wenn man K in einen (hinreichend hoch-dimensionalen) Raum R singularitätenfrei einbettet, so folgt aus dem Satz II und dem Dualitätssatz von Alexander:

$$p_r(N) > p_{n-r-1}(R^n - K) = p_r(K), \quad \text{w. z. b. w.}$$

Aus Kap. I, § 10, Hilfssatz II, folgt sodann, daß, *wenn man einen Komplex K als eine Punktmenge betrachtet und für dieselbe die Bettischen Zahlen $p_r, p'_r, \bar{p}_r, \bar{p}'_r$ vermöge der Definitionen der §§ 1–3 erklärt, so sind diese Zahlen der r-ten Bettischen Zahl des Komplexes K (im klassischen Sinne) gleich.*

13. Ein Spezialfall des Satzes II ist die Behauptung:

Wenn $F \subset R^n$ und $\dim F \leqq h$ ist, so ist jeder in $R^n - F$ gelegene höchstens $n-h-2$-dimensionale Zyklus daselbst homolog Null.

Diese Tatsache veranlaßt uns zu folgender

DEFINITION. *Es sei $F \subset R$ und $\gamma^r \subset R^n - F$; der Zyklus γ^r heißt irreduzibel verschlungen mit der Menge F, wenn er nicht homolog Null in $R^n - F$ ist und andererseits für jede abgeschlossene echte Teilmenge F' von F*

$$\gamma^r \sim 0 \quad (\text{in } R^n - F')$$

ist.

Es gilt nun folgender

SATZ IV. *Wenn $F \subset R$ und γ^r mit F irreduzibel verschlungen ist, so gilt für jede abgeschlossene Teilmenge F' von F, welche die letztere Menge zerlegt, die Ungleichung* $p_{n-r-2}(F') > 0$.

Es sei in der Tat

$$F = G + F' + G', \qquad H(G, G') = 0; \tag{18}$$[40]

dann ist

$$\bar{G} + F' \neq F \neq \bar{G}' + F', \quad \bar{G} \cdot \bar{G}' \subset F', \quad (\bar{G} + F') \cdot (\bar{G}' + F') = F'. \tag{19}$$

Es sei jetzt $p_{n-r-2}(F') = 0$; wenn $\gamma^r \subset R^n - F$ beliebig gewählt ist, so folgt aus unseren Voraussetzungen die Existenz zweier Komplexe $K^{r+1} \subset R^n - (\bar{G} + F')$ und $K'^{r+1} \subset R^n - (\bar{G}' + F')$, so daß

$$K^{r+1} \to \gamma^r, \qquad K'^{r+1} \to \gamma^r$$

ist; $K^{r+1} + K'^{r+1}$ ist ein Zyklus $\gamma^{r+1} \subset R - F'$, also ist $\gamma^{r+1} \sim 0$ (in $R^n - F'$), woraus man vermöge des Alexanderschen Additionssatzes[41] auf $\gamma^r \sim 0$ (in $R^n - F$) schließt, w. z. b. w.

[40] $H(M, N)$ bedeutet (im Falle, wo M und N beliebige Punktmengen sind) die Menge $\bar{M} \cdot N + M \cdot \bar{N}$, wobei $\bar{M}$ bzw. $\bar{N}$ die abgeschlossene Hülle von M bzw. N ist.

[41] Alexander, Transactions Amer. Math. Soc., 23 (1922), S. 342 (Corollary W').

Man kann den Satz IV auch in folgender Form aussprechen:

Wenn $\gamma^r \sim 0$ *in* $R^n - F_1$ *und in* $R^n - F_2$ *und* $p_{n-r-2}(F_1 \cdot F_2) = 0$ *ist, so ist* $\gamma^r \sim 0$ *in* $R^n - (F_1 + F_2)$.

14. Der wichtigste Spezialfall dieses Satzes ist der Fall $r = 0$: man beweist nämlich leicht folgende Behauptung:

F ist dann und nur dann die gemeinsame Grenze zweier zusammenhängender Gebiete des R, wenn es ein mit F irreduzibel verschlungenes Punktepaar gibt.[42]

Diese Tatsache ergibt zusammen mit dem Satz IV folgendes wichtiges Resultat:

VERALLGEMEINERTER URYSOHNSCHER SATZ: *Die gemeinsame Grenze zweier (oder mehrerer) zusammenhängender Gebiete des R^n ist eine $n-1$-dimensionale Cantorsche Mannigfaltigkeit, die überdies durch keine einfach-zusammenhängende $n-2$-dimensionale abgeschlossene Teilmenge zerlegt werden kann.*[43]

15. Die Methoden dieses Kapitels erlauben uns leider noch nicht, eine notwendige und hinreichende Bedingung dafür auszusprechen, daß F die gemeinsame Grenze wenigstens zweier zusammenhängender Gebiete des R ist; eine solche Bedingung wird erst im nächsten Kapitel gegeben. Dagegen können wir schon jetzt diejenigen abgeschlossenen Mengen des R^n charakterisieren, die als gemeinsame Grenzen aller und dabei mindestens zweier Komponenten ihres Komplementärraumes (wir sagen kurz: als *absolute Gebietsgrenzen*) auftreten. Es gilt nämlich folgender Satz:

SATZ V. *Unter allen $n-1$-dimensionalen Mengen des R^n sind die absoluten Gebietsgrenzen Φ dadurch ausgezeichnet, daß $p(\Phi) > 0$ ist, während für jede echte Teilmenge Φ' von Φ $p(\Phi') = 0$ ist.*

Dieser Satz folgt unmittelbar aus dem Satz I, wenn man sich an folgende (wohlbekannte) Tatsache erinnert:

Eine Menge F ist dann und nur dann eine absolute Gebietsgrenze, wenn F, jedoch keine echte Teilmenge F' von F den R^n zerlegt.

Der Beweis der letzteren Tatsache verläuft wie folgt:

A. Es sei F eine absolute Gebietsgrenze und F' eine echte Teilmenge von F'; es sei ferner $\bar{x} \subset F - F'$ und $U \subset R^n - F'$ eine sphärische Umgebung von $\bar{x}$; wir wollen zeigen, daß je zwei Punkte x und y von

[42] S. wegen des Beweises Kap. IV, § 14.

[43] Dieser Satz sowie die Sätze des § 13 folgen auch aus dem Satz V des Kap. IV.

Unter einer n-dimensionalen Cantorschen Mannigfaltigkeit versteht man nach Urysohn eine abgeschlossene n-dimensionale Menge F, die durch keine *höchstens $n-2$-dimensionale abgeschlossene Teilmenge* zerlegt werden kann (d. h. wenn $F' \subset F$ und $\dim F' \leq n-2$ ist, so ist $F - F'$ zusammenhängend). Da in jeder Menge M, die einen kompakten metrischen Raum F zerlegt eine, dieselbe Eigenschaft besitzende abgeschlossene Menge F' enthalten ist (siehe z. B. "Darstellung", S. 35), so kann eine n-dimensionale Cantorsche Mannigfaltigkeit überhaupt durch keine Menge von einer Dimension $\leqq n-2$ zerlegt werden.

$R^n - F'$ mittels eines Streckenzuges in $R^n - F'$ miteinander verbunden werden können. Es seien $U(x)$ und $U(y)$ zwei zu F' fremde sphärische Umgebungen von x und y, x' und y' zwei bzw. in diesen Umgebungen liegende Punkte von $R^n - F$ (die es sicher gibt, weil F nirgends dicht ist), $G(x')$ und $G(y')$ die Komponenten dieser Punkte in $R^n - F$, und x'' und y'' zwei zu $G(x')$ bzw. zu $G(y')$ gehörende Punkte von U; man verbinde: x mit x', y mit y', x'' mit $\bar{x}$ und y'' mit $\bar{x}$ durch die geradlinigen Wege xx', $y'y$, $x''\bar{x}$, $\bar{x}y''$ und x' mit x'', bzw. y' mit y'' durch Wege $x'x''$ bzw. $y'y''$ in $G(x')$ bzw. in $G(y')$; der Weg

$$xy = xx' + x'x'' + x''\bar{x} + \bar{x}y'' + y''y' + y'y$$

verbindet die Punkte x und y in $R^n - F'$.

B. Wenn F den R^n zerlegt, aber keine absolute Gebietsgrenze ist, dann gibt es stets eine echte Teilmenge F' von F, die den R^n zerlegt. Es sei in der Tat x ein Punkt von F, der nicht Häufungspunkt *jeder* Komponente von $R^n - F$ ist; dann gibt es eine Komponente G von $R^n - F$, deren Rand F' erstens zu x fremd, zweitens (wie der Rand jeder Komponente von $R^n - F$) in F enthalten ist; somit ist F' eine echte Teilmenge von F, die, wie leicht ersichtlich, den R^n dennoch zerlegt, w. z. b. w.

16. In den Sätzen I und V ist folgender Invarianzsatz enthalten:

Die Eigenschaften einer abgeschlossenen Menge F, den R^n nicht zu zerlegen bzw. ihn in genau k zusammenhängende Gebiete zu zerlegen bzw. eine absolute Gebietsgrenze im R^n zu sein, sind invariant gegenüber topologischen Transformationen der Menge F.

Viertes Kapitel.

Die Brouwerschen Zahlen.

1. Da jede abgeschlossene Menge F sich durch Spektra, insbesondere durch Projektionsspektra definieren läßt, so kann jede topologische Eigenschaft der Menge F als eine Eigenschaft aufgefaßt werden, die diesen Spektren möglicher- bzw. notwendigerweise zukommt. Eine Eigenschaft eines Spektrums kann aber entweder als eine Eigenschaft der dieses Spektrum bildenden (wir sagen kurz "approximierenden") Komplexe auftreten (also von den, diese Komplexe verbindenden Relationen — im Falle des Projektionsspektrums sind das die Projektionen — unabhängig sein), oder erst mit Hilfe der soeben genannten Relationen ausdrückbar sein. Eigenschaften der Menge F, die sich durch Eigenschaften der approximierenden Komplexe (ohne Zuhilfenahme irgendwelcher Relationen) ausdrücken lassen, habe ich

"*kombinatorische Eigenschaften*" der Menge genannt.[44] Das Hauptresultat des vorigen Kapitels kann somit folgendermaßen formuliert werden: *Die nullte Bettische Zahl von $R^n - F$ stellt eine kombinatorische Eigenschaft der Menge F dar.*

Die natürlicherweise entstehende Frage, *ob dasselbe auch von einer beliebigen Bettischen Zahl von $R^n - F$ gilt,* bleibt unbeantwortet; das zu vermutende Resultat ist $p^{n-r-1}(R^n - F) = p_r(F)$, wobei $p_r(F)$ gemäß einer der Vorschriften der §§ 1–3, Kap. III definiert ist; die Richtigkeit dieser Formel ist aber bis jetzt unentschieden, selbst in den einfachsten Fällen: man weiß nicht einmal, ob man aus $\gamma^{n-r-1} \sim 0$ (*für jedes* $\gamma^{n-r-1} \subset R^n - F$) schließen kann, daß F durch kleine Transformationen in Komplexe mit verschwindender r-ter Bettischer Zahl übergeführt werden kann.

2. *Trotzdem sind die Bettischen Zahlen des $R^n - F$ topologische Invarianten von F*: es gilt nämlich *der Dualitätssatz*

$$p^{n-r-1}(R^n - F) = B^r(F), \tag{1}$$

wobei $B^r(F)$ die r-te *Brouwersche Zahl* der Menge F ist, eine Invariante, die sich ohne Zuhilfenahme jeglicher metrischer Begriffe z. B. wie folgt ausdrücken läßt:

Es sei ein Projektionsspektrum

$$K_1, K_2, \cdots, K_m, \cdots \qquad K_m = \pi(K_{m+1}) \tag{2}$$

der Menge F gegeben. Ein Zyklus $\Gamma^r_{(m)}$ aus K_m heißt *wesentlich,* wenn er (in K_m) für jedes h der Projektion $\pi_m^{m+h}(\Gamma^r_{(m+h)})$ eines Zyklus $\Gamma^r_{(m+h)}$ aus K_{m+h} homolog ist. Die Maximalanzahl der (in K_m) unabhängigen wesentlichen r-dimensionalen Zyklen soll $B^r_m(F)$ heißen. Wie leicht ersichtlich, kann $B^r_m(F)$ mit zunehmenden m nicht abnehmen, so daß

entweder $B^r_m(F)$ mit m unendlich wächst — dann ist definitionsgemäß $B^r(F) = \infty$,

oder sind von einem bestimmten m an alle $B^r_m(F)$ untereinander gleich — der gemeinsame Wert dieser Zahlen ist dann definitionsgemäß $B^r(F)$.

Die Ungleichung $B^r_m(F) \leqq B^r_{m+1}(F)$ ergibt sich einfach daraus, daß die Homologien in K_{m+1} sich in Homologien in K_m projizieren (d. h., daß aus $Q^{r+1} \subset K_{m+1}$, $Q^{r+1} \to \Gamma^r$ folgt, daß $\overline{\pi}(Q^{r+1}) \to \overline{\pi}(\Gamma^r)$ in K_m ist);[45] wenn

[44] Simpl. Appr., § 19.

[45] Bei unseren früheren Betrachtungen haben wir simpliziale Abbildungen eines Komplexes K auf einen anderen Komplex Q von rein mengentheoretischem Standpunkt aus behandelt und es kam uns insbesondere garnicht darauf an, zu wissen, wie oft ein Simplex von Q als Bildsimplex der Simplexe von K auftritt. Jetzt stellen wir uns, wenigstens im Falle, in dem K ein homogener r-dimensionaler Komplex ist, auf den *kombinatorischen*

also $\Gamma^r_{(m),1}$, $\Gamma^r_{(m),2}$, $\cdots$, $\Gamma^r_{(m),k}$ in K_m unabhängig sind, und $\Gamma^r_{(m),i} \sim \bar{\pi}(\Gamma^r_{(m+1),i})$ ist, so sind auch die $\Gamma^r_{(m+1),i}$ in K_{m+1} unabhängig.[46]

Bemerkung. Da im folgenden öfters Zyklen mit mehreren Indizes versehen werden müssen, werden wir von jetzt an einen r-dimensionalen Zyklus kurz mit (r) bezeichnen; wenn dabei Indizes auftreten, wird auch auf die Klammer verzichtet und einfach etwa r^i_m geschrieben.

Jeden Zyklus r_{m+h} aus K_{m+h}, dessen Projektion $\bar{\pi}_m(r_{m+h})$ in K_m dem gegebenen Zyklus $r_m \subset K_m$ homolog ist, werden wir als einen *Nachfolger* von r_m (im Komplex K_{m+h}) bezeichnen.

3. Die ursprüngliche Definition der Brouwerschen Zahlen[47] unterscheidet sich der Form nach von der soeben gegebenen. Sie lautet wie folgt. Es sei eine abgeschlossene Menge F gegeben; eine aus $n+1$ Punkten von F bestehende Menge heißt ein *Simplex von der Dimension* n, die Punkte selbst — *Eckpunkte* des Simplexes. Wenn der Durchmesser dieser endlichen Menge kleiner als δ ist, liegt ein δ-Simplex vor; wenn einige unter den Eckpunkten des Simplexes geometrisch zusammenfallen, spricht man von singulären Simplexen. Die Begriffe eines Komplexes, seines Randes, eines Zyklus usw. ergeben sich in üblicher Weise; wenn die Simplexe eines Komplexes sämtlich δ-Simplexe sind, ist der Komplex ein δ-Komplex.

Ein δ-Zyklus (in F) heißt ε-homolog Null in F, wenn er der Rand eines ε-Komplexes (in F) ist. Hiermit sind auch die Begriffe der ε-Unabhängigkeit, einer ε-Basis von δ-Zyklen usw. definiert.

Es seien nun ε und δ, $\varepsilon \geqq \delta$, beliebig gegeben; die Maximalanzahl der ε-unabhängigen δ-Zyklen von der Dimension r soll $\beta^r_{\varepsilon,\delta}(F)$ heißen; wenn δ abnimmt, kann $\beta^r_{\varepsilon,\delta}(F)$ nur abnehmen bzw. stationär bleiben, so daß eine nicht negative ganze Zahl

$$\beta^r_\varepsilon(F) = \min_{\delta \to 0} \beta^r_{\varepsilon,\delta}(F)$$

existiert. Wenn ε abnimmt, kann $\beta^r_\varepsilon(F)$ nur zunehmen bzw. stationär bleiben, so daß ein endlicher oder unendlicher Limes

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Standpunkt, so daß das Bild des Komplexes K^r als eine Linearform in den r-dimensionalen Simplexen von Q auftritt; dabei soll (im Falle der Theorie modulo 2, in dem wir uns die ganze Zeit befinden) diese Linearform modulo 2 reduziert werden. Auf diese Weise entsteht das *reduzierte Bild* des Komplexes K^r bei der gegebenen simplizialen Abbildung f, welches wir mit $\bar{f}(K^r)$ bezeichnen werden.

[46] Da K_{m+1} nun endlich-viele Teilkomplexe hat, kann man die $\Gamma^r_{(m+1),i}$ so wählen, daß jeder von ihnen für unendlich-viele h mit $\bar{\pi}^{m+h}_{m+1}(\Gamma^r_{(m+h),i})$ identisch ist (während $\bar{\pi}^{m+h}_m \Gamma^r_{(m+h),i} \sim \Gamma^r_{m,i}$ ist); sodann sind die $\Gamma^r_{(m+1),i}$ wesentlich. (Vgl auch § 7, Bemerkung 2.)

[47] Siehe wegen Literaturangabe die Fußnote [15].

(3) $$\beta^r(F) = \lim_{\varepsilon \to 0} \beta_\varepsilon^r(F)$$

existiert. Dieser Limes ist die r-te Brouwersche Zahl in ihrer ursprünglichen Definition. Für unsere Zwecke ist folgende ("versteckte") Definition (die der durch die Formel (3) gegebenen unmittelbar äquivalent ist) am bequemsten:

Es ist dann und nur dann

$$\beta^r(F) \geqq k,$$

wenn es ein $\varepsilon > 0$ gibt, so daß man für jedes δ mindestens k ε-unabhängige r-dimensionale δ-Zyklen in F finden kann.

4. Es soll jetzt bewiesen werden, daß $\beta^r(F) = B^r(F)$ ist.

Beweis der Ungleichung $\beta^r(F) \geqq B^r(F)$. Es sei $B^r(F) \geqq k$; es ist zu beweisen, daß dann auch $\beta^r(F) \geqq k$ ist. Es sei

(4) $$K_1, K_2, \cdots, K_m, \cdots$$

ein Projektionsspektrum von F, welches so beschaffen ist, daß bereits in K_1 mindestens k unabhängige wesentliche r-dimensionale Zyklen vorkommen. Die Komplexe (4) können als Nerven von Überdeckungen

(5) $$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_m, \cdots, \qquad K_m = N(\mathfrak{P}_m),$$

aufgefaßt werden, wobei $\mathfrak{P}_m$ eine ε_m-Überdeckung von F und $\lim_{m \to \infty} \varepsilon_m = 0$ ist. Man setze $\varepsilon = d(\mathfrak{P}_1)$ und wähle $\delta > 0$ ganz beliebig. Es genügt zu zeigen, daß es mindestens k ε-unabhängige δ-Zyklen von der Dimension r gibt.

Man wähle zu diesem Zwecke h so groß, daß $2\varepsilon_h$ kleiner als δ und ε ist; es seien

(6) $$r_1^1, r_1^2, \cdots, r_1^k$$

unabhängige wesentliche Zyklen in K_1 und

(7) $$r_h^1, r_h^2, \cdots, r_h^k$$

deren Nachfolger in K_h; die Zyklen (7) sind $2\varepsilon_h$-, also δ-Zyklen. Ich behaupte, daß sie ε-unabhängig sind. Es sei in der Tat

(8) $$Q^{r+1} \to \lambda_i r_h^i,$$

wobei Q^{r+1} ein ε-Komplex ist. Man kann nun Q^{r+1} simplizial auf einen Teilkomplex Q_1^{r+1} von K_1 dadurch abbilden, daß man jedem Eckpunkt b_j von Q^{r+1} einen (dem Elemente F_i von $\mathfrak{P}_1$ entsprechenden) Eckpunkt a_i von K_1 unter der Bedingung

(9) $$b_j \subset F_i$$

zuordnet und dabei beachtet, daß jedem Eckpunkt b_j, der gleichzeitig ein Eckpunkt von K_h ist (*also insbesondere jedem Eckpunkt des Randes von* Q^{r+1}) dessen Projektion (auf K_1) zugeordnet sei. Diese simpliziale Abbildung führt r_h^i in $\pi_1(r_h^i)$ über,[48] so daß

(10) $$\bar{Q}_1^{r+1} \to \lambda_i \bar{\pi}_1(r_h^i),$$

also $\lambda_i \bar{\pi}_1(r_h^i) \sim 0$ (in K_1) ist; da aber r_h^i ein Nachfolger von r_1^i ist, so ist $\bar{\pi}_1(r_h^i) \sim r_1^i$ (in K_1), was zusammen mit (10) die unserer Voraussetzung widersprechende Homologie $\lambda_i r_1^i \sim 0$ ergibt.

5. Beweis der Ungleichung $B^r(F) \geqq \beta^r(F)$. *Vorbemerkung.* Eine Folge

(11) $$\mathfrak{r} = (r^{\delta_1}, r^{\delta_2}, \cdots, r^{\delta_m}, \cdots)$$

von δ_m-Zyklen soll ein (r-dimensionaler) *Vollzyklus* heißen, wenn für jedes $\varepsilon > 0$ man ein n so bestimmen kann, daß für $p > n$ $\delta_p < \varepsilon$ ist, während für je zwei Zahlen p, q, $p > n$, $q > n$, die Relation $r^{\delta_p} \underset{\varepsilon}{\sim} r^{\delta_q}$ gilt.

Zwei Vollzyklen $\mathfrak{r}_1$ und $\mathfrak{r}_2$ heißen homolog, wenn es zu jedem ε ein n gibt, so daß $r_1^{\delta_p'} \underset{\varepsilon}{\sim} r_2^{\delta_p''}$ für alle $p > n$ ist. Somit ist auch die Unabhängigkeit der r-dimensionalen Vollzyklen definiert.[49]

Auch der Begriff der Basis der r-dimensionalen Vollzyklen läßt sich in wenigen Worten feststellen; es gilt nämlich folgende Definition:

Ein endliches System von Vollzyklen

(12) $$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_k$$

heißt eine Basis, wenn die (12) linear unabhängig sind und wenn überdies jeder r-dimensionale Vollzyklus einer linearen Kombination der (12) homolog ist.

Ein unendliches System von Vollzyklen

(12′) $$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_m, \cdots$$

heißt eine Basis, wenn die (12′) linear unabhängig sind (in dem — übrigens einzig möglichen — Sinne, daß keine endliche lineare Kombination dieser

[48] Wir verzichten auf den oberen Index h bei der Projektion $\pi_1^h(r_h^i)$.

[49] Der Begriff des Vollzyklus ist (unter dem Namen "Fundamentalfolge") von Vietoris, a. a. O.[15], eingeführt worden. Im orientierten Falle hätte man in der Definition des Vollzyklus die Bedingung $r^{\delta_p} \underset{\varepsilon}{\sim} r^{\delta_q}$ durch $\mu_p r^{\delta_p} \underset{\varepsilon}{\sim} \mu_q r^{\delta_q}$ zu ersetzen; (Lefschetz, a. a. O.[16]). Übrigens könnte man diese Bedingung auch durch $r^{\delta_p} \underset{\varepsilon_p}{\sim} \mu_{p+1} r^{\delta_{p+1}}$ mit $\lim_{p \to \infty} \varepsilon_p = 0$ ersetzen.

Zyklen homolog Null ist) und wenn man überdies zu jedem ε ein endliches Teilsystem (12^{ε}) der $(12')$ finden kann, von der Eigenschaft, daß jeder r-dimensionale Vollzyklus einer linearen Kombination der (12^{ε}) ε-homolog ist.

Es ist leicht einzusehen, daß die r-te Brouwersche Zahl einer abgeschlossenen Menge F endlich bzw. unendlich ist, je nachdem F eine endliche oder unendliche Basis der r-dimensionalen Vollzyklen besitzt. Auf die nähere Untersuchung dieser Basen kommen wir erst im § 12 zurück.

6. Es sei jetzt $\beta^r(F) \geqq k$; man hat zu zeigen, daß auch $B^r(F) \geqq k$ ist. Aus der Ungleichung $\beta^r(F) \geqq k$ folgt zunächst leicht[50] die Existenz von k unabhängigen Vollzyklen

$$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_k, \tag{12}$$

$$\left.\begin{aligned} \mathfrak{r}_i = (r_i^{\delta_1}, r_i^{\delta_2}, \cdots, r_i^{\delta_m}, \cdots), \quad r_i^{\delta_m} \underset{\eta_m}{\sim} r_i^{\delta_{m+k}}, \quad 0 = \lim_{m\to\infty} \delta_m = \lim_{m\to\infty} \eta_m, \\ i = 1, 2, \cdots, k. \end{aligned}\right\} \tag{13$_i$}$$

Es existiert somit eine feste Zahl α von der Eigenschaft, daß für jedes m die $r_i^{\delta_m}$ α-unabhängig sind. Es seien jetzt (4) und (5) ein (beliebiges) Projektionsspektrum und die zugehörige Folge von Unterteilungen von F. Man kann offenbar annehmen, daß (in den Bezeichnungen des § 4) $2\varepsilon_1 < \alpha$ ist; durch eine eventuelle Weglassung endlich-vieler Anfangsglieder in jeder der Folgen (13_i), $i = 1, 2, \cdots, k$, kann man erreichen, daß sämtliche δ_m und η_m kleiner als $d = d(\mathfrak{P}_1)$ sind. Man wähle jetzt wieder für jeden Eckpunkt b_j von $r_i^{\delta_1}$ einen Eckpunkt a_h von K_1 unter der üblichen Bedingung $F_h \supset b_j$; auf diese Weise entsteht eine ε_1-Transformation des Zyklus $r_i^{\delta_1}$ in einen Zyklus $r_1^i \subset K_1$; da die $r_i^{\delta_1}$ α-unabhängig und die Simplexe von K_1 kleiner als $2\varepsilon_1 < \alpha$ sind, so sind die k Zyklen $r_1^1, r_1^2, \cdots, r_1^k$ in K_1 unabhängig; es bleibt also nur übrig, zu zeigen, daß die r_1^i wesentliche Zyklen sind. Es sei n so groß, daß für sämtliche $m \geqq n$ $\varepsilon_m < d$ ist. Es genügt, zu beweisen, daß jeder Zyklus r_1^i in K_m $(m \geqq n)$ einen Nach-

[50] Vgl. Vietoris, Math. Ann., 97, S. 464, Zeilen 11 u. w. von oben. Dieselbe Behauptung gilt auch im orientierten Fall für die hier gegebene (dem Inhalt nach von Lefschetz herrührende) Definition von Vollzyklen. Dagegen stimmt diese Behauptung (wie es Vietoris auch selbst bemerkt) für seine Zyklendefinition im orientierten Falle *nicht*. Da auch für den orientierten Fall der Dualitätssatz gilt, so erweisen sich die Brouwerschen Zahlen als *die* richtigen Zusammenhangszahlen, so daß die von Vietoris vorgeschlagene Modifikation derselben als unzweckmäßig zu betrachten ist. Was den Beweis des Dualitätssatzes im orientierten Fall betrifft, so ist ein solcher (und zwar sogar für eine allgemeinere Formel) zum erstenmal von Lefschetz [16] gegeben worden. Die hier angegebene Abänderung der Zyklendefinition genügt übrigens, um den Dualitätssatz auch im orientierten Falle unter Zuhilfenahme der entsprechenden Modifikationen des Alexanderschen bzw. des Frankl-Pontrjaginschen Satzes zu beweisen.

folger r_m^i besitzt (weil dann für alle m', $1<m'<n$ der Zyklus $\pi_{m'}(r_n^i)$ als Nachfolger von r_1^i genommen werden kann).

Zu diesem Zweck wähle man j so groß, daß $\delta_j<d(\mathfrak{P}_m)$ ausfällt und bilde in der üblichen Weise mittels einer ε_m-Transformation $r_i^{\delta_j}$ simplizial auf einen Zyklus $r_m^i \subset K_m$ ab; dadurch entsteht eine ε_m-Homologie

$$r_i^{\delta_j} \underset{\varepsilon_m}{\sim} r_m^i; \tag{14}$$

andererseits ist $r_i^{\delta_j} \underset{\eta_1}{\sim} r_i^{\delta_1}$, was zusammen mit (14) — angesichts der Ungleichungen $\varepsilon_m<d$, $\eta_1<d$ — die Relation

$$r_m^i \underset{d}{\sim} r_i^{\delta_1}, \tag{15}$$

und folglich die Existenz eines d-Komplexes Q^{r+1},

$$Q^{r+1} \to r_i^{\delta_1} + r_m^i \tag{16}$$

ergibt. Es seien c_l die Eckpunkte von Q^{r+1}; unter diesen Eckpunkten kommen sowohl Eckpunkte b_j der Zyklen $r_i^{\delta_1}$, als auch gewisse Eckpunkte $a_{i_1 \dots i_m}$ von K_m (nämlich die Eckpunkte der Zyklen r_m^i) vor. Den Eckpunkten b_j waren schon früher gewisse Eckpunkte a_h von K_1 zugeordnet (so daß dadurch $r_i^{\delta_1}$ auf r_1^i abgebildet wurde); man lasse jetzt jedem unter den Punkten c_l vorkommenden Eckpunkte $a_{i_1 \dots i_m}$ seine Projektion a_{i_1} entsprechen; den noch übrig bleibenden Punkten c_l ordne man einen beliebigen a_i unter der einzigen Bedingung $c_l \subset F_i$ zu; auf diese Weise entspricht dem Komplexe Q^{r+1} ein Teilkomplex $\bar{Q}_1^{r+1}$ von K_1, wobei die Berandungsrelation (16) in

$$\bar{Q}_1^{r+1} \to r_1^i + \pi_1(r_m^i) \tag{16'}$$

übergeht; somit ist $\pi_1(r_m^i) \sim r_1^i$ (in K_1), d. h. r_m^i ist ein Nachfolger von r_1^i, w. z. b. w.

7. Bemerkung I. Aus den obigen Überlegungen folgt, daß auch der Begriff des Vollzyklus sein von metrischen Begriffen unabhängiges Analogon in der folgenden Definition findet:

Definition des Projektionszyklus. *Wenn $K_1, K_2, \dots, K_m, \dots$ ein Projektionsspektrum von F ist und r_m ein Zyklus aus K_m und dabei ein Nachfolger von r_{m-1} ist, so heißt die Folge $(r_1, r_2, \dots, r_m, \dots)$ ein Projektionszyklus.*

Zwei Projektionszyklen

$$\varrho = (r_1, r_2, \dots, r_m, \dots) \text{ und } \varrho' = (r_1', r_2', \dots r_m', \dots)$$

heißen *homolog*, wenn für alle hinreichend großen m

$$r_m \sim r'_m \text{ (in } K_m)$$

ist.[51]

In den Überlegungen der §§ 6–7 ist der Beweis der folgenden Tatsache enthalten:

Die r-te Brouwersche Zahl von F ist gleich der Maximalzahl der unabhängigen r-dimensionalen Projektionszyklen (die in bezug auf *irgendein* Projektionsspektrum von F definiert sind).

Daraus folgt ohne weiteres: *Wenn* dim $F = \lambda$ *und* $r > \lambda$ *ist, so ist* $\beta^r(F) = 0$.

Es ist übrigens ohnehin klar, daß jeder Projektionszyklus als ein Vollzyklus im Sinne der Definition des § 5 aufgefaßt werden kann; aus § 6 folgt aber leicht, daß auch umgekehrt jeder Vollzyklus (als solcher) einem Projektionszyklus homolog ist.

Bemerkung II. Alle bisherigen Betrachtungen sind *mutatis mutandis* auf den "orientierten" Fall übertragbar (wie es jedesmal in Fußnoten erwähnt wurde). Wenn man sich prinzipiell auf den Fall der Begriffsbildungen "modulo 2" beschränkt, so kann der Begriff eines Projektionszyklus durch den noch schärferen Begriff eines *echten* Projektionszyklus ersetzt werden; man erhält diesen Begriff, wenn man in der obigen Definition des Projektionszyklus das Wort "Nachfolger" durch "*echter* Nachfolger" ersetzt und r_{m+h} (aus K_{m+h}) als *echten* Nachfolger von r_m (aus K_m) bezeichnet, wenn (mod. 2) $r_m = \overline{\pi}_m(r_{m+h})$ ist. Es ergibt sich dabei, daß die Brouwersche Zahl $B^r(F)$ mit der Maximalzahl der unabhängigen echten Projektionszyklen identisch ist.

Bemerkung III. Die in diesem Kapitel gegebene Definition der Brouwerschen Zahlen hat die Eigenschaft, von der speziellen Wahl des Projektionsspektrums der Menge F vollkommen unabhängig zu sein; die Brouwerschen Zahlen drücken also eine Eigenschaft der Menge F aus, die als eine *jedem* diese Menge approximierenden *Projektionsspektrum zukommende* Eigenschaft auftritt.

8. Es sei jetzt $F \subset R^n$ und

$$\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots) \tag{17}$$

ein Vollzyklus (der also insbesondere auch ein Projektionszyklus sein könnte); wenn γ^{n-r-1} irgendein in $R^n - F$ gelegener Zyklus ist, so sagen wir, daß γ^{n-r-1} mit $\mathfrak{r}$ *verschlungen* ist, wenn γ^{n-r-1} mit allen r_m von einem bestimmten m an verschlungen ist[51a]. In diesem Falle ist *die Verschlingungs-*

[51] Im orientierten Falle hätte man sich (analog wie in der Fußnote [49]) der Lefschetzschen Koeffizienten μ zu bedienen.

[51a] Vgl. wegen des Begriffes der Verschlingung (und der zugehörigen Literaturangaben) z. B. die unter [39] und [16] angegebenen Arbeiten von Pontrjagin und mir.

zahl $\mathfrak{v}(\mathfrak{r}, \gamma^{n-r-1})$ definitionsgemäß 1; man überzeugt sich leicht davon, daß im Falle, wenn (im Sinne der soeben gemachten Definition) $\mathfrak{r}$ nicht mit γ^{n-r-1} verschlungen ist, die Verschlingungszahl (im klassischen Sinne) der Zyklen r_m und γ^{n-r-1} stets (bis auf endlich viele eventuell auftretende Ausnahmswerte von m) Null ist, so daß man veranlaßt ist, in diesem Falle $\mathfrak{v}(\mathfrak{r}, \gamma^{n-r-1}) = 0$ zu setzen.

Es seien jetzt endlich-viele Vollzyklen

$$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_k \tag{18}$$

in F und ebenso viele Zyklen

$$\gamma_1^{n-r-1}, \gamma_2^{n-r-1}, \cdots, \gamma_k^{n-r-1} \tag{19}$$

in $R^n - F$ gegeben. Nach der soeben festgelegten Definition ist für jedes Indexpaar i, j, die Verschlingungszahl $\mathfrak{v}(\mathfrak{r}_i, \gamma_j^{n-r-1}) = 0$ oder 1 erklärt, so daß man von der *Verschlingungsmatrix* $\| \mathfrak{v}(\mathfrak{r}_i, \gamma_j^{n-r-1}) \|$ sprechen kann; wenn der Rang dieser Matrix der höchste (also k) ist, sagen wir, daß *die Systeme* (18) *und* (19) *verschlungen* sind; offenbar tritt das dann und nur dann auf, wenn jede (nicht identisch verschwindende) lineare Kombination der Zyklen des einen der beiden Systeme mit mindestens einem Zyklus des andern Systems verschlungen ist. Wenn die Zyklen der beiden Systeme (18) und (19) eine solche gegenseitige Lage haben, daß jeder Zyklus des einen Systems mit einem einzigen Zyklus des anderen verschlungen ist (d. h. wenn die Verschlingungsmatrix die Einheitsmatrix ist), sagen wir, daß (18) und (19) *eindeutig verschlungen sind.*

9. Der DUALITÄTSSATZ (1) ergibt sich nunmehr ohne weiteres aus folgenden beiden Tatsachen:

I. *Es seien*

$$\gamma_1^{n-r-1}, \gamma_2^{n-r-1}, \cdots, \gamma_k^{n-r-1} \tag{19}$$

in $R^n - F$ unabhängig und U eine zu allen Zyklen (19) *fremde Polyederumgebung von F. Dann gibt es in F k Vollzyklen*

$$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_k, \qquad \mathfrak{r}_i = (r_1^i, r_2^i, \cdots, r_m^i, \cdots) \tag{18}$$

so, daß die beiden Systeme (18) *und* (19) *eindeutig verschlungen sind und infolgedessen die* (18) *in U unabhängig sind.*

(Dabei heißen die (18) in U unabhängig, wenn für ein hinreichend großes m kein Komplex $Q^{r+1} \subset U$ einer Relation von der Gestalt $Q^{r+1} \to \lambda_i r_m^i$ genügt; in analoger Weise definiert man für Vollzyklen auch den Begriff der Homologie in U.)

II. *Wenn F, U und ein in U unabhängiges System* (18) *gegeben sind, so läßt sich in $R^n - U$ ein mit* (18) *eindeutig verschlungenes System* (19) *konstruieren. Die Zyklen* (19) *sind sodann unabhängig in $R^n - F$.*

Den Beweis dafür, daß (18) und (19) als verschlungene Systeme konstruiert werden können, habe ich in meiner unter [16] zitierten Note in aller erforderlichen Ausführlichkeit erbracht, so daß es wohl überflüssig ist, ihn hier zu wiederholen; eine elementare Matrizentransformation[52] erlaubt sodann die (18) (im Falle I) bzw. die (19) (im Falle II) durch solche lineare Kombinationen derselben Zyklen zu ersetzen, daß die Verschlingung eindeutig wird.

10. Aus den beiden Sätzen I, II des letzten Paragraphen ergeben sich weitere Konsequenzen, die einerseits erlauben, den Dualitätssatz durch einen verschärfenden Zusatz zu ergänzen, andererseits aber für unsere weiteren Betrachtungen, insbesondere von Gebietsgrenzen im R^n, von Wichtigkeit sind.

Vorbemerkung. Es seien $F \subset R^n$ und U gegeben. Eine Basis von n U unabhängigen Vollzyklen r-ter Dimension soll mit $\mathfrak{B}_U^r(F)$, die Anzahl ihrer Elemente mit $\beta_U^r(F)$ bezeichnet werden. Wie leicht ersichtlich, ist $\beta_U^r(F)$ stets eine endliche Zahl.[53]

Ein System von in $R^n - U$ liegenden Zyklen

$$\gamma_1^{n-r-1}, \gamma_2^{n-r-1}, \ldots, \gamma_k^{n-r-1} \tag{20}$$

soll eine $\mathfrak{B}_U^{n-r-1}(R^n - F)$ heißen, wenn folgende beide Bedingungen erfüllt sind:

a) die (20) sind unabhängig in $R^n - F$;

b) jeder *in* $R^n - U$ *liegende* Zyklus γ^{n-r-1} ist einer linearen Kombination der Zyklen (20) *in* $R^n - F$ *homolog.*

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[52] Es sei eine modulo 2 zu betrachtende quadratische Matrix mit s Zeilen und dem Range s gegeben. Es handelt sich um den Beweis der folgenden Behauptung: Man kann die Spalten dieser Matrix so durch lineare Kombinationen derselben ersetzen, daß (nach ihrer eventuellen Permutation) die Matrix sich in die Einheitsmatrix (immer modulo 2) verwandelt.

Diese bekannte Tatsache beweisen wir (mit Pontrjagin) am einfachsten wohl so:

Es seien a_{ij} die Elemente der Matrix (wobei i der Spalten- und j der Zeilenindex ist); unter den Unterdeterminanten der k-ten Spalte ist wenigstens eine von Null verschieden; es sei dies A_{km}; man betrachte jetzt das Gleichungssystem

$$\sum_{i,\, i \neq k} a_{ij} x_i = a_{kj}, \quad 1 \leqq j \leqq s, \quad j \neq m.$$

Es seien $x_i = c_i$ die Lösungen dieses Gleichungssystems. Wenn man jetzt die k-te Spalte S_k durch $S_k + c_i S^i$, $i \neq k$, ersetzt, so sind alle Elemente der resultierenden Spalte gleich 0, mit Ausnahme des m-ten, das gleich 1 ist. Die Wiederholung dieses Verfahrens führt uns zu einer Matrix, deren Spalten je ein einziges von Null verschiedenes Element enthalten; dabei liegen diese von Null verschiedenen Elemente sämtlich in verschiedenen Zeilen, so daß man nach einer Permutation der Spalten die Einheitsmatrix erhält.

[53] Vgl. Vietoris, a. a. O., S. 461.

Die Anzahl der Elemente einer $\mathfrak{B}_U^{n-r-1}(R^n - F)$ (die von der speziellen Wahl dieser $\mathfrak{B}_U^{n-r-1}(R^n - F)$ unabhängig ist) soll $\beta_U^{n-r-1}(R^n - F)$ heißen.

11. Es gelten folgende Sätze:

Hilfssatz I. Es ist $\beta_U^r(F) = \beta_U^{n-r-1}(R^n - F)$; insbesondere bilden die zu einer $\mathfrak{B}_U^r(F)$ bzw. $\mathfrak{B}_U^{n-r-1}(R^n - F)$ nach den Vorschriften des § 9 konstruierten Systeme (19) bzw. (18) eine $\mathfrak{B}_U^{n-r-1}(R^n - F)$ bzw. eine $\mathfrak{B}_U^r(F)$.

Der Beweis besteht aus einer unmittelbaren Anwendung der beiden Behauptungen des § 9.

Hilfssatz II′. Es seien F, U und in $R^n - U$ liegende, in $R^n - F$ unabhängige Zyklen

$$\gamma_1^{n-r-1}, \gamma_2^{n-r-1}, \cdots, \gamma_k^{n-r-1} \tag{21}$$

gegeben, deren Anzahl $k < p = \beta_U^{n-r-1}(R^n - F) = \beta_U^r(F)$ ist. Wenn die Vollzyklen

$$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_k \tag{22}$$

ein mit (21) eindeutig verschlungenes System bilden, so kann man (22) zu einer $\mathfrak{B}_U^r(F)$ dadurch ergänzen, daß man Vollzyklen hinzufügt, von denen keiner mit einem der Zyklen (21) verschlungen ist.

Der Beweis verläuft ganz kurz wie folgt. Da die (21) in $R^n - U$ liegen und in $R^n - F$ unabhängig sind, lassen sie sich in eine $\mathfrak{B}_U^{n-r-1}(R^n - F)$ aufnehmen, die aus den Zyklen

$$\gamma_1^{n-r-1}, \cdots, \gamma_k^{n-r-1}, \quad \gamma_{k+1}^{n-r-1}, \cdots, \gamma_p^{n-r-1} \tag{23}$$

bestehen möge. Zu den Zyklen (23) kann man ein System von Vollzyklen

$$\mathfrak{r}_1', \mathfrak{r}_2', \cdots, \mathfrak{r}_p' \tag{24}$$

so wählen, daß (23) mit (24) eindeutig verschlungen ist. Ich behaupte, daß die Zyklen

$$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_k, \mathfrak{r}_{k+1}', \cdots, \mathfrak{r}_p' \tag{25}$$

eine $\mathfrak{B}_U^r(F)$ bilden; da ihre Anzahl p ist, genügt es zu zeigen, daß die (25) in U unabhängig sind. Es sei nun, im Gegenteil dazu, eine Homologie

$$\lambda^i \mathfrak{r}_i + \lambda'^j \mathfrak{r}_{k+j}' \sim 0 \qquad (\text{in } U) \tag{26}$$

vorhanden; da sowohl die (22) als auch die (24) in U unabhängig sind, muß wenigstens ein λ^i, z. B. λ^1, und wenigstens ein λ'^j von Null verschieden sein; da einerseits $\mathfrak{r}_1$, andererseits aber keiner von den Zyklen $\mathfrak{r}_i$, $i = 2, 3, \cdots, k$, sowie keiner von den Zyklen $\mathfrak{r}_{k+j}'$, $j = 1, 2, \cdots, p - k$, mit γ_1^{n-r-1} verschlungen ist, so ist die linke Seite von (26) ebenfalls mit γ_1^{n-r-1} verschlungen, so daß die Homologie (26) widerspruchsvoll ist.

In einer völlig analogen Weise beweist man auch den

Hilfssatz II''. Unter denselben Voraussetzungen wie im Hilfssatz II' kann man (21) zu einer $\mathfrak{B}_U^{n-r-1}(R^n - F)$ dadurch ergänzen, daß man Zyklen hinzufügt, von denen keiner mit einem der Vollzyklen (22) verschlungen ist.

12. Wenn man den Begriff der eindeutigen Verschlingung auch auf unendliche Systeme von Zyklen (bzw. Voll- oder Projektionszyklen) ausdehnt, indem man sagt, daß

(26) $$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_m, \cdots$$

mit

(27) $$\gamma_1^{n-r-1}, \gamma_2^{n-r-1}, \cdots, \gamma_m^{n-r-1}, \cdots$$

eindeutig verschlungen ist, falls für $i \neq j$ stets $\mathfrak{v}(\mathfrak{r}_i, \gamma_j^{n-r-1}) = 0$ und für jedes i $\mathfrak{v}(\mathfrak{r}_i, \gamma_i^{n-r-1}) = 1$ ist, so bekommt man folgende Verschärfung des Dualitätssatzes (1):

ZUSATZ. *Die r-dimensionale Basis von F und die $n-r-1$-dimensionale Basis von $R^n - F$ können stets so gewählt werden, daß die beiden Basen miteinander eindeutig verschlungen sind.*

Beweis. Man denke sich eine Folge von unendlich fein werdenden Polyederumgebungen

(28) $$U_1, U_2, \cdots, U_m, \cdots$$

von F und fange z. B. mit einer $\mathfrak{B}_{U_1}^r(F)$ an. Zu dieser $\mathfrak{B}_{U_1}^r(F)$ konstruiere man eine mit ihr eindeutig-verschlungene $\mathfrak{B}_{U_1}^{n-r-1}(R^n - F)$.

Die Elemente von $\mathfrak{B}_{U_1}^r(F)$, die ja in U_1 unabhängig sind, sind es erst recht in U_2, so daß die ganze $\mathfrak{B}_{U_1}^r(F)$ jedenfalls (als ein echtes oder unechtes) Teilsystem in die zu konstruierende $\mathfrak{B}_{U_2}^r(F)$ aufgenommen werden kann; auf Grund des Hilfssatzes II' kann man die $\mathfrak{B}_{U_1}^r(F)$ zur $\mathfrak{B}_{U_2}^r(F)$ durch solche neue Zyklen ergänzen, daß keiner dieser neuen Zyklen mit einem Zyklus aus $\mathfrak{B}_{U_1}^{n-r-1}(R^n - F)$ verschlungen ist. Zu der auf diese Weise gewonnenen $\mathfrak{B}_{U_2}^r(F)$ konstruiere man eine mit ihr eindeutig verschlungene $\mathfrak{B}_{U_2}^{n-r-1}(R^n - F)$ und ersetze in ihr die ersten $\beta_{U_1}^{n-r-1}(R^n - F)$ Elemente durch die Elemente der schon konstruierten $\mathfrak{B}_{U_1}^{n-r-1}(R^n - F)$. Genau wie beim Beweise des Hilfssatzes II' überzeugt man sich davon, daß das soeben erhaltene Zyklensystem wiederum eine $\mathfrak{B}_{U_2}^{n-r-1}(R^n - F)$ bildet, die offenbar mit unserer $\mathfrak{B}_{U_2}^r(F)$ eindeutig verschlungen ist.

Auf diese Weise fortfahrend erhält man zwei Folgen von Basen:

(29) $$\mathfrak{B}_{U_1}^r(F), \mathfrak{B}_{U_2}^r(F), \cdots, \mathfrak{B}_{U_m}^r(F), \cdots$$

und

(30) $\mathfrak{B}^{n-r-1}_{U_1}(R^n-F),\ \mathfrak{B}^{n-r-1}_{U_2}(R^n-F),\ \cdots,\ \mathfrak{B}^{n-r-1}_{U_m}(R^n-F),\ \cdots$

von der Eigenschaft, daß bei jedem m $\mathfrak{B}^r_{U_m}(F)$ ein Teilsystem der $\mathfrak{B}^r_{U_{m+1}}(F)$, $\mathfrak{B}^{n-r-1}_{U_m}(R^n-F)$ ein Teilsystem von $\mathfrak{B}^{n-r-1}_{U_{m+1}}(R^n-F)$ und $\mathfrak{B}^r_{U_m}(F)$ mit $\mathfrak{B}^{n-r-1}_{U_m}(R^n-F)$ eindeutig verschlungen ist.

Zwei Fälle sind nun möglich: entweder sind von einem bestimmten m an alle $\mathfrak{B}^r_{U_m}(F)$ bzw. alle $\mathfrak{B}^{n-r-1}_{U_m}(R^n-F)$ untereinander identisch, — oder wächst die Anzahl der Elemente von $\mathfrak{B}^r_{U_m}(F)$ und von $\mathfrak{B}^{n-r-1}_{U_m}(R^n-F)$ mit m über alle Grenzen. Im ersten Fall ist die r-te Brouwersche Zahl von F endlich und man erhält eine endliche r-dimensionale Basis von F und eine mit ihr eindeutig verschlungene $n-r-1$-dimensionale Basis von R^n-F.

Im zweiten Fall erhält man zwei eindeutig verschlungene Systeme von Zyklen: $\mathfrak{B}^r(F)$ und $\mathfrak{B}^{n-r-1}(R^n-F)$, wobei das erste System aus allen in den Basen (29) und das zweite aus allen in den Basen (30) vorkommenden Zyklen besteht. Es mögen

(31) $$\mathfrak{r}_1,\ \mathfrak{r}_2,\ \cdots,\ \mathfrak{r}_m,\ \cdots$$

bzw.

(32) $$\gamma_1^{n-r-1},\ \gamma_2^{n-r-1},\ \cdots,\ \gamma_m^{n-r-1},\ \cdots$$

die Elemente von $\mathfrak{B}^r(F)$ bzw. $\mathfrak{B}^{n-r-1}(R^n-F)$ sein. Wie leicht ersichtlich, ist keine endliche lineare Kombination der (31) in F, der (32) in R^n-F homolog Null. Jeder zu F fremde Zyklus γ^{n-r-1} ist bei hinreichend großem m in R^n-U_m enthalten und also einer *endlichen* linearen Kombination der (32) in R^n-F homolog. Somit ist (32) eine $n-r-1$-dimensionale Basis von R^n-F.

Da jeder Vollzyklus $\mathfrak{r}$ in jedem U_m einer Linearkombination der in $\mathfrak{B}^r_{U_m}(F)$ vorkommenden $\mathfrak{r}_m$ homolog ist, so ist (31) eine r-dimensionale Basis von F, womit der Zusatz bewiesen ist.

13. Bemerkung. Den Beweis des Zusatzes konnte man ebensogut mit der Betrachtung einer $\mathfrak{B}^{n-r-1}_{U_1}(R^n-F)$ anfangen; da aber jeder γ^{n-r-1}, der in R^n-F nicht homolog Null ist, bei einer passend gewählten U_1 in die $\mathfrak{B}^{n-r-1}_{U_1}(R^n-F)$ aufgenommen werden kann, hat man folgendes

Korollar. Es sei γ^{n-r-1} ein beliebiger Zyklus, der in R^n-F gelegen und daselbst nicht ~ 0 ist; man kann dann die den Bedingungen des Zusatzes genügenden $\mathfrak{B}^r(F)$ und $\mathfrak{B}^{n-r-1}(R^n-F)$ so wählen, daß γ^{n-r-1} ein Element von $\mathfrak{B}^{n-r-1}(R^n-F)$ ist.

Eine analoge Behauptung gilt natürlich auch für einen beliebigen Vollzyklus $\mathfrak{r}$, der in F nicht homolog Null ist.

14. Die obigen Betrachtungen sollen jetzt auf das Problem der *irreduziblen Verschlingung*[54] angewandt werden: dieses Problem besteht darin, *topologische Eigenschaften einer im R^n liegenden Menge F aufzustellen, die für die Existenz eines mit der Menge F irreduzibel-verschlungenen $n-r-1$-dimensionalen Zyklus notwendig und hinreichend sind.*

Die Wichtigkeit dieser Frage beruht auf folgendem bekannten

Satz. Eine abgeschlossene Menge F ist dann und nur dann die gemeinsame Grenze von zwei oder mehr zusammenhängenden Gebieten im R^n, wenn es einen mit dieser Menge irreduzibel-verschlungenen nulldimensionalen Zyklus gibt.

Beweis. Es sei γ^0 mit F irreduzibel-verschlungen. Man kann stets annehmen, daß γ^0 aus einem Punktepaar a, b besteht. Es seien nun A und B die Komponenten der Punkte a und b in $R^n - F$, und F_a und F_b die Grenzen von A und B. Dann ist $F_a + F_b \subset F$; zu zeigen ist, daß $F_a = F = F_b$ ist. Letztere Identität folgt aber daraus, daß jeder Streckenzug $\overline{ab}$ sowohl F_a als auch F_b trifft, so daß γ^0 mit F_a sowie mit F_b verschlungen ist, was (da γ^0 mit F irreduzibel-verschlungen ist) nur dann möglich ist, wenn die obige Identität gilt.

Es sei jetzt F die gemeinsame Grenze zweier zusammenhängender Gebiete A und B des R^n; man wähle die Punkte $a \subset A$ und $b \subset B$; der Zyklus $\gamma^0 = (a, b)$ ist natürlich nicht ~ 0 in $R^n - F$. Es sei jetzt F' eine beliebige echte abgeschlossene Teilmenge von F und x ein Punkt von $F - F'$; man betrachte eine zu F' fremde sphärische Umgebung V von x und wähle in derselben zwei Punkte: $a' \subset A$ und $b' \subset B$. Man verbinde a und a' in A, b und b' in B durch Wege aa' und bb', man verbinde endlich a' mit b' durch die geradlinige Strecke $a'b'$. Der Weg $ab = aa' + a'b' + b'b$ liegt sodann in $R^n - F'$, so daß $\gamma^0 \sim 0$ (in $R^n - F'$) ist, w. z. b. w.

15. Um die am Anfang des vorigen Paragraphen gestellte Frage zu beantworten, bedarf man einiger neuer Definitionen.

Definition I. Es sei in der abgeschlossenen Menge F ein Vollzyklus

$$\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots) \tag{33}$$

gegeben. Jede abgeschlossene Teilmenge F' von F von der Eigenschaft, daß $\mathfrak{r}$ ein Vollzyklus von F' ist, heißt ein *Träger* von $\mathfrak{r}$. Wie leicht beweisbar,[55]

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[54] Kap. III, § 13 (S. 153).

[55] Vgl. Anhang I.

ist der Durchschnitt aller Träger von $\mathfrak{r}$ ebenfalls ein Träger von $\mathfrak{r}$; dieser soll der *kleinste* Träger heißen.[56]

Definition II. Es sei eine abgeschlossene Menge F und daselbst eine Homologie

$$\mathfrak{r} \sim 0, \quad \mathfrak{r} = (r_1^{\delta_1}, r_2^{\delta_2}, \cdots, r_m^{\delta_m}, \cdots), \quad \lim_{m\to\infty} \delta_m = 0, \tag{34}$$

gegeben; es sei ferner

$$Q_1^{r+1}, Q_2^{r+1}, \cdots, Q_m^{r+1}, \cdots, \tag{35}$$

$$Q_m^{r+1} \to r_m, \tag{36}$$

eine "die Homologie (34) realisierende Folge von Komplexen" (wobei die Q_m^{r+1} η_m-Komplexe in F, $\lim_{m\to\infty} \eta_m = 0$, sind). Jede Menge F', die als topologischer Limes einer konvergenten Teilfolge von (36) auftritt, soll ein *Träger der Homologie*[55] (34) heißen.[56] Dabei heißt der Zyklus $\mathfrak{r}$ *ein kombinatorischer Rand der Menge* F' und der kleinste Träger F'' dieses Zyklus *eine kombinatorische Randmenge der Menge* F'.

16. An diese Definitionen, die auch ein allgemeineres Interesse darbieten dürften, schließt sich folgende, speziellere an:

Definition III. Eine r-dimensionale Basis

$$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_m, \cdots \tag{37}$$

heißt *ausgezeichnet*, wenn sie ein durch folgende Eigenschaft *ausgezeichnetes Element*, etwa $\mathfrak{r}_1$, enthält:

Es gibt eine positive Zahl α von der Beschaffenheit, daß jeder Träger eines Zyklus $\mathfrak{r}$ mit der ganzen Menge F identisch ist, sobald dieser Zyklus einer Homologie $\mathfrak{r} \underset{\alpha}{\sim} \lambda^i \mathfrak{r}_i$ *mit* $\lambda^1 \neq 0$ genügt.

Man überzeugt sich leicht davon, daß bei einer Homöomorphie $\overset{*}{F} = f(F)$ eine ausgezeichnete Basis von F in eine ausgezeichnete Basis von $\overset{*}{F}$ übergeht (als $\overset{*}{\alpha}$ kann man eine beliebige Zahl wählen, die so klein ist, daß aus $\varrho(x, y) < \overset{*}{\alpha}$ (in $\overset{*}{F}$) folgt, daß die den Punkten x und y entsprechenden Punkte x und y von F eine Entfernung $< \alpha$ haben).

Somit ist die Lösung des am Anfang des § 14 aufgestellten Problems durch folgenden Satz gegeben:

VERSCHLINGUNGSSATZ. *Es sei F eine im R^n liegende abgeschlossene Menge. Die notwendige und hinreichende Bedingung dafür, daß es einen mit F*

[56] Ohne Einschränkung der Allgemeinheit könnte man voraussetzen, daß die r_m (als endliche Punktmengen betrachtet) stets eine konvergente Folge (Hausdorff, Mengenlehre, S. 146, § 2) bilden, d. h. man kann nur *konvergente Zyklen* betrachten, die dieser Bedingung genügen. Offenbar ist dann $\lim \operatorname{top} r_m \subset F'$, wenn F' ein Träger von $\mathfrak{r}$ ist. Analoges gilt für (35).

irreduzibel-verschlungenen $n-r-1$-dimensionalen Zyklus gibt, besteht in der Existenz einer ausgezeichneten r-dimensionalen Basis der Menge F.

Korollar I. *Die Existenz eines mit F irreduzibel-verschlungenen Zyklus ist eine topologisch-invariante Eigenschaft der Menge F.*

Korollar II. *Die Eigenschaft einer abgeschlossenen Menge, die gemeinsame Grenze von mindestens zwei zusammenhängenden Gebieten des R^n zu sein, ist invariant gegenüber topologischen Transformationen dieser Menge.*

17. Beweis des Verschlingungssatzes.

a) Die Bedingung ist notwendig. Es sei in der Tat γ^{n-r-1} irreduzibel-verschlungen mit F. Infolge des Korollars I des § 13 existieren zwei miteinander eindeutig verschlungene Basen

$$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_m, \cdots \tag{38}$$

und

$$\gamma_1^{n-r-1}, \gamma_2^{n-r-1}, \cdots, \gamma_m^{n-r-1}, \cdots \tag{39}$$

von F bzw. $R^n - F$, *wobei* $\gamma_1^{n-r-1} = \gamma^{n-r-1}$ *ist.*

Es sei jetzt $3\alpha = \varrho(F, \gamma^{n-r-1})$ und U eine der Bedingung

$$S(F, \alpha) \subset U \subset S(F, 2\alpha)$$

genügende Polyederumgebung von F. Ich behaupte, daß (38) eine ausgezeichnete Basis von F, und zwar, daß $\mathfrak{r}_1$ ihr ausgezeichnetes Element ist. Es sei in der Tat $\mathfrak{r}$ irgendein Vollzyklus, der einer endlichen Linearkombination der Zyklen (38) α-homolog ist:

$$\mathfrak{r} \underset{\alpha}{\sim} \lambda^i \mathfrak{r}_i, \quad i = 1, 2, \cdots, k; \qquad \lambda^1 \neq 0. \tag{40}$$

Es ist zu zeigen, daß jeder Träger des Zyklus $\mathfrak{r}$ mit der ganzen Menge F identisch ist; dazu genügt aber wiederum zu beweisen, daß, wenn F' ein solcher Träger ist, γ^{n-r-1} in $R^n - F'$ nicht homolog Null sein kann.

Da γ^{n-r-1} mit $\mathfrak{r}_1$ und mit keinem der übrigen Zyklen (38) verschlungen ist, so ist γ^{n-r-1} mit der rechten Seite von (40) und folglich (da die Homologie (40) in U stattfindet) mit $\mathfrak{r}$ verschlungen. Es bleibt uns also nur übrig, folgende Tatsache zu beweisen:

Hilfssatz. Wenn γ^{n-r-1} mit $\mathfrak{r}$ verschlungen und F' ein Träger von $\mathfrak{r}$ ist, so kann γ^{n-r-1} in $R^n - F'$ nicht homolog Null sein.

Es sei

$$\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots); \tag{41}$$

ohne Einschränkung der Allgemeinheit kann man voraussetzen, daß die Folge $r_1, r_2, \cdots, r_m, \cdots$ eine konvergente Folge ist. Man bezeichne mit

Q^{n-r} einen beliebigen in R^n durch γ^{n-r-1} begrenzten Komplex; da $\mathfrak{r}$ mit γ^{n-r-1} verschlungen ist, so hat bei jedem noch so großen m Q^{n-r} mit r_m gemeinsame Punkte, d. h. bei jedem; beliebig kleinen ε ist $Q^{n-r} \cdot S(F', \varepsilon) \neq 0$; dann ist aber auch $Q^{n-r} \cdot F' \neq 0$, womit unser Hilfssatz und folglich die ganze Behauptung a) bewiesen sind.

b) Die Bedingung ist hinreichend. Es sei in der Tat (38) eine ausgezeichnete Basis und $\mathfrak{r}_1$ ihr ausgezeichnetes Element; man wähle eine solche Umgebung U von F, daß $\mathfrak{r}_1$ in U nicht homolog Null ist und überdies $U \subset S\left(F, \frac{\alpha}{3}\right)$ ist, wo α die in der Definition III vorkommende Zahl ist. Man kann dann $\mathfrak{r}_1$ in eine aus Elementen von (38) konstruierte $B^r_U(F)$ aufnehmen; es sei

(42) $$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_k$$

diese $B^r_U(F)$ und

(43) $$\gamma_1^{n-r-1}, \gamma_2^{n-r-1}, \cdots, \gamma_k^{n-r-1}$$

eine mit (42) eindeutig verschlungene $\mathfrak{B}^{n-r-1}_U(R^n - F)$. Ich behaupte, daß γ_1^{n-r-1} mit F irreduzibel verschlungen ist.

Zunächst ist γ_1^{n-r-1} in $R^n - F$ gewiß nicht ~ 0; es genügt also zu zeigen, daß bei jeder Wahl der echten Teilmenge F' von F

(44) $$\gamma_1^{n-r-1} \sim 0 \qquad (\text{in } R^n - F')$$

ist. Es sei F' eine Teilmenge von F, die so gewählt ist, daß die Homologie (44) nicht zutrifft. Dann gibt es in F' einen Vollzyklus $\mathfrak{r}$ mit dem γ_1^{n-r-1} verschlungen ist. Als Vollzyklus von F' betrachtet, ist $\mathfrak{r}$ in U einer Linearkombination der (42) homolog

(45) $$\mathfrak{r} \sim \lambda^i r_i \qquad (\text{in } U),$$

woraus folgt, daß γ_1^{n-r-1} mit der rechten Seite von (45) verschlungen ist; da γ_1^{n-r-1} mit $\mathfrak{r}_1$ und keinem anderen Zyklus (42) verschlungen ist, muß $\lambda^1 \neq 0$ sein; zufolge der Wahl von U können wir annehmen, daß die Homologie (45) eine α-Homologie ist,[57] so daß jeder Träger des Zyklus $\mathfrak{r}$

[57] Es sei in der Tat

$$\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots)$$
$$\mathfrak{r}_i = (r_1^i, r_2^i, \cdots, r_m^i, \cdots)$$

und $Q_m^{r+1} \to r_m + \lambda_i r_m^i$ (in U); man nehme an, daß alle vorkommenden Komplexe in $\frac{\alpha}{3}$-Simplexen gegeben sind und wähle für jeden Eckpunkt von Q_m^{r+1} einen der zu diesem Eckpunkte am nächsten liegenden Punkte von F; dadurch gehen die r_m bzw. r_m^i in sich und Q_m^{r+1} in einen α-Komplex $\bar{Q}_m^{r+1}$ über, wobei $\bar{Q}_m^{r+1} \to r_m + \lambda_i r_m^i$ ist.

mit F identisch ist; dies ist mit der Tatsache, daß $\mathfrak{r}$ ein Vollzyklus der Menge $F' \subset F$ ist, nur dann verträglich, wenn $F' = F$ ist, w. z. b. w.

18. Wenn man in der Richtung der Definition III (§ 16) noch einen Schritt weiter machen wollte, würde man Mengen F zu betrachten haben, deren jeder r-dimensionale Vollzyklus, insofern er in F nicht ~ 0 ist, die ganze Menge F als seinen einzigen Träger besitzt. Es ist leicht zu zeigen, daß, wenn $F \subset R^n$ ist, die obige Eigenschaft nichts anderes besagt, als daß jeder $n-r-1$-dimensionale Zyklus $\gamma^{n-r-1} \subset R^n - F$ entweder in $R^n - F$ homolog Null, oder mit F irreduzibel-verschlungen ist. Wie leicht konstruierbare Beispiele zeigen, folgt daraus im allgemeinen nicht, daß für jede echte Teilmenge F' von F die r-te Brouwersche Zahl verschwindet. Dagegen gilt folgender Satz:

SATZ I. *Wenn dim $F = r$ und jeder r-dimensionale Vollzyklus entweder in F homolog Null ist, oder letztere Menge für seinen einzigen Träger hat, so hat jede echte Teilmenge F' von F eine verschwindende r-te Brouwersche Zahl.*

Dieser Satz folgt aus einer, nach meiner Ansicht viel interessanteren Tatsache, zu deren Formulierung ich eine neue Definition brauche.

Definition IV. Man sagt, daß ein Vollzyklus $\mathfrak{r}$ von F *identisch verschwindet*, wenn er in jedem seiner Träger homolog Null ist.

19. Der Satz I folgt ohne weiteres aus dem

SATZ II. *Wenn ein r-dimensionaler nicht identisch verschwindender Vollzyklus einer abgeschlossenen Menge F daselbst homolog Null ist, so ist jeder Träger dieser Homologie mindestens $r+1$-dimensional.*

Wir beweisen diesen Satz zunächst für den Fall, in dem $F \subset R^n$ ist. Es sei F' ein Träger des gegebenen Vollzyklus $\mathfrak{r}$, wobei wir annehmen, daß F' so gewählt ist, daß $\mathfrak{r}$ in F' nicht ~ 0 ist. Es gibt sodann einen mit $\mathfrak{r}$ verschlungenen $\gamma^{n-r-1} \subset R^n - F'$. Es sei ferner F'' ein beliebiger Träger der Homologie

$$\mathfrak{r} \sim 0. \tag{46}$$

Somit kann eine konvergente Folge von komplexen Q_m gewählt werden:

$$F'' = \lim \operatorname{top} Q_m, \tag{47}$$

wobei, wenn $\mathfrak{r} = (r_1, r_2, \ldots, r_m, \ldots)$ ist, für jedes m

$$Q_m \to r_m \tag{48}$$

gilt. Da γ^{n-r-1} mit $\mathfrak{r}$ verschlungen ist, kann man annehmen, daß für jedes m

$$\mathfrak{v}(r_m, \gamma^{n-r-1}) = 1 \tag{49}$$

ist. Die untere Grenze aller Entfernungen $\varrho(r_m, \gamma^{n-r-1})$ ist eine positive Zahl 2η und die Relationen (49) bleiben erhalten, wenn man r_m durch einen

aus r_m mittels einer η-Transformation hervorgehenden Zyklus r'_m ersetzt. Es sei jetzt dim $F'' \leqq r$. Vermöge Kap. I, Satz II, kann Q_m (bei hinreichend großem m) in einen höchstens r-dimensionalen Komplex Q'_m η-übergeführt werden, von dem man annehmen kann, daß Q'_m zu γ^{n-r-1} fremd ist; bei dieser η-Überführung geht r_m in einen Zyklus r'_m über, der einerseits mit γ^{n-r-1} verschlungen sein muß, andererseits es sicher nicht ist, weil er den zu γ^{n-r-1} fremden Komplex Q'_m begrenzt. Durch diesen Widerspruch ist der Satz II im Spezialfall $F \subset R^n$ bewiesen.

20. Der allgemeine Fall folgt sodann aus einem von Menger bewiesenen "Einbettungssatze", der besagt, daß jede endlich-hoch dimensionale Menge in einen Euklidischen Raum topologisch eingebettet werden kann,[58] in wenigen Worten: um ihn auf den schon erledigten Spezialfall zurückzuführen, genügt es, den Homologieträger F'', der nur im Falle dim $F'' \leqq r$ zu untersuchen ist, in einen R^n einzubetten. Da aber der Mengersche Satz ein viel tiefer liegendes Ergebnis darstellt, als unser Satz II, scheint mir wünschenswert, einen elementaren Beweis auf einem anderen Wege zu suchen.

21. Dieser Weg ist durch folgende Bemerkung gegeben:

Hilfssatz. Ein nicht identisch verschwindender Vollzyklus kann nicht mittels k. T. seines Trägers in identisch-verschwindende Zyklen übergehen.

Wir nehmen für einen Augenblick an, daß dieser Hilfssatz bereits bewiesen wäre. Es sei sodann dim $F'' \leqq r$, woraus folgt, daß F'' mittels k. T. in höchstens r-dimensionale Komplexe K und $\mathfrak{r}$ in auf diesen Komplexen liegende r-dimensionale Vollzyklen $\mathfrak{r}_K$ übergeht; diese Vollzyklen sind überdies in den Komplexen K homolog Null; da aber jeder Komplex als eine Teilmenge eines R^n betrachtet werden kann, so verschwinden sie (vermöge des schon bewiesenen Spezialfalles des Satzes II) identisch; dies widerspricht jedoch dem Hilfssatze.

22. Beweis des Hilfssatzes. Es sei

$$\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots) \tag{50}$$

ein konvergenter Vollzyklus, von dem man weiß, daß er mittels ε_k-Transformationen eines seiner Träger — also jedenfalls seines kleinsten Trägers F' — in identisch verschwindende Vollzyklen

$$\mathfrak{r}^k = (r_1^k, r_2^k, \cdots, r_m^k, \cdots), \qquad \lim \varepsilon_k = 0 \tag{51}$$

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[58] Menger, Proceed. Akad. Amsterdam, 29 (1926), S. 482; Jahresbericht d. D. M. V., 35 (1926), S. 149 und 36 (1927), S. 12; Wiener Akademie, Akademischer Anzeiger Nr. 1 (Stzg. der math. nat. Kl., 12. Jan. 1928); ausführlicher Beweis in Mengers Buch *Dimensionstheorie* (Leipzig, Teubner, 1928), Kap. IX, S. 287–303.

übergeht; bei unserer ε_k-Transformation geht F' in einen Träger F'_k von $\mathfrak{r}^k$ über; ferner sind die $\mathfrak{r}^k$ konvergente Zyklen, und es ist:

$$52) \qquad F''_k = \lim_{m\to\infty} \operatorname{top} r^k_m \subset F'_k.$$

Da die (51) identisch verschwinden, so ist für jedes k $\mathfrak{r}^k \sim 0$ (in F'_k), so daß es $\delta_{k,m}$-Komplexe $Q_{k,m}$ in F'_k mit

$$(53) \qquad Q_{k,m} \to r^k_m, \qquad \lim_{m\to\infty} \delta_{k,m} = 0$$

gibt. Man betrachte jetzt die Folge

$$(54) \qquad Q_{1,m_1}, Q_{2,m_2}, \cdots, Q_{k,m_k}, \cdots,$$

wobei m_k unter der Bedingung $\delta_{k,m_k} < \varepsilon_k$ gewählt ist; indem man eventuell (50) durch eine Teilfolge ersetzt, kann man voraussetzen, daß (54) konvergiert und $m_k = k$ ist; dann aber ist notwendig

$$(55) \qquad F'' = \lim \operatorname{top} Q_{k,k} \subset F'.$$

Man beachte jetzt die Tatsache, daß r^k_k aus r_k mittels einer ε_k-Transformation entstanden ist; infolgedessen sind r^k_k und r_k miteinander ε_k-homolog, und zwar beranden sie zusammen einen ("ringförmigen") Komplex q_k, dessen sämtliche Eckpunkte sich unter den Eckpunkten von $r^k_k + r_k$ befinden; wenn man den ε_k-Komplex $Q_{k,k} + q_k$ mit Q'_k bezeichnet, so ist

$$(56) \qquad Q'_k \to r_k.$$

Man lasse jetzt jedem Eckpunkt von Q'_k einen unter den zu diesem Punkte am nächsten liegenden Punkten von F' entsprechen, wodurch Q'_k in einen zu F' gehörenden $3\varepsilon_k$-Komplex Q''_k und r_k in sich übergeht; dabei bleiben die Berandungsrelationen erhalten, so daß

$$(57) \qquad Q''_k \to r_k \qquad (\text{in } F')$$

ist; da aber $\lim_{k\to\infty} \varepsilon_k = 0$ ist, so bedeuten die Relationen (57) nichts anderes, als daß $\mathfrak{r} \sim 0$ (in F'), womit der Hilfssatz und der Satz II bewiesen sind.

23. Aus dem Satz II folgt (unter Benutzung der Resultate des § 7):

Korollar I. *In einer λ-dimensionalen Menge F gibt es keinen nicht identisch verschwindenden Zyklus von einer Dimension $> \lambda$.*

Außerdem ergeben sich ohne weiteres:

Korollar II. *Wenn ein in einem kompakten metrischen Raume F gelegener singularitätenfreier r-dimensionaler Zyklus Γ^r einer stetigen Deformation unter-*

worfen ist, deren Endresultat $\overset{*}{\Gamma}$ *wenigstens einen Punkt von* Γ^r *nicht enthält, so ist die Spur*[59] *von* Γ^r *bei dieser Deformation mindestens* $r+1$*-dimensional.*

Es genügt eine Kreislinie in die aus derselben Kreislinie und eines ihrer Radien bestehende Figur stetig zu deformieren, um sich zu überzeugen, daß das obige Korollar im allgemeinen nicht verschärft werden kann.

Korollar III. *Es sei* F *ein topologischer Raum, der ein stetiges Bild eines* n*-dimensionalen Komplexes* K^n *ist:* $F = f(K^n)$; *wenn* K^n *einen nicht verschwindenden Rand hat und die Abbildung* f *daselbst eineindeutig ist, so ist der Bildraum* F *mindestens* n*-dimensional.*

An die soeben bewiesenen Resultate schließt sich überdies folgende Tatsache an:

Satz IIa. *Ein Träger eines nicht identisch verschwindenden* r*-dimensionalen Vollzyklus ist eine mindestens* r*-dimensionale abgeschlossene Menge.*

Es sei in der Tat $\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots)$ ein Vollzyklus, dessen Träger F' eine höchstens $r-1$-dimensionale Menge ist. Durch kleine Transformationen der r_m (denen eine Homologie in F' entspricht) kann man erreichen, daß die r_m Teilkomplexe der die Menge F' approximierenden höchstens $r-1$-dimensionalen Komplexe K_m sind; dann sind aber die r_m *eo ipso* in diesen K_m homolog Null, womit auch die Homologie $\mathfrak{r} \sim 0$ (in F'), also der Satz IIa, bewiesen ist.

24. Wir kehren zu den irreduziblen Verschlingungen und verwandten Begriffen zurück. Aus den Überlegungen der §§ 18–23 folgt, *daß im Falle einer* r*-dimensionalen Menge* F *jede der beiden Bedingungen:*

a) ein r-dimensionaler Vollzyklus kann nur dann eine echte Teilmenge von F als seinen Träger besitzen, wenn er in F homolog Null ist —

und

b) ein r-dimensionaler Vollzyklus kann nur dann eine echte Teilmenge von F als seinen Träger besitzen, wenn er identisch verschwindet —

gleichwertig mit der Bedingung:

c) die r-te Brouwersche Zahl einer beliebigen echten Teilmenge F' von F ist Null —

ist.

Wenn überdies $F \subset R^n$ ist, so ist jede der Bedingungen a), b), c) mit folgender weiterer gleichwertig:

d) jeder $n-r-1$-dimensionale Zyklus in $R^n - F$ ist entweder homolog Null (in $R^n - F$) oder irreduzibel verschlungen mit F.

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[59] Einen Spezialfall dieses Satzes (für im R^n gelegene Mengen) habe ich in Fund. Math., 13 bewiesen. Dort befinden sich auch alle notwendigen Definitionen bezüglich stetiger Deformationen.

Folgende Frage bleibt dagegen unbeantwortet:

Kann es vorkommen, daß in einer n-dimensionalen Menge F für ein bestimmtes $r<n$ alle nicht identisch verschwindenden r-dimensionalen Zyklen die Menge F als ihren einzigen Träger besitzen?

25. Bereits im Kapitel III haben wir gesehen, daß eine Menge $F \subset R^n$, zu der es einen irreduzibel verschlungenen γ^{n-r-1} gibt, *mindestens* r-dimensional ist und sogar durch keine $r-2$-dimensionale Menge zerlegt werden kann. *Wenn also eine solche Menge genau r-dimensional ist, ist sie eine r-dimensionale Cantorsche Mannigfaltigkeit;*[60] sie kann aber im allgemeinen auch von einer höheren Dimension sein: es ist leicht eine *zweidimensionale* $F \subset R^3$ zu konstruieren, deren einzige von Null verschiedene Brouwersche Zahl, nämlich die erste, den Wert 1 hat und die Eigenschaft besitzt, daß jeder eindimensionale Zyklus in $R^3 - F$ entweder homolog Null oder irreduzibel verschlungen mit F ist. Im Falle $r = n-1$ (d. h. im Falle, wo F eine nulldimensionale irreduzible Verschlingung zuläßt, also die Grenze zweier Gebiete ist) ist dagegen $\dim F = n-1$.

26. Diese Überlegungen lassen folgende Definition als berechtigt erscheinen:

Definition einer geschlossenen Cantorschen Mannigfaltigkeit: *Eine Cantorsche Mannigfaltigkeit von der Dimension m heißt geschlossen, wenn ihre m-te Brouwersche Zahl von Null verschieden ist, während die m-te Brouwersche Zahl jeder echten abgeschlossenen Teilmenge von F den Wert Null hat.*

Eine m-dimensionale geschlossene Cantorsche Mannigfaltigkeit heiße *regelmäßig* geschlossen, wenn ihre m-te Brouwersche Zahl den Wert 1 hat.

Wir können dann die letzterwähnten Tatsachen wie folgt zusammenfassen:

Satz III. *Eine abgeschlossene Menge $F \subset R^n$ hat dann und nur dann die Eigenschaft, den Raum zu zerlegen und gleichzeitig die Grenze aller durch sie bestimmten zusammenhängenden Gebiete zu sein, wenn sie eine geschlossene $n-1$-dimensionale Cantorsche Mannigfaltigkeit ist. Wenn sie regelmäßig geschlossen ist, und nur in diesem Falle, ist die Anzahl der Gebiete 2. Eine im R^n gelegene r-dimensionale geschlossene Cantorsche Mannigfaltigkeit hat die Eigenschaft, daß es in ihrem Komplementärraume $n-r-1$-dimensionale Zyklen gibt, die daselbst nicht homolog Null sind; alle solche Zyklen sind mit der gegebenen Mannigfaltigkeit irreduzibel verschlungen, wobei durch diese Eigenschaft die geschlossenen r-dimensionalen Cantorschen Mannigfaltigkeiten unter allen im R^n gelegenen r-dimensionalen Mengen F charakterisiert sind.*

27. Wir beweisen jetzt folgende wesentliche Verschärfung des Satzes IV des Kap. III:

[60] Die Definition einer Cantorschen Mannigfaltigkeit ist in der Fußnote [43] (Kap. III) wiedergegeben.

SATZ IV. *Wenn eine $r+1$-dimensionale geschlossene Cantorsche Mannigfaltigkeit F als Summe zweier echter abgeschlossener Teilmengen F' und F'' dargestellt ist, so enthält die Durchschnittsmenge $\Phi = F' \cdot F''$ mindestens einen r-dimensionalen Vollzyklus $\mathfrak{r}$, welcher zwar nicht in Φ, aber sowohl in F' als auch in F'' homolog Null ist.*

Beweis. Es sei

$$F = F' + F'' \tag{58}$$

und

$$\mathfrak{s} = (s_1, s_2, \cdots, s_m, \cdots), \qquad s = r+1, \tag{59}$$

ein $r+1$-dimensionaler Vollzyklus, der in F nicht ~ 0 ist. Es möge dabei s_m ein δ_m-Zyklus und $\lim_{m\to\infty} \delta_m = 0$ sein.

Man bezeichne durch K'_m den Teilkomplex von s_m, welcher aus allen zu F' nicht fremden Grundsimplexen besteht; K''_m bestehe dann definitionsgemäß aus allen übrigen Grundsimplexen von s_m. Die beiden Komplexe K'_m und K''_m haben denselben Rand r_m und man kann annehmen, daß (nach einer eventuellen δ_m-Verrückung) r_m zu Φ gehört.

Ich behaupte, daß

$$(r_1, r_2, \cdots, r_m, \cdots) \tag{60}$$

ein Vollzyklus in Φ ist. Da r_m ein δ_m-Zyklus in Φ ist, genügt es zu zeigen, daß man die ε_m unter der Bedingung $\lim_{m\to\infty} \varepsilon_m = 0$ so wählen kann, daß $r_m \underset{\varepsilon_m}{\sim} r_{m+h}$ ist. Man nehme zu diesem Zwecke die durch

$$s_m \underset{\varepsilon_m}{\sim} s_{m+h} \text{ (in } F) \tag{61}$$

bestimmten ε_m. Es sei $\bar{K}_{m,h} \to s_m + s_{m+h}$ und $\bar{K}'_{m,h}$ die Gesamtheit aller zu F' nicht fremden Grundsimplexe von $\bar{K}_{m,h}$; der Rand von $\bar{K}'_{m,h}$ ist $K'_m + K'_{m+h} + q$, wobei q ein eventuell (und zwar im Falle $r_m = r_{m+h}$) verschwindender ε_m-Komplex in Φ ist; jedenfalls ist aber $q \to r_m + r_{m+h}$, d. h. $r_m \underset{\varepsilon_m}{\sim} r_{m+h}$ (in Φ).

Den Vollzyklus (60) bezeichnen wir durch $\mathfrak{r}$; aus seiner Definition folgt, daß $\mathfrak{r}$ sowohl in F' als auch in F'' homolog Null ist; es bleibt nur noch übrig, zu zeigen, daß $\mathfrak{r}$ in Φ nicht homolog Null ist.

Es sei im entgegengesetzten Falle

$$Q_m \underset{\eta_m}{\rightarrow} r_m \text{ (in } \Phi), \qquad \lim_{m\to\infty} \eta_m = 0.$$

Da die $r+1$-te Brouwersche Zahl für F' und F'' verschwindet, gibt es für jedes ε ein solches m_ε, daß für alle $m > m_\varepsilon$ die Zyklen

$$s'_m = Q_m + K'_m \quad \text{bzw.} \quad s''_m = Q_m + K''_m$$

in F' bzw. in F'', also jedenfalls in F ε-homolog Null sind. Dann müßte aber auch $s'_m + s''_m = s_m$ (für alle $m > m_\varepsilon$) ε-homolog Null, also (da ε beliebig ist) $\mathfrak{s} \sim 0$ in F sein, was unserer Voraussetzung widerspricht.

28. Genau auf dieselbe Weise wie den Satz IV beweist man folgende Tatsache: *Wenn* $\gamma^{n-r-2} \sim 0$ *in* $R^n - F'$ *und* $R^n - F''$, *aber nicht in* $R^n - (F' + F'')$ *ist, so gibt es einen Vollzyklus* $\mathfrak{r}$ *in* $F' \cdot F''$, *welcher in* F' *sowie in* F'', *aber nicht in* $F' \cdot F''$ *homolog Null ist.*

Man konstruiert in der Tat zuerst einen mit γ^{n-r-2} verschlungenen Vollzyklus (59) und sodann den durch (60) gegebenen $\mathfrak{r}$, der in F' und F'' homolog Null ist. Wenn nun angenommen wird, daß $\mathfrak{r} \sim 0$ (in $F' \cdot F''$) ist, so existieren wieder die Zyklen s'_m und s''_m; da aber γ^{n-r-2} mit $\mathfrak{s}$, also mit allen s_m verschlungen ist, muß γ^{n-r-2} auch mit unendlich vielen s'_m oder s''_m verschlungen sein; wenn dementsprechend für unendlich viele m z. B. $\mathfrak{v}(s'_m, \gamma^{n-r-2}) = 1$ und $F''' \subset F'$ der topologische Limes der s'_m ist, so kann γ^{n-r-2} in $R^n - F'''$, also erst recht nicht in $R^n - F'$ homolog Null sein, womit unsere Annahme als widerspruchsvoll erwiesen ist.

29. In der soeben bewiesenen Tatsache ist der Beweis der einen Hälfte des folgenden Satzes enthalten:

Satz V (verallgemeinerter Phragmén-Brouwerscher Satz). *Wenn jeder* $n - r - 1$*-dimensionale Zyklus in* $R^n - F'$ *bzw. in* $R^n - F''$ *daselbst homolog Null ist, so gibt es dann und nur dann einen* γ^{n-r-1} *der in* $R^n - (F' + F'')$ *nicht homolog Null ist, wenn ein* $r - 1$*-dimensionaler Vollzyklus in* $F' \cdot F''$ *existiert, welcher in* F' *und* F'', *aber nicht in* $F' \cdot F''$ *homolog* 0 *ist.*

Die wichtigste Folgerung aus diesem Satz ist

Der verallgemeinerte Phragmén-Brouwersche Zerlegungssatz. *Wenn keine der beiden Mengen* F' *und* F'' *den* R^n *zerlegt, so zerlegt die Menge* $F' + F''$ *den Raum dann und nur dann, wenn es einen* $n - 2$*-dimensionalen Vollzyklus in* $F' \cdot F''$ *gibt, welcher in* F' *und* F'', *aber nicht in* $F' \cdot F''$ *homolog Null ist.*

Um den Satz V zu beweisen, bleibt es nach dem Resultat des § 28 nur übrig zu zeigen, daß aus der Existenz eines $\mathfrak{r}$ in $F' \cdot F''$, welcher in F' und F'', aber nicht in $F' \cdot F''$ homolog Null ist, die Existenz eines γ^{n-r-2} folgt, welcher in $R^n - (F' + F'')$ nicht homolog Null ist.

Das geschieht aber mit Hilfe einer Lebesgue-Alexanderschen Methode[61] in wenigen Worten. Zunächst folgt aus unseren Voraussetzungen die Existenz eines in $R^n - F' \cdot F''$ gelegenen, mit $\mathfrak{r}$ verschlungenen γ^{n-r-1}; *da* $\mathfrak{r}$ *in* F' *sowie in* F'' *homolog Null ist, hat jeder mit* $\mathfrak{r}$ *verschlungene Zyklus*

[61] Lebesgue, Comptes Rendus, 154 (séance du 27 mars 1911); Alexander, Trans. Amer. Math. Soc., 23 (1922), S. 343.

mit jeder der beiden Mengen F' und F'' gemeinsame Punkte. Man betrachte jetzt die beiden zueinander fremden abgeschlossenen Mengen $\gamma^{n-r-1} \cdot F'$ und $\gamma^{n-r-1} \cdot F''$ und eine so feine Unterteilung $\mathfrak{z}$ des Komplexes γ^{n-r-1}, daß kein Simplex dieser Unterteilung gleichzeitig mit F' und F'' gemeinsame Punkte hat; es seien sodann K_1^{n-r-1} der aus allen zu F' nicht fremden Simplexen dieser Unterteilung gebildete Teilkomplex von γ^{n-r-1}, K_2^{n-r-1} der aus den übrigen Simplexen von $\mathfrak{z}$ bestehende Komplex und γ^{n-r-2} der gemeinsame Rand von K_1^{n-r-1} und K_2^{n-r-1}; ich behaupte, daß γ^{n-r-2} in $R^n - (F' + F'')$ nicht homolog Null sein kann.

Es sei in der Tat $Q^{n-r-1} \to \gamma^{n-r-2}$ (in $R^n - (F' + F'')$); da $K_1^{n-r-1} + Q^{n-r-1}$ zu F'' und $K_2^{n-r-1} + Q^{n-r-1}$ zu F' fremd ist, kann keiner dieser Zyklen mit $\mathfrak{r}$ verschlungen sein; dann kann aber auch ihre Summe (mod. 2) nicht mit $\mathfrak{r}$ verschlungen sein, was unmöglich ist, weil diese Summe $K_1^{n-r-1} + K_2^{n-r-1} = \gamma^{n-r-1}$ ist.

Der Satz V ist hiermit bewiesen.

30. An die Definitionen des § 15 schließen sich unmittelbar einige Betrachtungen an, welche das mehrdimensionale Analogon des Begriffes eines zwischen zwei gegebenen Punkten irreduziblen Kontinuums darstellen.

Definition IV. *Es sei F' eine echte oder unechte Teilmenge von F, die als Homologieträger eines ihrer nicht identisch verschwindenden Vollzyklen $\mathfrak{r} \sim 0$ (in F) auftritt; wenn keine echte Teilmenge von F' als Träger derselben Homologie betrachtet werden kann, heißt F' eine in bezug auf $\mathfrak{r}$ irreduzible Membran.*

Ein Punktepaar a, b kann stets als ein nulldimensionaler Vollzyklus einer Menge $F \supset a + b$ betrachtet werden, und man sieht sofort ein, daß F dann und nur dann eine in bezug auf diesen Zyklus irreduzible Membran darstellt, wenn F ein zwischen a und b irreduzibles Kontinuum ist. Die irreduziblen Membranen treten also als eine naturgemäße Verallgemeinerung der irreduziblen Kontinuen auf.

Es ist leicht zu zeigen, daß, wenn die Mengen $F_1, F_2, \cdots, F_m, \cdots$ Träger einer und derselben Homologie $\mathfrak{r} \sim 0$ (in F) sind und überdies eine gegen eine Menge F' konvergierende Folge bilden, so ist F' ebenfalls ein Träger der Homologie $\mathfrak{r} \sim 0$[55]; da andererseits eine abnehmende Folge abgeschlossener Mengen immer gegen den Durchschnitt dieser Mengen konvergiert, so ist der Durchschnitt einer abnehmenden Folge von Trägern einer und derselben Homologie wiederum ein Träger dieser Homologie, m. a. W., die Eigenschaft einer abgeschlossenen Menge $F' \subset F$ ein Träger einer *festen* Homologie $\mathfrak{r} \sim 0$ (in F) zu sein, ist eine *induktive* Eigenschaft. Daraus folgt aber der

Satz VI. *Jeder Träger einer Homologie $\mathfrak{r} \sim 0$ (in F) enthält eine in bezug auf den Zyklus $\mathfrak{r}$ irreduzible Membran.*

31. Unter den Vollzyklen $\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots)$ sind solche besonders ausgezeichnet, bei denen unendlich viele Zyklen r *irreduzible*[62] Zyklen sind. Solche Vollzyklen sollen *irreduzible Vollzyklen* genannt werden. Von ihnen gilt der

SATZ VII. *Eine in bezug auf einen r-dimensionalen irreduziblen Vollzyklus $\mathfrak{r}$ irreduzible Membran kann durch keine höchstens $r-1$-dimensionale Menge zerlegt werden.* Eine solche Membran ist also (wenn sie nicht von einer höheren Dimension als $r+1$ ist) *eine $r+1$-dimensionale Cantorsche Mannigfaltigkeit.*

Beweis. Es sei $\mathfrak{r} \sim 0$ (in F), F''' ein Träger dieser Homologie, welcher durch eine höchstens $r-1$-dimensionale Menge Φ in zwei zueinander fremde offene Mengen G' und G'' zerlegt wird; wir wollen zeigen, daß diese Voraussetzungen mit der Irreduzibilität von $\mathfrak{r}$ unverträglich sind.

Es sei in der Tat

$$\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots), \tag{62}$$

wobei die r_m sämtlich irreduzible Zyklen sind. Es seien ferner die η_m-Komplexe ($\lim_{m\to\infty} \eta_m = 0$)

$$Q_m \to r_m \tag{63}$$

mit $\lim \operatorname{top} Q_m = F'''$ gegeben. Man bezeichne mit Q'_m den aus allen zu $F' = G' + \Phi$ nicht fremden Grundsimplexen von Q_m bestehenden Komplex, mit Q''_m die Gesamtheit aller übrigen Grundsimplexe von Q_m.

Durch eine "kleine Transformation" der Komplexe Q_m (die die Konvergenz nicht stört) kann man erreichen, daß sämtliche den beiden Komplexen Q'_m und Q''_m gemeinsame Elemente bei jedem m auf den, die Menge Φ approximierenden, höchstens $r-1$-dimensionalen Komplexen liegen. Somit haben die beiden $r+1$-dimensionalen Komplexe Q'_m und Q''_m einen höchstens $r-1$-dimensionalen Durchschnitt, woraus folgt, daß der Rand von Q'_m sowie der Rand von Q''_m im Rande von Q_m, d. h. in r_m enthalten ist; da aber r_m irreduzibel ist, und die Ränder von Q'_m und Q''_m nicht identisch sein können (ohne daß r_m verschwindet, was ja nicht der Fall ist), so muß der Rand des einen unter den beiden Komplexen Q'_m und Q''_m mit r_m identisch sein, während der Rand des anderen verschwindet. Es sei dementsprechend für unendlich viele m etwa $Q'_m \to r_m$, $Q''_m \to 0$; diese Q'_m approximieren dann eine in F' enthaltene Menge $\overset{*}{F}$, die ein Träger unserer Homologie ist.

Der Satz VII ist hiermit bewiesen.

32. An die hier gegebene Definition der irreduziblen Membran schließen

[62] Ein Zyklus heißt nach Veblen irreduzibel ("circuit"), wenn er keinen echten Teilzyklus von derselben Dimension, wie der gegebene Zyklus selbst, enthält.

sich zahlreiche Probleme an, die sich alle in die Fragestellung einordnen, inwiefern die Theorie der irreduziblen Kontinua auf allgemeine irreduzible Membranen sich übertragen läßt.[63]

PRINCETON, N. J.,
15. Januar bis 15. April 1928.

Anhang I.

BEMERKUNGEN ÜBER DEN BEGRIFF EINES VOLLZYKLUS UND SEINES TRÄGERS.

Wenn man einen Vollzyklus

$$\mathfrak{r} = (r_1, r_2, \cdots, r_m, \cdots) \tag{1}$$

hat, so ist es natürlich, als *erlaubte Operationen* folgende beide Operationen einzuführen, die den Vollzyklus $\mathfrak{r}$ in einen Vollzyklus $\mathfrak{r}'$ transformieren, welchen wir als einen von $\mathfrak{r}$ nur *unwesentlich* verschiedenen Vollzyklus betrachten (insbesondere ist stets $\mathfrak{r} \sim \mathfrak{r}'$):

a) Übergang von (1) zu

$$\mathfrak{r}' = (r'_1, r'_2, \cdots, r'_k, \cdots),$$

wobei $r'_k = r_{m_k}$ ist (und die r'_k also eine Teilfolge der Folge r_m bilden).

Diese Operation erlaubt stets einen Vollzyklus durch einen *konvergenten* Vollzyklus zu ersetzen; (vgl. Fußnote [56]).

b) Die sogenannten *unendlich-kleinen Abänderungen* eines Vollzyklus (1). Diese bestehen darin, daß man eine gegen Null konvergierende Folge positiver Zahlen

$$\vartheta_1, \vartheta_2, \cdots, \vartheta_m, \cdots \tag{2}$$

wählt und jedes r_m in ein r'_m dadurch verwandelt, daß man die Eckpunkte von r_m höchstens um ϑ_m verschiebt.

Am bequemsten erscheint es, folgende (von Lefschetz herrührende) Modifikation des Begriffes eines Vollzyklus einer im (kompakten metrischen) Raume F liegenden abgeschlossenen Menge F' einzuführen: eine Folge (1)

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[63] Für eine solche Untersuchung dürfte insbesondere folgender Begriff des *r-dimensionalen lokalen Zusammenhanges* einer abgeschlossenen Menge von Wichtigkeit sein: F heißt r-dimensional lokal zusammenhängend, wenn für jedes ε man ein solches δ finden kann, daß jeder in einer δ-Umgebung eines beliebigen Punktes von F liegende höchstens r-dimensionale Vollzyklus in einer ε-Umgebung desselben Punktes homolog Null ist.

Wie ich persönlichen Mitteilungen Herrn Alexanders entnehme, ist auch ihm dieser Begriff bekannt. Als eine natürliche Verallgemeinerung des entsprechenden eindimensionalen Satzes, könnte man sich z. B. folgende Frage stellen: Kann man behaupten, daß jede in bezug auf eine einfache geschlossene Kurve irreduzible Membran, deren sämtliche Brouwerschen Zahlen verschwinden und die zweidimensional lokal zusammenhängend ist, notwendig ein zweidimensionales Element darstellt?

ist ein Vollzyklus in F', wenn die Zyklen r_m Zyklen *in* F sind (im Sinne der Definitionen des § 5 des Kap. IV), die folgender Bedingung genügen:

zu jedem $\varepsilon > 0$ läßt sich eine natürliche Zahl k bestimmen, so daß für alle $h = 0, 1, 2, \cdots$ die Zyklen r_{k+h} ε-Zyklen sind, die sämtlich in $S(F', \varepsilon)$ liegen und dortselbst untereinander ε-homolog sind.

Offenbar geht jeder Vollzyklus von F' (in diesem neuen Sinne) in einen Vollzyklus von F' (in unserem früheren Sinne) mittels einer unendlich kleinen Abänderung über.

Es ist bequem, sich auch für den Begriff einer Homologie

$$\mathfrak{r} \sim 0 \text{ (in } F', F' \subset F)$$

einer analogen Modifikation zu bedienen.

Für die so modifizierten Begriffe beweist man mühelos folgende Tatsachen:

1. Der Durchschnitt zweier Träger von $\mathfrak{r}$ ist ein Träger von $\mathfrak{r}$.
2. Wenn jede der abnehmenden abgeschlossenen Mengen

$$F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots \tag{3}$$

Träger des Vollzyklus $\mathfrak{r}$ ist, so ist auch der Durchschnitt F_ω aller Mengen (3) Träger desselben Zyklus (es genügt ja zu bemerken, daß die Mengen F_n zu ihrem Durchschnitt *konvergieren*). Aus diesen beiden Eigenschaften folgt, daß der Durchschnitt aller Träger von $\mathfrak{r}$ wiederum ein Träger (und zwar der *kleinste* Träger) von $\mathfrak{r}$ ist. Wenn man bei unserer ursprünglichen Definition bleiben will, so ist letztere Behauptung dahin zu modifizieren, daß es für jeden Vollzyklus $\mathfrak{r}$ eine kleinste Menge gibt, die als Träger eines von $\mathfrak{r}$ durch unendlich kleine Abänderung entstandenen Vollzyklus auftritt.

Analoges ist *mutatis mutandis* auch für Homologieträger zu bemerken. Es wäre eigentlich am bequemsten, jede Menge $F' \subset F$, in welcher der Vollzyklus $\mathfrak{r}$ homolog Null ist, als Träger der Homologie $\mathfrak{r} \sim 0$ zu bezeichnen und dabei nur vorauszusetzen, daß die Komplexe $Q_m^{r+1} \to r_m$ in $S(F', \varepsilon_m)$, $\lim_{m\to\infty} \varepsilon_m = 0$, liegen. Ebenso wie im Falle der Vollzyklen selbst könnte man von kleinsten Trägern einer Homologie sprechen (das sind dann gerade die "irreduziblen Menbranen"); nur besitzt im allgemeinen eine Homologie mehrere kleinste Träger.

Für die so gewonnene neue Definition eines Homologieträgers gilt ohne weiteres der Satz II des Kap. IV sowie die Theorie der §§ 16–30. Man sieht übrigens, daß jeder Homologieträger in diesem neuen Sinne einen Homologieträger im Sinne der Def. II des § 15 enthält, während jeder

Homologieträger im alten Sinne erst recht ein solcher im neuen Sinne ist. Deshalb ist die ganze Abänderung der Definition — abgesehen vielleicht von ihren methodologischen Vorzügen — nur von geringer Bedeutung.

Anhang II.*

Der geometrische Dimensionsbegriff.

Die obige Darstellung regt zur Einführung einer (wenigstens a priori) neuen topologischen Invariante ein, welcher wir den Namen der *geometrischen Dimension* einer abgeschlossenen Menge geben.

Definition der geometrischen Dimension. *Unter der geometrischen Dimension der abgeschlossenen Menge F wird die größte Zahl $\Delta(F) = r+1$ verstanden von der Eigenschaft, daß es in F einen nicht identisch verschwindenden r-dimensionalen Vollzyklus $\mathfrak{r}$ gibt, der in F homolog Null ist. Wenn es einen solchen Zyklus für kein r gibt, ist definitionsgemäß $\Delta(F) = 0$; wenn zu jedem noch so großen r ein in F berandender*[64] *nicht identisch verschwindender r-dimensionaler Vollzyklus existiert, heißt F unendlichdimensional.*

1. Wir werden im folgenden unter $\Delta(F)$ stets die geometrische, unter $\dim F$ die gewöhnliche ("mengentheoretische") Dimension der Menge F verstehen. Aus dem Satz II des Kap. IV folgt sodann ohne weiteres, daß $\dim F \geqq \Delta(F)$ für jedes F ist. Daraus folgt wieder, daß die geometrische Dimension eines (im gewöhnlichen Sinne) n-dimensionalen Komplexes gleich n ist. Ebenso leicht überzeugt man sich davon, daß für eine im R^n liegende abgeschlossene Menge F dann und nur dann $\Delta(F) = n$ gilt, wenn F innere Punkte enthält. Aus dem Hilfssatz des § 21 (Kap. IV) folgt ferner, daß der soeben eingeführte Dimensionsbegriff *dem Brouwerschen Invarianzprinzip genügt,* d. h., daß die geometrische Dimension nicht durch k. T. erniedrigt werden kann.

Weitere Sätze beziehen sich auf Teilmengen Euklidischer Räume. Man beweist zunächst leicht (mit Hilfe von Kap. IV, Satz V):

I. *Wenn F einen nicht identisch verschwindenden $r+1$-dimensionalen*

* In diesem Anhang sollen einige Resultate zusammengestellt werden, welche unmittelbar an die vorliegende Arbeit (insbesondere an das Kap. IV) anschließen und einerseits für diese Arbeit einen gewissen Abschluß bilden, andererseits aber zu ganz neuen Fragestellungen führen. Hier sollen nur kurz die Resultate selbst skizziert werden; eine ausführliche Darstellung (insbesondere die Beweise) findet der Leser in meiner Arbeit "*Zum allgemeinen Dimensionsproblem*" (Göttinger Nachrichten, Sitzung 6. Juli 1928). Es sei noch bemerkt, daß die in letzterer Arbeit (Fußnote [12]) irrtümlicherweise gegebene Definition des Trägers eines Zyklus durch unsere Definition I (Kap. IV, § 15) zu ersetzen ist (beide Definitionen sind übrigens äquivalent — bis auf erlaubte Abänderungen von Vollzyklen im Sinne des Anhanges I).

[64] Berandet = homolog Null.

Zyklus enthält ($r \geqq 0$), *so enthält* F *auch einen berandenden nicht identisch verschwindenden r-dimensionalen Zyklus.*

Daraus ergibt sich unmittelbar:

II. *Wenn die r-te Brouwersche Zahl von* F *von Null verschieden ist, so ist* $\Delta(F) \geqq r$.

III. *Wenn* $\Delta(F) = m$ *ist, so gibt es zu jedem* r, $0 \leqq r \leqq m-1$, *einen nicht identisch verschwindenden berandenden r-dimensionalen Vollzyklus.*

Aus dem Satz I folgt, daß, wenn $F \subset R^n$ ist und der Zyklus γ^{n-r-1} in $R^n - F$ liegt und dort nicht berandet, $\Delta(F) \geqq r$ ist. Insbesondere hat die Begrenzung eines beschränkten Gebietes des R^n stets die geometrische Dimension $n-1$.

2. Definition I. *Es sei* $F \subset R^n$ *und* a *ein Punkt von* F. *Wir sagen, daß* F *in der Nähe von* a *ein r-dimensionales Hindernis darstellt, wenn es eine positive Zahl* ε *von der Beschaffenheit gibt, daß bei jedem noch so kleinen* δ *in* $(R^n - F) \cdot S(a, \delta)$ *Zyklen* γ^{n-r-1} *liegen, welche in* $(R^n - F) \cdot S(a, \varepsilon)$ *nicht beranden.*

Als *Rechtfertigungssatz* für den neuen Dimensionsbegriff ist folgender Satz zu betrachten:

IV. *Es sei* $F \subset R^n$; $\Delta(F)$ *ist dann und nur dann gleich r, wenn* F *in der Nähe mindestens eines Punktes ein r-dimensionales Hindernis und nirgends ein Hindernis höherer Dimension darstellt.*

Beim Beweise dieses Satzes spielt der Begriff einer *ε-Modifikation* eines gegebenen Komplexes K^m eine wesentliche Rolle. Darunter wird eine Abänderung des Komplexes K^m verstanden, die darin besteht, daß man jedes Simplex T_i^r von K^m ($0 \leqq r \leqq m$) durch einen (eventuell singulären) Polyederkomplex Z_i^r ersetzt, und zwar so, daß dabei folgende Bedingungen erfüllt seien:

a) wenn $T_i^r \to \lambda^j T_j^{r-1}$ ist, so ist $Z_i^r \to \lambda^j Z_j^{r-1}$;

b) es gibt für jedes r und jedes i einen Punkt $x_i^r \subset T_i^r$, so daß $T_i^r + Z_i^r \subset S(x_i^r, \varepsilon)$ ist; (der Komplex K^m ist in einer so feinen Simplizialzerlegung zu denken, daß seine sämtlichen Simplexe T_i^r kleiner als ε sind).

Der Beweis von IV stützt sich im wesentlichen auf den verallgemeinerten Phragmén-Brouwerschen Satz (Satz V des Kap. IV) und ergibt gleichzeitig mit dem Rechtfertigungssatz folgendes Ergebnis:

IV′. *Eine im* R^n *gelegene abgeschlossene Menge* F *hat dann und nur dann die geometrische Dimension r, wenn bei jedem* ε *jeder höchstens* $n-r-1$*-dimensionale Komplex mittels einer ε-Modifikation von Punkten der Menge* F *befreit werden kann, während ein solcher* $n-r$*-dimensionaler Komplex* K^{n-r} *und ein solches* ε *gefunden werden können, daß* K^{n-r} *und jede ε-Modifikation desselben mit* F *gemeinsame Punkte hat.*

Zum letzteren Satz ist zu bemerken, daß — wie man dem bekannten Beispiel von Antoine mühelos entnimmt — in seiner Behauptung die

ε-Modifikationen der Komplexe nicht durch ε-Deformationen ersetzt werden können. Dagegen gilt folgendes Resultat:

IV″. *Wenn $F \subset R^n$ ist, so ist dann und nur dann $\dim F = r$, wenn man bei jedem ε jeden höchstens $n - r - 1$-dimensionalen (aber nicht jeden $n - r$-dimensionalen) Komplex mittels ε-Deformationen der Menge F von Punkten letzterer Menge befreien kann.*

3. Die weitere Untersuchung der geometrischen Dimension geschieht auf Grund folgender Begriffsbildungen.

Definition II. *Es sei $\Delta(F) = m$ und a ein Punkt von F; a heißt ein direkter Kernpunkt von F, wenn es ein solches ε gibt, daß bei jeder ε-Aussonderung*[65] *$F = A + B + D$ des Punktes a die Menge B einen $m - 1$-dimensionalen Vollzyklus enthält, welcher in $A + B$, aber nicht in B homolog Null ist. Ein direkter Kernpunkt einer abgeschlossenen Teilmenge von F heißt schlechthin Kernpunkt von F.* Ein Punkt von F, der kein Kernpunkt ist, heißt ein *absoluter Randpunkt* von F; die Gesamtheit aller solchen Punkte bildet den *absoluten Rand* von F.

Jede abgeschlossene Menge (endlicher geometrischer Dimension) enthält Kernpunkte; es gilt sogar mehr: der absolute Rand einer m-dimensionalen Menge kann höchstens $m - 1$-dimensionale abgeschlossene Mengen enthalten. Das alles ist in einem für die ganze Theorie grundlegenden Satze enthalten, zu dessen Formulierung wir wieder eine neue Definition brauchen:

Definition III. Eine abgeschlossene Menge Φ heißt eine *Kernmenge*, wenn $\Phi = \bar{G}$ ist, wobei G eine aus lauter direkten Kernpunkten von Φ bestehende (in Φ) offene Menge ist.

Sodann lautet unser Satz folgendermaßen:

V. *Jede abgeschlossene Menge von der geometrischen Dimension m enthält eine m-dimensionale Kernmenge, und zwar eine solche, die durch keine höchstens $m - 2$-dimensionale abgeschlossene Teilmenge zerlegt werden kann.*

Auch beim Beweise dieses Satzes bildet der verallgemeinerte Phragmén-Brouwersche Satz das wesentliche Hilfsmittel; es zeigt sich überdies beim Beweise, daß die m-dimensionale Kernmenge $\Phi = \bar{G}$, deren Existenz die Behauptung des Satzes IV bildet, folgende fundamentale Eigenschaft hat:

Wenn man in der abgeschlossenen Menge Φ die Teilmenge $\Psi = \bar{G} - G$ durch einen Punkt ξ ersetzt[66]**, so entsteht ein m-dimensionales geschlossenes Gebilde $\Phi^* = G + \xi$.**

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[65] Eine Zerlegung $F = A + B + D$ der abgeschlossenen Menge F in drei paarweise zueinander fremde Teilmengen heißt eine ε-Aussonderung des Punktes a, wenn A und D offen sind (also B abgeschlossen ist) und überdies $a \subset A$, $A + B \subset S(a, \varepsilon)$ ist; wichtig ist, daß im Fall $\varepsilon' < \varepsilon$ jede ε'-Aussonderung zugleich eine ε-Aussonderung ist.

[66] Der Punkt ξ erhält als Umgebungen die Mengen $U(\xi) = \xi + (\Gamma - \Psi)$, wo Γ eine beliebige, die Menge Ψ enthaltende, offene Teilmenge von Φ ist.

(Dabei sagen wir, daß eine abgeschlossene Menge F ein *m-dimensionales geschlossenes Gebilde* ist, wenn $\Delta(F) = m$ ist und wenn überdies man F in einen R^n so einbetten kann, daß es einen mit F irreduzibel verschlungenen Zyklus γ^{n-r-1} in $R^n - F$ gibt; eine invariante Charakterisierung der geschlossenen Gebilde ist in den §§ 16—18 des Kap. IV gegeben.)

Ein unmittelbares Korollar des Satzes V ist:

Wenn G im Kleinen kompakt ist, so ist die geometrische Dimension des (eindeutig bestimmten) kompakten Raumes $F = G + \xi$, in den G durch topologische Hinzufügung eines Punktes übergeht[67], *gleich dem Maximum der Dimensionen der kompakten abgeschlossenen Teilmengen von G.*

Auch folgender Additionssatz ergibt sich leicht aus dem Satz V:

VI. *Wenn* $F = \sum_{k=1}^{\infty} F_k$ *und* $\Delta(F) = m$ *ist, so ist für mindestens ein* k $\Delta(F_k) = m$.

4. Die fundamentale Frage: *ob stets $\Delta(F) = \dim F$ ist,* bleibt bis jetzt unentschieden. Sie ordnet sich in folgende Fragestellung ein, die ich als *das allgemeine Dimensionsproblem* bezeichne und als die vielleicht aktuellste gegenwärtige Frage der mengentheoretischen Topologie (im weitesten Sinne des Wortes!) betrachte:

Gegeben seien eine abgeschlossene Menge F, eine natürliche Zahl n und eine ganze Zahl r, $0 \leqq r \leqq n-1$; man nehme an, daß zu jedem r-dimensionalen Vollzyklus $\mathfrak{r}$ der Menge F man eine zum Träger von $\mathfrak{r}$ fremde höchstens $n-r-1$-dimensionale Menge Φ finden kann, so daß $\mathfrak{r}$ in $F-\Phi$ nicht berandet[68] *und daß es Vollzyklen $\mathfrak{r}$ gibt, für die man letzteres Ergebnis mit keiner Menge Φ von einer Dimension $< n-r-1$ erreichen kann. In welchen Fällen läßt sich aus obiger Sachlage schließen, daß F (im gewöhnlichen bzw. in unserem Sinne) n-dimensional ist?*

Anhang III.*

Geschlossene Cantorsche Mannigfaltigkeiten und irreduzible Membranen.

Aus dem verallgemeinerten Phragmén-Brouwerschen Satz (Kap. IV, § 29, Satz V), ergibt sich leicht folgender, wie mir scheint, wichtiger

Satz VIII. *Jede m-dimensionale geschlossene Cantorsche Mannigfaltigkeit F kann stets (und zwar auf unendlich viele Weisen) als Vereinigungsmenge*

[67] Vgl. Alexandroff, Math. Ann. 92, (1924), S. 294.

[68] Es gibt verschiedene Möglichkeiten zu definieren, wann ein Vollzyklus eines *im Kleinen kompakten* Raumes nicht berandet; vgl. hierzu die schon zitierte Arbeit "Zum allgemeinen Dimensionsproblem", Fußnoten [29] und [8].

* Zu diesem Anhang bin ich von der Druckerei angeregt worden, welche auf diese Weise die durch die vorigen Anhänge zerstörte Parität der Seitenzahl des Heftes herstellen wollte.

zweier (in bezug auf denselben $m-1$-dimensionalen Vollzyklus $\mathfrak{s}$) irreduzibler Membranen F' und F'' dargestellt werden, so daß überdies $F' = \bar{G}_1$ und $F'' = \bar{G}_2$ ist, wobei G_1 und G_2 zwei zueinander fremde nicht leere offene Teilmengen von F sind (und $F' \cdot F''$ in F nirgends dicht ist).[69]

Beweis. Vorbemerkung. Auf Grund des Mengerschen Einbettungssatzes können und wollen wir annehmen, daß F eine Teilmenge von R^n ist (n hinreichend groß).

Es sei Φ eine beliebige nirgends dichte abgeschlossene Teilmenge von F, durch die letztere Menge zerlegt wird:

$$F = G_1 + \Phi + G_2, \qquad F = \bar{G}_1 + \bar{G}_2, \qquad G_1 \cdot G_2 = 0.$$

Vermöge des verallgemeinerten Phragmén-Brouwerschen Satzes existiert in $\bar{G}_1 \cdot \bar{G}_2$ ein $m-1$-dimensionaler Vollzyklus $\mathfrak{s}$, welcher in $F_1 = \bar{G}_1$ und $F_2 = \bar{G}_2$, jedoch nicht in $F_1 \cdot F_2$ berandet. Vermöge des Satzes VI (Kap. IV) existiert sodann eine in bezug auf $\mathfrak{s}$ irreduzible Membran $F' \subset F_1$ und eine ebensolche Membran $F'' \subset F_2$. Die abgeschlossenen Mengen F' und F'' haben folgende Eigenschaften:

1. $F' \subset R^n$; $F'' \subset R^n$;
2. jeder $n-m-1$-dimensionale Zyklus in $R^n - F'$ und jeder ebensolche Zyklus in $R^n - F''$ ist dortselbst homolog Null;
3. es gibt in $F' \cdot F''$ einen $m-1$-dimensionalen Vollzyklus (nämlich $\mathfrak{s}$), welcher in F' und in F'', aber nicht in $F' \cdot F''$ berandet.

Auf Grund des Satzes V existiert also ein $\gamma^{n-m-1} \subset R^n - (F' + F'')$, welcher dort nicht berandet; also ist $\beta^m (F' + F'') > 0$ und mithin $F' + F'' = F$, also (da $F' \subset \bar{G}_1$, $F'' \subset \bar{G}_2$ und $G_1 \cdot G_2 = 0$ ist) $F' = \bar{G}_1$, $F'' = \bar{G}_2$, womit alles bewiesen ist.

Es wäre interessant festzustellen, ob man in der Formulierung des Satzes VIII verlangen kann, daß $F' \cdot F''$ mit dem kleinsten Träger des Vollzyklus $\mathfrak{s}$ identisch sei.

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[69] Man vergleiche diesen Satz mit den bekannten Sätzen über irreduzible Kontinua und ebene Gebietsgrenzen.

SEMI-PARALLEL TRANSFORMATIONS OF LINES OF CURVATURE.*

By M. M. Slotnick.†

1. Let S and $\bar{S}$ be two surfaces in one-to-one point correspondence such that the tangent planes to the two surfaces at corresponding points, P and $\bar{P}$, are not parallel. If there exist two families of curves, one on each surface, which have the property that they correspond and have parallel tangents at corresponding points, the surfaces are said to correspond by a semi-parallel map.‡ We propose to discuss the case in which the curves corresponding by parallel tangents are lines of curvature on the surfaces, and the lines of curvature on the two surfaces correspond.

Let the two surfaces be represented parametrically so that corresponding points have the same curvilinear coördinates; i. e.,

$$(1) \qquad S\colon\ x = x(u, v),\S \qquad \bar{S}\colon\ \bar{x} = \bar{x}(u, v).$$

The families of lines of curvature, one on each surface, which correspond by parallel tangents shall be the u-curves; the other families of the lines of curvature shall be the v-curves.

2. Since the lines of curvature of a surface form a conjugate system, we must have

$$(2) \qquad x_u = aU, \qquad \bar{x}_u = \bar{a}U,$$

where U represents a triple of functions of u alone.|| For the same reason, $f = 0$; and because the lines of curvature form an orthogonal system, $F = 0$.¶ The point equation of the lines of curvature of S is

$$\theta_{uv} = \frac{\partial \log \sqrt{E}}{\partial v}\theta_u + \frac{\partial \log \sqrt{G}}{\partial u}\theta_v.$$

Consequently, as seen from the first of equations (2), G must be a function of v alone. The same is also true for the surface $\bar{S}$.

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† National Research Fellow.

‡ W. C. Graustein, Semi-parallel maps of surfaces, Annals of Math. (2), vol. 27, pp. 271–278.

§ I. e., $x_i = x_i(u, v)$, $i = 1, 2, 3$.

|| Loc. cit. § 3.

¶ The first and second fundamental quadratic forms of S are written:

$$E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad e\,du^2 + 2f\,du\,dv + g\,dv^2.$$

The v-curves on both surfaces are seen to be geodesics, and being at the same time lines of curvature, are therefore plane curves. Hence the surfaces S and $\bar{S}$ are surfaces of Monge or molding surfaces.*

Since the tangents to corresponding u-curves at corresponding points are parallel, the planes of corresponding v-curves are either parallel or coincident. Hence corresponding rulings of the director-developables of both surfaces are either parallel or coincident.

A necessary and sufficient condition that two surfaces be related by a semi-parallel map in which the lines of curvature correspond and such that the curves which correspond by parallel tangents are lines of curvature on their respective surfaces is that the surfaces be either surfaces of Monge, the edges of regression of whose director-developables are in a Combescurian relation or identical, or molding surfaces whose director-cylinders have parallel rulings or are identical.† Corresponding plane generating curves (which are lines of curvature on the surfaces) are those lying in corresponding parallel or coincidental planes; and the lines of curvature corresponding by parallel tangents are the orthogonal trajectories of these plane curves.

The sufficiency of this condition is at once evident from the definitions of sufaces of Monge and molding surfaces, when surfaces of these types are related to one another as indicated in the theorem.

3. The point equation of the lines of curvature on S is

$$x_{uv} = \frac{\partial \log \sqrt{E}}{\partial v} x_u. \tag{3}$$

Hence

$$x_u = \sqrt{E}\, U; \tag{4}$$

and therefore the triple of functions U of u alone are such that

$$U_1^2 + U_2^2 + U_3^2 = 1;$$

i. e., U represents the direction cosines of the tangential directions to the u-curves.

In order that a surface $\bar{S}: \bar{x}$ be in the relation of a semiparallel map with S of the type under discussion, the following equations must be fulfilled:

$$\bar{x}_u = \lambda U, \quad \bar{x}_v = \beta x_v + \gamma \zeta, \tag{5}$$

* Cf. L. P. Eisenhart, Differential Geometry, pp. 305 and 307.

† If corresponding generating plane curves of the two surfaces in the map are in a Combescurian relation, the map will be a parallel one. We shall include this case, and only this case, of a parallel map, as a special type of the semi-parallel maps under discussion.

where ζ represents the direction cosines of the normal to S. These equations are compatible if and only if

$$\frac{\lambda_v}{\lambda} = \frac{\beta E_v - 2\gamma e}{2\sqrt{E}}. \tag{6}$$
$$\beta_u = 0, \quad \gamma_u = 0.$$

The fact that β and γ are functions of v alone leads to the fact that $\overline{G}$ is a function of v alone (cf. equations (5)), which has already been noted.

Given, now, any arbitrary functions β and γ of v only. Then any function λ of u and v, satisfying the first of equations (6), will render equations (5) compatible; and so determine surfaces $\overline{S}:\overline{x}$ which are mapped on S by a semi-parallel map in the manner we are discussing. The existence of such maps and the relations of the surfaces in the map to one another have already been made evident geometrically above.

4. Consider now two surfaces or two molding surfaces $S:x$ and $\overline{S}:\overline{x}$, with the same director-developable, whose plane generating curves are in any one-to-one point correspondence. The lines of curvature on the surfaces will be in the relation of a semi-parallel transformation of the type being discussed here. The normal to S, whose direction cosines we indicate by ζ, lies in the plane of the generating plane curve. Consequently the coördinates $\overline{x}$ are of the form

$$\overline{x} = x + \varrho x_v + \sigma\xi.$$

From geometric considerations, ϱ and σ must be functions of v alone. This follows analytically when we demand that equations (5) be fulfilled.*

It is evident that two surfaces of Monge or two molding surfaces with the same director-developable and mapped by a semi-parallel map of our type will have their lines of curvature corresponding by a fundamental transformation.† That is to say, the developables of the congruence of the lines joining corresponding points of the two surfaces intersect the surfaces in their lines of curvature. The point of intersection, P, of the tangents to the plane generating curves at corresponding points also traces a surface of Monge or a molding surface—according as the case may be—with the same director-developable and in the relation of a semi-parallel

* If $\frac{d\sigma}{dv} = \varrho g$ we have the case in which the map of S on $\overline{S}$ is a parallel one; i. e., the correspondence between corresponding plane generating curves is Combescurian. We have agreed to include this case as a special type of our semi-parallel maps.

† L. P. Eisenhart, Transformations of Surfaces, Princeton (1923), p. 34.

map of our type with both S and $\bar{S}$. The lines of curvature of the surface traced by P is a Levy transform* of those of S and $\bar{S}$.

In order that a semi-parallel map of our type be also fundamental, the tangents to each pair of corresponding plane generating curves at corresponding points must intersect one another. If the plane generating curves lie in parallel planes, this is impossible. Hence the only case of a fundamental, semi-parallel transformation of the lines of curvature is that in which both surfaces have the same director-developable.

The lines of curvature of any two surfaces of Monge or molding surfaces having the same director-developable are related to one another by a fundamental transformation which is also semi-parallel. Conversely, if the lines of curvature of two surfaces are related to one another by a semi-parallel, fundamental transformation, the surfaces are either surfaces of Monge or molding surfaces with the same director-developable.

5. We shall now consider the case of two surfaces of Monge (not molding surfaces) with the same director-developable in more detail.

Let C be a skew curve: $y = y(u)$, the parameter u being chosen so that $du = d\sigma/\varrho$, where σ is the arc of C and $1/\varrho$ its curvature. Let α, β, γ each be triples which represent the direction cosines of the tangent, principal normal and binormal of C, respectively.

A surface S: $x = x(u, v)$, traced by a curve in the osculating plane of C may be parametrically represented by

$$x = y(u) + \lambda(u, v)\,\alpha(u) + \nu(u, v)\,\beta(u). \tag{7}$$

In order that every point x trace an orthogonal trajectory of the osculating plane of C, the coefficients of α and β in

$$x_u = (\varrho + \lambda_u - \nu)\,\alpha + (\lambda + \nu_u)\,\beta + \frac{\nu\varrho}{\tau}\,\gamma \tag{8}$$

obtained by the use of the Frenet-Serret formulas, must vanish. Here $1/\tau$ is the torsion of the curve C. The surface is a surface of Monge on which the lines of curvature are parametric, the v-curves being plane. The equation of the surface S may now be written

$$x = y - \nu_u\,\alpha + \nu\,\beta, \tag{9}$$

where $\nu_{uu} + \nu - \varrho = 0$. The equation of a second surface of Monge, $\bar{S}$, with the same director-developable is of the form:

$$\bar{x} = y - \bar{\nu}_u\,\alpha + \bar{\nu}\,\beta. \tag{10}$$

* Loc. cit., p. 18 and p. 39.

We have then

$$x_u = \frac{\nu\varrho}{\tau}\gamma, \qquad \bar{x}_u = \frac{\bar{\nu}\bar{\varrho}}{\tau}\gamma. \tag{11}$$

From these equations the torsions of corresponding u-curves are found to be

$$\frac{1}{T} = \frac{\tau}{\nu\varrho}, \qquad \frac{1}{\bar{T}} = \frac{\tau}{\bar{\nu}\bar{\varrho}}; \tag{12}$$

i. e., from (11)

$$x_u = T\gamma, \qquad \bar{x}_u = \bar{T}\gamma.^* \tag{13}$$

The tangents to corresponding v-curves of (9) and (10) at corresponding points intersect in a point whose coördinates can be written

$$x + \mu x_v = \bar{x} + \bar{\mu}\bar{x}_v. \tag{14}$$

We assume that the plane generating curves of both surfaces are not in a Combescurian relation.† The differentiation of (14) with respect to u yields the fact that μ and $\bar{\mu}$ are functions of v alone.

We have seen in the preceding section that the semi-parallel map existing between the lines of curvature of two surfaces of Monge with the same director-developable is also fundamental. The invariant of a fundamental transformation may be defined as the cross ratio in which a pair of corresponding points — one on each surface of the map — is divided by the focal points of the line joining them.‡ In our case it is found to be

$$I = \frac{\bar{T}\bar{\mu}}{T\mu}.$$

Since both μ and $\bar{\mu}$ are functions of v alone, I will be constant for points along the curves mapped by parallel tangents if and only if $\bar{T}/T$ is a function of v alone.

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* It is of interest to point out that with the parameter u chosen in the manner here indicated and v chosen as the arc of the plane generating curve, the square of the linear element of a surface of Monge can be put into the form

$$ds^2 = T^2du^2 + dv^2.$$

The functions T and γ here are the same as λ and U respectively in equations (5).

† If such were the case, the map would be a parallel one. We have heretofore included this case in our discussion; and the following theorems, in fact, hold in this case also. This follows from the fact that if the relation between the two plane generating curves in corresponding positions is Combescurian, we shall have $\bar{x}_v = Vx_v$ where V is a function of v alone.

‡ Cf. W. C. Graustein, An invariant of a general transformation of surfaces, Bull. Amer. Math. Soc. vol. 32 (1926), p. 359.

A necessary and sufficient condition that the invariant of the semi-parallel, fundamental transformation existing between the lines of curvature of two surfaces of Monge with the same director-developable be constant along each pair of curves — one on each surface — corresponding by parallel tangents is that the ratio of the torsions of the curves of each such pair be constant.

From (12) we see that $\overline{T}/T = \overline{\nu}/\nu$. Consequently, if I is to be a function of v alone, $\overline{\nu} = V(v)\nu$. Equations (9) and (10) become respectively

$$x = y - (\nu_u \alpha - \nu \beta), \qquad \overline{x} = y - V(v)(\nu_u \alpha - \nu \beta);$$

i. e., corresponding points $x, \overline{x}$, and y are collinear. Conversely, if this is the case, the ratio $\overline{\nu}/\nu$ is a function of v only.

A necessary and sufficient condition that the invariant of the semi-parallel, fundamental transformation existing between the lines of curvature of two surfaces of Monge with the same director-developable be constant along each pair of curves — one on each surface — corresponding by parallel tangents is that corresponding points of the two plane generating curves be collinear with the point of contact of their plane with the edge of regression of the director-developable.

We shall now discuss the case of two molding surfaces with the same director-cylinder, Γ. Let C: $y = y(u)$ be the parametric representation of any right section of Γ, where u is chosen so that $du = d\sigma/\varrho$, σ being the arc of C and $1/\varrho$ its curvature.

A point in a tangent plane to Γ has coördinates of the form

$$x = y + \lambda \alpha + \nu \gamma.$$

Here α is a triple representing the direction cosines of the tangent to C, and γ one representing those of the binormal. The direction γ is constant and is that of the rulings of Γ.

Using the Frenet-Serret formulas and demanding that x trace an orthogonal trajectory of the tangent planes to Γ, we have

$$x_u = \lambda \beta, \tag{15}$$

$$\varrho + \lambda_u = 0, \qquad \nu_u = 0. \tag{16}$$

Hence a molding surface with Γ as its director-developable is parametrically represented by

$$S\colon\ x = y(u) + \lambda(u, v)\alpha + \nu(v)\gamma(u), \tag{17}$$

and the parametric curves are the lines of curvature. The function λ is such that it satisfies (16).

Let $\bar{S}$ be another molding surface with Γ as its director-cylinder:

$$\bar{S}: \quad \bar{x} = y(u) + \bar{\lambda}(u, v)\alpha + \bar{r}(v)\gamma(u), \tag{18}$$

where

$$\varrho + \bar{\lambda}_u = 0. \tag{19}$$

Assuming that the generating plane curves of S and $\bar{S}$ in corresponding positions are not in a Combescurian relation,* the tangents to those curves at corresponding points will intersect in a point whose coördinates are of the form (14), and here, too, μ and $\bar{\mu}$ are functions of v alone.

The invariant of the semi-parallel, fundamental transformation established between the parametric curves of S and $\bar{S}$ is, in this case,

$$I = \frac{\bar{\lambda}\bar{\mu}}{\lambda\mu}.$$

Hence I is a function of v alone if and only if $\bar{\lambda}/\lambda$ is a function of v alone. From (16) and (19), it follows that $(\lambda - \bar{\lambda})_u = 0$. Hence both λ and $\bar{\lambda}$ are functions only of the parameter v. In that case, however, ϱ must vanish (cf. equation (16)); i. e., Γ is a line and the molding surfaces S and $\bar{S}$ reduce to surfaces of rotation.

The invariant of the semi-parallel, fundamental transformation existing between the lines of curvature of two molding surfaces with the same director-cylinder is constant along each pair of curves—one on each surface—corresponding by parallel tangents if and only if the director-cylinder is a line. The two molding surfaces are then surfaces of rotation with a common axis, and corresponding plane generating curves are those coplanar with the common axis of rotation.

* Cf. second footnote in § 5.

PRINCETON UNIVERSITY,
February 22, 1928.

ON THE LINEAR ORDINARY q-DIFFERENCE EQUATION.*

By C. Raymond Adams.

Introduction. The equation

$$\sum_{i=0}^{n} a_i(x) f(q^{n-i}x) = 0, \tag{1}$$

in which the $a_i(x)$ are known functions of the complex variable x, q is a constant $\neq 0$, and $f(x)$ is the function to be determined, is called the homogeneous linear ordinary q-difference equation of the nth order. We assume the coefficient functions $a_i(x)$ to be analytic or to have poles of finite order at the origin; then there is no lack of generality in supposing

$$a_i(x) = a_{i0} + a_{i1}x + a_{i2}x^2 + \cdots \qquad (i = 0, 1, \cdots, n) \tag{2}$$

for $|x| < R$, with at least one of the constants a_{i0} different from zero. The algebraic equation

$$a_{00}\varrho^n + a_{10}\varrho^{n-1} + \cdots + a_{n-1,0}\varrho + a_{n0} = 0 \tag{3}$$

is termed the "characteristic equation for $x = 0$" of (1). We distinguish *two essentially different cases* according as the roots of the characteristic equation are or are not all finite and different from zero. In the first of these cases (when $|q|$ is $\neq 1$) the existence of a set of n power series formally satisfying (1), which converge for $|x| < R$ and thereby provide a full set of solutions analytic in the vicinity of the origin, may be inferred from the general theorems of Grévy† on functional equations. This case has also been studied by Ryde‡ and, in part, by Carmichael.§ In the second case, which from a reasonable point of view is the more general, the question of existence of formal and analytic solutions has been left practically open.‖

This paper aims especially to present a new treatment of the first case, which has quite clear advantages of simplicity and brevity over the earlier studies of the problem, and to investigate the second case. In detail the

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† "Étude sur les équations fonctionnelles", Thèse (Paris), 1894; reprinted in large part in the *Annales de l'Éc. Norm. Sup.*, (3), vol. 11 (1894), pp. 249–323.

‡ "A contribution to the theory of linear homogeneous geometric difference equations (q-difference equations)", Dissertation (Lund), 1921, 45 pp.

§ In his paper, "The general theory of linear q-difference equations", *Amer. Journ. Math.*, vol. 34 (1912), pp. 147–168, Carmichael imposes the restriction that no root of (3) shall equal the product of another by an integral power of q (including q^0).

‖ In a very limited way it has been touched on by Mason, "On properties of linear q-difference equations with entire function coefficients", *Amer. Journ. of Math.*, vol. 37 (1915), pp. 439–444.

points of particular interest are as follows: (a) the method of power series with suitable convergence proofs is used throughout; (b) in both cases a full set of n series expansions formally satisfying the equation (1) is found* when $|q|$ is $\neq 1$ (§ 1); (c) by appropriate convergence proofs the existence of a full set of analytic solutions is established in the first case (§ 2); (d) the third and seventh paragraphs of § 2 provide a notably simple and concise proof of the existence theorems obtained by Carmichael;† (e) a study is made of the convergence of the formal series solutions in the second case (§ 3); and (f) a formal solution is found for the non-homogeneous q-difference equation of order n and inferences are drawn concerning its convergence.

A parallel situation is found at infinity when the functions $a_i(x)$ are analytic or have poles there; the analogy is so complete that we leave the corresponding theorems to be inferred by the reader.

1. **Formal series solutions.** In this section we shall merely state the facts concerning the series developments which formally satisfy the equation (1) in both cases. That the series do satisfy (1) formally may readily be verified by substitution and calculation of the coefficients.

CASE 1: *the roots of* (3) *are all finite and different from zero* ($a_{00} \neq 0$, $a_{n0} \neq 0$). In this case the entire situation as regards formal series may be summed up thus briefly.

Let $\varrho_j = q^{r_j}$ $(j = 1, 2, \cdots, p)$ be a set of p roots $(p \geqq 1)$ of the equation (3) satisfying the following conditions: (a)

$$r_j = r_1 - m_j \qquad (j = 2, 3, \cdots, p), \tag{4}$$

where m_j is a positive integer or zero and the subscripts of the r's are so chosen as to make

$$0 \leqq m_2 \leqq m_3 \leqq \cdots \leqq m_p;$$

(b) no root of (3) outside this group is equal to q^{r_1+m} for m a positive or negative integer or zero. Then there correspond to the roots of this set p formal solutions of (1):

$$\begin{aligned} &x^{r_1} P(x), \\ &x^{r_2} [P(x) + t P(x)], \\ &x^{r_3} [P(x) + t P(x) + t^2 P(x)], \\ &\cdots\cdots\cdots\cdots\cdots \\ &x^{r_p} [P(x) + t P(x) + t^2 P(x) + \cdots + t^{p-1} P(x)], \end{aligned} \tag{5}$$

* Immediate application may be made here of a method introduced by the author in a recent paper, "On the irregular cases of the linear ordinary difference equation", *Trans. Amer. Math. Soc.*, vol. 30 (1928), pp. 507–541. This paper will hereafter be referred to as I.

† Loc. cit., pp. 148–161.

where we have

$$t = \frac{\log x}{\log q}$$

and $P(x)$ is used in a generic sense to denote a power series in x. Throughout this paper those determinations of $\log x$ and $\log q$ will be taken in which the coefficient of $\sqrt{-1}$ is positive or zero and less than 2π.

Some of the coefficients in the power series of (5) are arbitrary; their distribution is worthy of note. In the first solution the constant term in $P(x)$ is arbitrary and subsequent coefficients are determined uniquely in terms of it. In the second solution, if m_2 is zero, the constant term in each $P(x)$ is arbitrary and coefficients that follow are uniquely determined in terms of them; if m_2 is not zero, the constant term and the coefficient of x^{m_2} in the first $P(x)$ are arbitrary and all other coefficients of both power series are uniquely determined in terms of them. Though the situation becomes more involved as one proceeds, similar statements can be made for the other solutions.

It should be mentioned also that if m_j is zero for $j = 2, 3, \cdots, k \leqq p$, none of the power series in the first k solutions (5) that follows a power of t may disappear entirely; in the contrary case this is not true. For example, in the second solution the second $P(x)$ would vanish identically if m_2 were not zero and the following condition were satisfied:

$$\begin{vmatrix} 0 & 0 & \cdot\ \cdot & L_{1,2,0} \\ 0 & 0 & \cdot\ \cdot & L_{2,2,0} \\ \cdot & \cdot & \cdot\ \cdot & \cdot\ \cdot \\ 0 & 0 & \cdot\ \cdot & L_{m_2-3,2,0} \\ 0 & L_{0,2,m_2-2} & \cdot\ \cdot & L_{m_2-2,2,0} \\ L_{0,2,m_2-1} & L_{1,2,m_2-2} & \cdot\ \cdot & L_{m_2-1,2,0} \\ L_{1,2,m_2-1} & L_{2,2,m_2-2} & \cdot\ \cdot & L_{m_2,2,0} \end{vmatrix} = 0,$$

where

$$L_{ijl} = a_{0i}\, q^{n(r_j+l)} + a_{1i}\, q^{(n-1)(r_j+l)} + \cdots + a_{n-1,i}\, q^{r_j+l} + a_{ni}. \tag{6}$$

CASE 2: *some of the roots of* (3) *are infinite, or some are zero, or both* (one or both of a_{00}, $a_n{}^0 = 0$). This case may be treated as in I.*) We denote by a_{ij_i} the first non-zero coefficient in $a_i(x)$ $(i = 0, 1, \cdots, n)$, and choosing i and j axes plot the points (i, j_i). A portion of a Newton polygon, which "roofs over" the sets of points (i, j_i) and is denoted by L, is then constructed.

* Loc. cit., pp. 511–513.

In this case it is a misnomer to call (3) *the characteristic equation of* (1), *for there are in fact several characteristic equations, one associated with each segment of* L. If one of the segments is horizontal, the characteristic equation associated with that segment (obtained from (3) by suppressing infinite and zero roots) picks out its quota of formal solutions precisely as (3) distinguishes n solutions in Case 1. In the case of a segment of slope $\mu \neq 0$, the transformation

$$f(x) = q^{\frac{\mu}{2}(t^2-t)} g(x) \tag{7}$$

plays the rôle of (8) in I and has the effect of bringing the segment of slope μ into a horizontal position.

In the first place let μ be an integer and let the characteristic equation associated with the segment of L having this slope possess a set of roots $\varrho_j = q^{r_j}$ $(j = 1, 2, \cdots, p)$ satisfying the same conditions as does the set described under Case 1. Then there corresponds to this set of roots a set of formal solutions exactly like (5) except for the additional factor $q^{\frac{\mu}{2}(t^2-t)}$ in each one.

Secondly, let μ be a fraction, r/s in lowest terms and s positive; then after making the transformation (7) and suppressing the factors common to all the terms of the equation (1) we have an equation of the same sort except that certain of the power series coefficients are multiplied by a factor of the type $x^{w/s}$, w being a positive integer. Let further the characteristic equation associated with the segment of L in question have a set of roots $\varrho_j = q^{r_j}$ $(j = 1, 2, \cdots, p)$ satisfying conditions like (a) and (b) under Case 1 except for the substitution of m_j/s for m_j and m/s for m; to this set of roots there corresponds a set of solutions just like (5) except that each solution contains the additional factor $q^{\frac{\mu}{2}(t^2-t)}$ and each power series $P(x)$ is replaced by one proceeding according to the sth root of x, $P(x^{1/s})$.

2. **Analytic solutions in Case 1.** We propose to show that in Case 1 a full quota of analytic solutions is provided by the formal series, which converge for x in the vicinity of the origin, when $|q|$ is different from 1.

Without loss of generality we may clearly assume $a_0(x) \equiv 1$. It is evident also that $|q|$ may be taken > 1: for, if $|q|$ were < 1, we might set $f(q^n x) = h(x)$ and $q^{-1} = q'$; then (1) would be unchanged in type and would still come under Case 1, and $|q'|$ would be > 1. For the present it will be assumed further that all of the power series $a_i(x)$ $(i = 1, 2, \cdots, n)$ have radii of convergence > 1; a little later we shall prove that this hypothesis is not essential.

We begin by proving* directly the convergence of the series in the first of solutions (5),

$$x^{r_1}(s_0 + s_1 x + s_2 x^2 + \cdots). \tag{8}$$

The coefficient s_k is determined by the relation (cf. (6))

$$s_k = -\frac{\sum_{i=0}^{k-1} s_i L_{k-i,1,i}}{L_{0,1,k}}. \tag{9}$$

Considering first the numerator of this quotient, we observe that the power series $a_i(x)$ are convergent for $x = 1$; therefore we have

$$|a_{ij}| < M' \quad (i = 1, 2, \cdots, n;\ j = 0, 1, \cdots).$$

The quantities $|q^{ir_1}|\,(i = 0, 1, \cdots, n-1)$ have an upper bound M''. As for the denominator, we may write

$$|L_{0,1,k}| = |q^{n(r_1+k)}| \cdot \left|1 + \frac{a_{10}}{q^{r_1+k}} + \cdots + \frac{a_{n0}}{q^{n(r_1+k)}}\right|.$$

k may be chosen so large that the second factor on the right will be $> \frac{1}{2}$; for such a value of k, say k_1, and for all larger values the modulus of the denominator will exceed $\frac{1}{2}|q^{n(r_1+k)}|$. For $k < k_1$, we have $|L_{0,1,k}| = C_k|q^{n(r_1+k)}|$, where C_k is $\neq 0$. Let the least of the positive numbers $C_1, C_2, \cdots, C_{k_1-1}, \frac{1}{2}$ be denoted by C. Then we have

$$|s_k| < \frac{n M' M'' |q|^{(n-1)(k-1)} \sum_{i=0}^{k-1} |s_i|}{C|q^{nr_1}|\,|q|^{nk}}. \tag{10}$$

Defining $\bar{s}_0$ as $|s_0|$, we obtain the following upper bound for s_k:

$$|s_k| < \frac{nM\sum_{i=0}^{k-1}\bar{s}_i}{|q|^{n+k-1}} = \bar{s}_k \qquad (k = 1, 2, \cdots), \tag{11}$$

where

$$M = \frac{M' M''}{C|q^{nr_1}|}.$$

As k becomes infinite the limit of

$$\frac{\bar{s}_k}{\bar{s}_{k+1}} = \frac{|q|\sum_{i=0}^{k-1}\bar{s}_i}{\bar{s}_k + \sum_{i=0}^{k-1}\bar{s}_i} = \frac{|q|}{\frac{nM}{|q|^{n+k-1}} + 1}$$

is $|q|$; we therefore infer the convergence of (8) for $|x| < |q|$.

* This first proof is essentially the same as that of Mason, loc. cit., pp. 440–442.

If one of the subsequent solutions of (5) should reduce to the same type as the first through the identical vanishing of the power series which follow powers of t, the above proof covers the convergence of this series. It need only be observed that in this event the denominator of (9) is zero for some positive value (or values) of k, but that whenever it is, the numerator is also zero.

Let us next consider the second solution of the set (5),

$$x^{r_2}[(u_0+u_1x+u_2x^2+\cdots)+t(s_0+s_1x+s_2x^2+\cdots)]; \tag{12}$$

these s_k are of course not to be confused with those of (9). Indeed we have

$$s_k = -\frac{\sum_{i=0}^{k-1} s_i L_{k-i,2,i}}{L_{0,2,k}},$$

and these constants satisfy inequalities exactly like (10) and (11); we define their bounds $\bar{s}_k$ as in (11) and infer the convergence of $\sum_{i=0}^{\infty}\bar{s}_i x^i$, and therefore of $\sum_{i=0}^{\infty} s_i x^i$, for $|x|<|q|$. Setting

$$L'_{ijl} = n\,a_{0i}q^{n(r_j+l)}+(n-1)\,a_{1i}q^{(n-1)(r_j+l)}+\cdots+a_{n-1,i}q^{r_j+l},$$

we find that the u_k satisfy the relations

$$u_k = -\frac{\sum_{i=0}^{k-1}[u_i L_{k-i,2,i}+s_i L'_{k-i,2,i}]+s_k L'_{0,2,k}}{L_{0,2,k}}.$$

Allowing M'' now to stand for the bound of $|q^{ir_2}|$ $(i=0,1,\cdots,n)$ and proceeding as before for (9), we find

$$|u_k|<\frac{nM\sum_{i=0}^{k-1}|u_i|}{|q|^{n+k-1}}+\frac{n(n-1)M\sum_{i=0}^{k-1}|s_i|}{|q|^{n+k-1}}+\frac{n(n+1)M|s_k|}{2}.$$

Since, however, $|s_k|$ satisfies an inequality like (10), the sum on the right is less than

$$\frac{n^2(n+1)M^2\sum_{i=0}^{k-1}[|u_i|+2|s_i|]}{|q|^{n+k-1}}.$$

Defining $\overline{u}_0$ as the modulus of u_0, we obtain an upper bound $\overline{u}_k$ for $|u_k|$ $(k = 1, 2, \cdots)$ by replacing $|u_i|$ by $\overline{u}_i$ and $|s_i|$ by $\overline{s}_i$ in this expression. Let us examine the quotient

$$\frac{\overline{u}_k}{\overline{u}_{k+1}} = |q| \frac{\sum_{i=0}^{k-1} [\overline{u}_i + 2\overline{s}_i]}{\overline{u}_k + 2\overline{s}_k + \sum_{i=0}^{k-1} [\overline{u}_i + 2\overline{s}_i]}$$

$$= |q| \frac{1}{\dfrac{n^2(n+1)M^2}{|q|^{n+k-1}} + 1 + \dfrac{2nM}{|q|^{n+k-1}} \dfrac{\sum_{i=0}^{k-1} \overline{s}_i}{\sum_{i=0}^{k-1} [\overline{u}_i + 2\overline{s}_i]}}. \tag{13}$$

The series $\sum_{i=0}^{\infty} \overline{s}_i$ is convergent. As k becomes infinite, $\sum_{i=0}^{k-1} \overline{u}_i$ must either approach a limit or become infinite; in either event the limit of the quotient (13) is $|q|$. Hence it follows that the series $\sum_{i=0}^{\infty} u_i x^i$ converges for $|x| < |q|$. This proof clearly establishes likewise the convergence of the series in any of the later solutions (5) which may perchance reduce to the type (12).

It is manifestly possible to extend the above proof to cover the convergence of the series $P(x)$ in all the other solutions of the set (5).

We may now show that the hypothesis that all the power series $a_i(x)$ have radii of convergence greater than 1 is not necessary. Assume the smallest radius to be $R \leqq 1$. The power series $a_i(x/q^\alpha)$, where α denotes a positive integer, are then all convergent for $|x| < |q^\alpha| R$, which exceeds 1 for α sufficiently large. If α is so chosen, the equation

$$\sum_{i=0}^{n} a_i(x/q^\alpha) f(q^{n-i-\alpha} x) = 0, \tag{14}$$

by the above discussion, has a full quota of formal power series solutions $s_i(x)$ $(i = 1, 2, \cdots, n)$ all convergent for $|x| < |q|$. But $s_i(q^\alpha x)$ $(i = 1, 2, \cdots, n)$ is then a formal solution of (1), of the type of those in (5), and convergent for $|x| < |q^{1-\alpha}|$.

By repeated use of the equation (1) it follows at once that *if the coefficient functions $a_i(x)$ are meromorphic, there exist n solutions of* (1), $s_i(x)$ $(i = 1, 2, \cdots, n)$, *each of which is analytic over the entire finite plane except perhaps at the origin and at points which are congruent externally**

* $|q|$ being > 1, the points $qx, q^2x, \cdots [q^{-1}x, q^{-2}x, \cdots]$ are said to be *externally* [*internally*] *congruent* to the point x.

to the poles of $a_i(x)$ $(i = 1, 2, \cdots, n)$; *at these latter points the solutions have poles or are analytic; in the vicinity of* $x = 0$ *they are given by the convergent series developments* (5). In the same way it may be concluded that, *whatever the character of the functions* $a_i(x)$ *may be away from the origin, the formal series* (5) *converge and represent solutions* $s_i(x)$ *which are analytic for* $|x| < |q| R$ *(with the possible exception of* $x = 0$*), where* R *is the minimum distance from the origin to a singularity of* $a_i(x)$ $(i = 1, 2, \cdots, n)$. *The general solution of* (1) *is*

$$f(x) = \sum_{i=1}^{n} p_i(x)\, s_i(x),$$

in which the $p_i(x)$ *are arbitrary functions satisfying the relation* $p(qx) = p(x)$.

If conditions like those here imposed upon the nature of the coefficient functions $a_i(x)$ at the origin hold also at infinity, there exists a second set of n solutions $s_i'(x)$ $(i = 1, 2, \cdots, n)$ which are given in the neighborhood of infinity by expansions like (5) except that all the power series proceed according to $1/x$. If, further, the $a_i(x)$ are polynomials [rational functions], the solutions of the first set are given in terms of those of the second set by relations

$$s_i(x) = \sum_{j=1}^{n} p_{ij}(x)\, s_j'(x),$$

in which the functions $p_{ij}(x)$ satisfy the relation $p(qx) = p(x)$ and may be proved, as in the regular case, analytic [except for poles] away from zero and infinity.

3. **Analytic solutions in Case 2.** It is readily shown by the methods of § 2 that *in this case the formal series associated with the segment of* L *farthest to the left* [*right*] *are convergent when* $|q|$ *is* > 1 [< 1]. An examination of the coefficients of the series associated with other segments of L leads one to suspect that those series may be divergent. Nevertheless it is quite clear that they are sometimes convergent, for a q-difference equation (1) may easily be constructed which has for solutions any specified set of convergent series of the type described under Case 2, § 1. We proceed to a detailed investigation of the equation of second order.

There is no loss of generality in taking the equation in the form

$$(15)\quad f(q^2x) + (a_{01} + a_{11}x + \cdots) f(qx) + x^{\mu}(a_{2,\mu} + a_{2,\mu+1}x + \cdots) f(x) = 0$$

with μ a positive integer and

$$a_{10},\ a_{2,\mu} \neq 0.$$

This equation has two formal solutions

$$s_1(x) = x^{r_1}(s_{10} + s_{11}x + \cdots), \tag{16}$$

$$s_2(x) = q^{\frac{\mu}{2}(t^2-t)} x^{r_2}(s_{20} + s_{21}x + \cdots), \tag{17}$$

the first of which is convergent in the vicinity of $x = 0$. We propose to make use of (16) to reduce the order of (15). Defining the operator Δ by the relation

$$\Delta g(x) = g(qx) - g(x)$$

and denoting its inverse by Σ, we set

$$f(x) = s_1(x)\,\Sigma g(x).$$

Equation (15) then becomes

$$\begin{aligned} &q^{2r_1}(s_{10} + s_{11}q^2x + \cdots)g(qx) + [q^{2r_1}(s_{10} + s_{11}q^2x + \cdots) \\ &\quad + (a_{10} + a_{11}x + \cdots)q^{r_1}(s_{10} + s_{11}qx + \cdots)]g(x) = 0. \end{aligned} \tag{18}$$

By virtue of the equations determining the s_{1i} $(i = 0, 1, \cdots)$, the terms in the bracket of degree less than μ disappear; the term in x^μ, however, has a coefficient not zero since $a_{2,\mu}$ is $\neq 0$.

Equation (18) has a solution

$$g(x) = q^{\frac{\mu}{2}(t^2-t)} x^r (s_0 + s_1 x + \cdots), \tag{19}$$

the series in parentheses being convergent for $|x| < R$. To evaluate $\Sigma g(x)$ is our next concern; if we set

$$\Sigma g(x) = q^{\frac{\mu}{2}(t^2-t)} x^r\, l(x),$$

$l(x)$ must be determined to satisfy the relation

$$x^\mu q^r\, l(qx) - l(x) = s_0 + s_1 x + \cdots. \tag{20}$$

A formal series expansion for $l(x)$,

$$l_0 + l_1 x + \cdots, \tag{21}$$

can readily be found; in fact the coefficients are determined to be

$$l_i = \begin{cases} -s_i & (i = 0, 1, \cdots, \mu - 1), \\ -s_i + l_{i-\mu}\, q^{r+i-\mu} & (i = \mu, \mu+1, \cdots). \end{cases}$$

We now have formally

$$s_2(x) \equiv k s_1(x) q^{\frac{\mu}{2}(t^2-t)} x^r (l_0 + l_1 x + \cdots) \qquad (k = \text{const.})$$

since the constants (after μ) in the formal solution (17) are uniquely determined save for an arbitrary constant multiplying the entire expression. It follows that the series in (17) will have a radius of convergence $\geqq 0$ according as that of (21) is $\geqq 0$.

One readily sees that the power series (21) formally satisfying a non-homogeneous q-difference equation like (20) has a radius of convergence that may be zero or not zero. An illustration of the first possibility is obtained when the series on the right in (20) reduces to a single term $s_m x^m$; this is easily proved by direct examination of the coefficients l_i. The second possibility is clearly shown by letting the series on the right be determined by substituting for $l(x)$ in the left member any power series with a positive radius of convergence.

But let us examine the series

$$\sum_{i=0}^{\infty} l_{i\mu+j}\, x^{i\mu+j} \qquad (j = 0, 1, \cdots, \mu-1). \tag{22}$$

We may write

$$l_{m\mu+j} = -q^{m(r+j)+(1+2+\cdots+m-1)\mu}\, Q_{m,j} \tag{23}$$

where

$$Q_{m,j} = s_j + \frac{s_{\mu+j}}{q^{r+j}} + \frac{s_{2\mu+j}}{q^{2(r+j)+\mu}} + \cdots + \frac{s_{m\mu+j}}{q^{m(r+j)+(1+2+\cdots+m-1)\mu}}$$

and hence

$$\left| \frac{l_{m\mu+j}}{l_{(m+1)\mu+j}} \right| = \left| \frac{Q_{m,j}}{q^{r+j+m\mu}\, Q_{m+1,j}} \right|. \tag{24}$$

As m becomes infinite $Q_{m,j}$ becomes a series which is evidently convergent, since the ratio of the $(n+1)$th term to the nth is numerically equal to

$$\frac{1}{|q^{r+j+n\mu}|} \left| \frac{s_{n\mu+j}}{s_{(n+1)\mu+j}} \right|;$$

the limit of the first factor is zero while that of the second is $1/R^{\mu}$, which is finite. The determination of whether a limit is approached by (24), and if so what limit, is not likely to be easy in general; it may be remarked, however, that *a sufficient [necessary] condition that the radius of convergence of* (21), *and therefore of the series in* (17), *be zero [positive] is the non-vanishing [vanishing] of the sum [sums] of any one [all] of the series which the* $Q_{m,j}$ *become as* m *increases indefinitely.*

The method here employed of reducing the order of the equation by the aid of known solutions may be used in studying the same problem for equations of higher order. We shall not attempt here a detailed analysis of the general problem; suffice it to say that *the series associated*

*with segments of L other than that farthest to the left when $|q|$ is >1 [right when $|q|$ is <1] may either diverge or converge.**

These results give rise to two interesting questions which we are not now prepared to answer, namely: When not all the formal solutions in Case 2 have positive radii of convergence, does there exist a full complement of n solutions analytic in the neighborhood of the origin? In case such a set does exist, do the formal solutions which converge only for $x = 0$ provide an asymptotic representation of corresponding analytic solutions?

4. The non-homogeneous equation. In this section we consider very briefly the equation obtained by adding to the left member of (1) a function $b(x)$ of the same character as the $a_i(x)$:

$$\sum_{i=0}^{n} a_i(x) f(q^{n-i}x) + b(x) = 0. \tag{25}$$

Without loss of generality we may assume the $a_i(x)$ to be as given by (2) and

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots \text{ for } |x| < R, \qquad b_0 \neq 0.$$

If among the quantities q^i $(i = 0, 1, \cdots)$ there are exactly p roots $(p \geqq 0)$ of (3), each root being counted according to its multiplicity, the equation (25) has a formal solution

$$s(x) = P(x) + tP(x) + \cdots + t^p P(x), \tag{26}$$

in which the coefficients of the power series $P(x)$ may be calculated by substitution in (25). If $|q|$ is $>1[<1]$ and the equation (3) has no infinite [zero] roots, we may further suppose $a_0(x) \equiv 1$ $[a_n(x) \equiv 1]$ and all the power series in (26) may be proved convergent in the vicinity of $x = 0$ by the methods of § 2. If $|q|$ is $>1[<1]$ and (3) has infinite [zero] roots, the series in (26) may or may not converge in the vicinity of $x = 0$. The last conclusion is drawn from the analysis of equation (20) in § 3.

The general solution of (25) is of course given by the sum of a particular solution and the general solution of the associated homogeneous equation (1).

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* Equations with simple coefficients illustrating the two possibilities for the equation (15) are

$$f(q^2x) + qf(qx) + qxf(x) = 0,$$

of which the solution (17) is divergent, and

$$(1 - qx^2) f(q^2x) - (1 + x - qx^2 + q^3x^3) f(qx) + x(1 - q^3x^2) f(x) = 0,$$

of which the solution (17) is convergent.

BROWN UNIVERSITY, PROVIDENCE, R. I.,
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A GENERALIZATION OF THE CAUCHY-RIEMANN EQUATIONS.†

By K. P. Williams.

1. In the customary development of the theory of functions of a complex variable the Cauchy-Riemann equations are derived as a necessary and sufficient condition that $u+iv$ possess a derivative with regard to $x+iy$. The remarkable significance of this property leads to the conception of an analytic function of the complex variable. One of the important results, continually used, but not always commented upon, is that the sum, difference, product, and quotient of two analytic functions are analytic. If we reflect upon this property we see that the Cauchy-Riemann equations possess peculiar group properties with regard to certain operations. Or, if we please, we can say that those equations are related in an essential way to the formal rules for differentiating a sum, product, and quotient of two functions. This leads us to raise the question whether the possession of such properties is sufficient to completely identify the Cauchy-Riemann equations. One would hardly expect that it was. If indeed it is not, we will be led to seek a generalization of those equations, and so see the complex variable theory as a peculiarly significant and particularly simple case of a broader theory.

2. The general considerations that precede are formulated with the precision necessary to an actual development in the following way.

Let $u(x, y)$ and $v(x, y)$ satisfy the equation

$$au_x+bu_y+cv_x+dv_y = 0, \tag{1}$$

where a, b, c, d are real constants, not satisfying any of the four cases:

$$a = 0, \quad b = 0; \qquad b = 0, \quad d = 0;$$
$$c = 0, \quad d = 0; \qquad a = 0, \quad c = 0.$$

We propose the following problems:

(i) Let it be required to annex to (1) another relation (or relations) such that if u^* and v^* are other functions satisfying (1) and the annexed relation, so also are U and V, where $U+iV$ is the sum, product, or quotient of $u+iv$ and u^*+iv^*, it being understood that U and V take the place of u and v, respectively, in (1).

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(ii) If an undesirable generality remains, let it be required to find some other natural and significant condition to impose, which will either lead to the Cauchy-Riemann equations, or to a simple generalization of them.

(iii) Let it also be required to find functions $f(x, y)$, $g(x, y)$ such that we can say in the general case that $u+iv$ is an analytic function of $f+ig$.†

3. A necessary condition to annex to (1) in order to satisfy the requirement (i) is easily obtained. We observe that $u^* = 0$, $v^* = 1$ is a solution of (1). It follows that U and V must satisfy (1) where

$$U+iV = i(u+iv).$$

That is,

$$U = -v, \qquad V = u$$

must satisfy (1). Hence u and v must not only satisfy (1), but must also satisfy

$$-av_x - bv_y + cu_x + du_y = 0.$$

This equation is distinct from (1); for the relations

$$c = ka, \quad d = kb, \quad a = -kc, \quad b = -kd$$

give $k^2 = -1$. This contradicts the fact that all coefficients in (1) are assumed to be real.

It is seen now that we can assume that one of the coefficients in (1) is zero without lessening the generality of the problem. If $d \neq 0$, $b \neq 0$, we can eliminate v_y from (1) and (2). We therefore replace (1) by the equation

$$Au_x + Bu_y + Cv_x = 0.‡ \tag{3}$$

The equation to annex to this as necessary for the possession of property (i) by solutions of (3) is seen to be

$$-Av_x - Bv_y + Cu_x = 0. \tag{4}$$

Equation (3) is obtained from (4) by the same formal process that gives (4) from (3). Thus the relation between the equations is reciprocal.

4. Equation (4) is also a sufficient condition to annex to (3) in order to satisfy (i). So far as the sum of solutions is concerned this is obvious. We have then to consider multiplication and division.

(a) Let u^* and v^* be other solutions of (3) and (4).

Put

$$U+iV = (u+iv)(u^*+iv^*),$$

† The sense in which this is to be understood is implied in the theorem of § 6.

‡ If $d = 0$, (1) is of the form (3). If, however, $b = 0$, we cannot obtain (3); but (1) is of the form (3) with u and v interchanged. The alteration in the subsequent work is obvious.

so that

$$U = uu^* - vv^*, \qquad V = u^*v + uv^*.$$

It is a matter of simple calculation to prove that U and V satisfy (3) and (4).

(b) Next let

$$U + iV = \frac{1}{u + iv},$$

so that

$$U = \frac{u}{u^2 + v^2}, \qquad V = \frac{-v}{u^2 + v^2}.$$

It can again be proved that U and V satisfy (3) and (4).

(c) It follows from (a) and (b) that if we put

$$U + iV = \frac{u + iv}{u^* + iv^*},$$

the functions U and V satisfy (3) and (4).

5. We next proceed, in accordance with (ii), to seek a new condition which will lessen somewhat the generality of the functions defined by (3) and (4).

We have from (3) and (4)

$$A u_{xx} + B u_{xy} + C v_{xx} = 0, \tag{5}$$

$$A u_{xy} + B u_{yy} + C v_{xy} = 0, \tag{6}$$

$$-A v_{xx} - B v_{xy} + C u_{xx} = 0, \tag{7}$$

$$-A v_{xy} - B v_{yy} + C u_{xy} = 0. \tag{8}$$

If we multiply (5), (6), and (7) by A, B, and C, respectively, and add, we find

$$(A^2 + C^2) u_{xx} + B^2 u_{yy} + 2AB u_{yx} = 0. \tag{9}$$

If we multiply (5), (7), and (8) by C, $-A$, and $-B$, respectively, and add, we find,

$$(A^2 + C^2) v_{xx} + B^2 v_{yy} + 2AB v_{xy} = 0.$$

The functions u and v therefore satisfy the same second order equation. The discriminant of the equation is B^2C^2 so the equation is of the elliptic type.

5. Let us now assume that u is an arbitrary solution of (9) and determine v so that u and v satisfy (3) and (4), if this can be done by imposing a condition on the constants in the equations.

Put

$$v = \int_{0,0}^{x,y} [(A^2 + C^2) u_x + AB u_y]\, dy - [B^2 u_y + AB u_x]\, dx. \tag{10}$$

The function so determined is independent of the path in view of (9).

We have

$$v_x = -(B^2 u_y + AB u_x),$$
$$v_y = (A^2 + C^2) u_x + AB u_y.$$

Substituting in (3) and (4) we find

$$A(1 - BC)u_x + B(1 - BC)u_y = 0,$$
$$C(1 - BC)u_x = 0.$$

We are therefore led to impose the condition

$$BC = 1.$$

It is clear that we can put $B = 1$, $C = 1$. We therefore have the equations

$$\begin{aligned} Au_x + u_y + v_x &= 0, \\ u_x - Av_x - v_y &= 0, \end{aligned} \tag{11}$$

as a natural generalization of the Cauchy-Riemann equations. The associated second order equation is

$$(1 + A^2) f_{xx} + 2A f_{xy} + f_{yy} = 0, \tag{12}$$

and is the corresponding generalization of the equation of LaPlace.

If now we impose the condition that the second order equation be symmetric, we arrive at the familiar equations of complex variable theory. Thus the case which is of fundamental intrinsic significance is obtained by imposing a purely formal property.

6. We come now to the question raised in (iii).

We shall reduce (12) to normal form by a linear substitution. This can be done in various ways, and the question will be developed with sufficient generality to show how different substitutions can be obtained.

We put

$$\xi = ax + by, \qquad \eta = cx + dy,$$

and substitute in (12). The necessary and sufficient conditions that the resulting equation reduces to

$$f_{\xi\xi} + f_{\eta\eta} = 0, \tag{13}$$

are

$$(a^2 - c^2)(1 + A^2) + (ab - cd)2A + (b^2 - d^2) = 0,$$
$$2ac(1 + A^2) + (ad + bc)2A + 2bd = 0.$$

These equations can be combined so as to produce the following equations equivalent to them:

$$[(a+ic)A+(b+id)]^2+(a+ic)^2 = 0,$$
$$[(a-ic)A+(b-id)]^2+(a-ic)^2 = 0.$$

We shall not consider all solutions of these equations but will use the one set of relations

$$(a+ic)A+(b+id) = i(a+ic),$$
$$(a-ic)A+(b-id) = -i(a-ic).$$

From these equations we find

$$aA+b = -c, \qquad cA+d = a.$$

The last equations are satisfied by $a = 1$, $b = -A$, $c = 0$, $d = 1$. Thus the substitution

$$\xi = x-Ay, \qquad \eta = y \tag{14}$$

will reduce (12) to the form (13).

Other substitutions can be obtained from the relations above.

The substitution (14) reduces (3) and (4) to

$$u_\eta+v_\xi = 0, \qquad u_\xi-v_\eta = 0.$$

But these are the Cauchy-Riemann equations. It follows that $u(\xi, \eta)+iv(\xi, \eta)$ is an analytic function of $\xi+i\eta$.

We can now state the following theorem, omitting familiar statements as to regions, etc.

THEOREM. *If u and v satisfy* (11) *then $u+iv$ is an analytic function of $Z = (x+iy)-Ay$.*

As an example, consider the function

$$u = -\frac{A}{1+A^2}x^2+xy,$$

which is seen to satisfy (12). From (10) we have

$$v = \int_{0,0}^{x,y}[-Ax+(1+A^2)y]\,dy-\left[\frac{1-A^2}{1+A^2}x+Ay\right]dx$$
$$= (1+A^2)\frac{y^2}{2}-\frac{1-A^2}{1+A^2}\frac{x^2}{2}-Axy.$$

It is not difficult to show that

$$u+iv = -\frac{1}{1+A^2}\left[A+\frac{i}{2}(1-A^2)\right]Z^2.$$

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CONTACT TRANSFORMATIONS.*

By Luther Pfahler Eisenhart.

It is the purpose of this paper to give an exposition of the theory of contact transformations and to apply the theory to Riemannian spaces and to dynamical systems. Lie† in his treatment of the subject begins with what may be called the non-homogeneous case, since one of the coördinates is distinguished from the others; he develops the theory of the homogeneous case therefrom. In this paper the order is reversed, so that the equations may be written in symmetrical form.

In §§ 1–4 discontinuous homogeneous transformations of various types are treated and in § 5 infinitesimal transformations and continuous groups of one parameter. The reader is supposed to be familiar with the elements of the theory of continuous groups. The theory of non-homogeneous transformations both discontinuous and continuous is developed in § 6, and in § 7 the subclass of these transformations which is generally used in the transformation of the Hamiltonian equations of mechanics. In fact it is shown in what sense this application to mechanics is part of the general theory of this restricted type of contact transformation.

Of particular interest is the case when the hessian of the characteristic function (§ 5) of a group of homogeneous transformations is of maximum rank. This case is studied in § 8 and in § 9 the geometrical properties of these transformations are discussed. In particular we have in this theory waves and wavefronts associated in accordance with Huygen's principle.‡ In § 10 an application is made to the geodesics of a Riemannian space and in § 11 to the trajectories of a holonomic conservative dynamical system for which the time does not enter in the constraints and the potential energy. In the latter the dynamical trajectories coincide with the trajectories of the associated contact transformation and the wave surfaces of the latter are orthogonal to the trajectories, orthogonal in the Riemannian metric defined by means of the expression for kinetic energy. The trajectories may also be identified with the paths of light

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† Theorie der Transformationsgruppen, zweiter Abschnitt, Leipzig, 1888.

‡ In some respects the treatment is similar to that given by Vessiot, Sur l'interprétation mécanique des transformations de contact infinitésimals, Bull. Soc. Math. de France, vol. 34 (1906), pp. 230–269; also by Levi-Civita, Lezioni de meccanica razionale, vol. 2, part 2, pp. 452–469, Bologna, 1927.

through an isotropic non-homogeneous medium in accordance with Fermat's principle, as Hamilton pointed out long ago. However, the time is not the same for the dynamical trajectories and the paths of light. All of these ideas are important because of their bearing on the new wave-mechanics.

1. **Point transformations of a space.** Consider an n-dimensional space V_n of coördinates x^i; throughout this paper latin indices take the values 1 to n. If φ^i are n functions of the x's, the equations

$$\bar{x}^i = \varphi^i(x^1, \cdots, x^n) \tag{1.1}$$

may be looked upon as defining a transformation of the points of V_n among themselves; that is, a given point of coördinates x^i is transformed into the point of coördinates $\bar{x}^i$, given by (1.1), in the given coördinate system. From (1.1) it follows that the point of coördinates $x^i + dx^i$ is transformed into the point of coördinates $\bar{x}^i + d\bar{x}^i$, where

$$d\bar{x}^i = \frac{\partial \varphi^i}{\partial x^j} dx^j; \tag{1.2}$$

throughout the paper a repeated index indicates the sum of terms obtained on assigning to the index all of its values.

If we require, as will always be done, that the jacobian of the φ's is of rank n, there exists for a neighborhood of a point an inverse of (1.1) in accordance with the fundamental theorem of implicit functions.* If we write the inverse in the form

$$x^i = \bar{\varphi}^i(\bar{x}^1, \cdots, \bar{x}^n), \tag{1.3}$$

any function $f(x)$ is transformed into a function $\bar{f}$ of the $\bar{x}$'s defined by

$$\bar{f}(\bar{x}^1, \cdots, \bar{x}^n) = f(\bar{\varphi}^1, \cdots, \bar{\varphi}^n); \tag{1.4}$$

also we have

$$\bar{f}(\varphi^1, \cdots, \varphi^n) = f(x^1, \cdots, x^n). \tag{1.5}$$

If we put

$$p_i = \frac{\partial f}{\partial x^i}, \qquad \bar{p}_i = \frac{\partial \bar{f}}{\partial \bar{x}^i}, \tag{1.6}$$

we have from (1.4) und (1.6)

$$\bar{p}_i = p_j \frac{\partial \bar{\varphi}^j}{\partial \bar{x}^i}, \tag{1.7}$$

* Cf. Fine, Calculus, New York, 1927, p. 253.

and from (1.5)

$$p_i = \bar{p}_j \frac{\partial \varphi^j}{\partial x^i}. \tag{1.8}$$

Equations (1.7) define the induced transformation of the functions p_i at a point $P(x^i)$ into the functions $\bar{p}_i$ at $\bar{P}(\bar{x}^i)$, and (1.8) define the inverse transformation.

From the foregoing it follows that a family of hypersurfaces

$$f(x^1, \cdots, x^n) = a, \tag{1.9}$$

where a is a parameter, is transformed into the family

$$\bar{f}(\bar{x}^1, \cdots, \bar{x}^n) = a. \tag{1.10}$$

For any displacement in one of the hypersurfaces (1.9) at $P(x^i)$ we have

$$p_i \, dx^i = 0. \tag{1.11}$$

Accordingly p_i are the components of the normal to the hypersurfaces if the x's are cartesian coördinates in a euclidean space. If we make any change of coördinates x^i into x'^i, we have that the p's defined by (1.6) are transformed according to the covariant law

$$p'_i = p_j \frac{\partial x^j}{\partial x'^i}.$$

Consequently we call the functions p_i in any coördinate system the components of the *covariant normal* to the hypersurface.* From (1.2) and (1.7) we have

$$\bar{p}_i \, d\bar{x}^i = p_i \, dx^i, \tag{1.12}$$

since

$$\frac{\partial \bar{\varphi}^j}{\partial \bar{x}^i} \frac{\partial \varphi^i}{\partial x^k} = \delta^j_k, \tag{1.13}$$

where

$$\delta^j_k = 1 \text{ or } 0, \text{ as } j = k \text{ or } j \neq k. \tag{1.14}$$

Comparing (1.11) and (1.12) we see that when a hypersurface (1.9) is transformed into the hypersurface defined by (1.10) for the same value of a, the covariant normal at a point of the former is transformed into the covariant normal to the latter at the corresponding point.

If in place of taking a set of functions p_i as defined by (1.6), we take any functions of the x's as the components of a covariant vector-field,

* A covariant normal is a covariant vector and determines $n-1$ directions at a point by equation (1.11), that is the tangent hyperplane at the point.

the equations (1.7) define a second covariant vector-field, of components $\bar{p}_i$ at each point in space, into which we say that the given field is transformed, when the points are transformed in accordance with (1.1). Moreover, equation (1.12) holds also in this case. At a point $P(x^i)$ for a given set of functions p_i, equations (1.11) define $n-1$ independent directions; so that we may say that p_i define an elemental V_{n-1}, or a hyperplane at P. From (1.12) it follows that when a point P is transformed into a point $\bar{P}$ by (1.1), an elemental V_{n-1} at P is transformed into one at $\bar{P}$, the determining functions $\bar{p}_i$ at $\bar{P}$ being given by (1.7). Consequently equations (1.1) and (1.7) define a transformation of points and elements at them into points and elements.

From the foregoing it is evident that, if two hypersurfaces are tangent at a point P, their transforms are tangent at the transformed point $\bar{P}$, and the covariant normal to the two hypersurfaces at P is transformed into the common covariant normal to their transforms at $\bar{P}$.

2. **Contact transformations.** We consider a generalization of the transformations (1.1) and (1.7) of the form

$$\bar{x}^i = \varphi^i(x^1, \cdots, x^n;\ p_1, \cdots, p_n), \quad \bar{p}_i = \psi_i(x^1, \cdots, x^n;\ p_1, \cdots, p_n) \tag{2.1}$$

in the $2n$ variables x^i and p_i, such that equations (1.12) hold, that is

$$\psi_j\left(\frac{\partial \varphi_j}{\partial x^i}\,dx^i + \frac{\partial \varphi^j}{\partial p_i}\,dp_i\right) = p_k\,dx^k, \tag{2.2}$$

for arbitrary values of the differentials dx^i and dp_i. A transformation (2.1) satisfying this condition is called a *contact transformation*; the significance of this term will appear later. From (2.2) it follows that a necessary and sufficient condition is that φ^i and ψ_i satisfy the equations

$$\psi_j \frac{\partial \varphi^j}{\partial x^i} = p_i, \qquad \psi_j \frac{\partial \varphi^j}{\partial p_i} = 0. \tag{2.3}$$

From the second set of these equations it follows that the rank of the jacobian $\left|\frac{\partial \varphi^j}{\partial p_i}\right|$ must be less than n. If the rank is $n-r$, on the elimination of the p's from the first set of equations (2.1) we obtain r independent equations

$$F_\alpha(\bar{x}^1, \cdots, \bar{x}^n;\ x^1, \cdots, x^n) = 0 \quad (\alpha = 1, \cdots, r). \tag{2.4}$$

From (2.4) it follows that the differentials at a transformed point corresponding to differentials at a point $P(x^i)$ are given by the system of equations

$$\frac{\partial F_\alpha}{\partial \bar{x}^i}\,d\bar{x}^i + \frac{\partial F_\alpha}{\partial x^j}\,dx^j = 0. \tag{2.5}$$

Since the conditions (1.12) must be satisfied as a consequence of (2.5), we must have

$$\bar{p}_i = \varrho_\alpha \frac{\partial F_\alpha}{\partial \bar{x}^i}, \qquad p_i = -\varrho_\alpha \frac{\partial F_\alpha}{\partial x^i}, \tag{2.6}$$

where the ϱ's are parameters such that, when these expressions for p_i are substituted in the first set of equations (2.1), the latter are satisfied identically because of (2.4). From the second set of equations (2.6) it follows that the rank of the jacobian matrix $\left\| \frac{\partial F_\alpha}{\partial x^i} \right\|$ must be r. Consequently r of these equations can be solved for the ϱ's, which as thus obtained are homogeneous functions of r of the p's of degree one, and when these are substituted in the remaining $n-r$ equations, we obtain $n-r$ equations involving x^i, $\bar{x}^i$ and p_i, the latter entering homogeneously of degree zero. Since on solving these new equations and (2.4) for the $\bar{x}$'s, we obtain the first set of (2.1), it follows that φ^i are homogeneous of degree zero in the p's, and consequently

$$p_i \frac{\partial \varphi^j}{\partial p_i} = 0. \tag{2.7}$$

Also from the first set of equations (2.6) we have that the functions $\psi_i (= \bar{p}_i)$ are homogeneous of degree one in the p's.

In order that the first set of equations (2.3) admit a uniqe solution in the ψ's, it is necessary and sufficient that the jacobian of the φ's with respect to the x's be of rank n, that is,

$$\left| \frac{\partial \varphi^j}{\partial x^i} \right| \not\equiv 0. \tag{2.8}$$

When this condition is satisfied and the solutions ψ_i of these equations are substituted in the second set, we have the n equations of condition

$$\begin{vmatrix} \frac{\partial \varphi^1}{\partial p_i} & \cdots\cdots & \frac{\partial \varphi^n}{\partial p_i} & 0 \\ \frac{\partial \varphi^1}{\partial x^1} & \cdots\cdots & \frac{\partial \varphi^n}{\partial x^1} & p_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \frac{\partial \varphi^1}{\partial x^n} & \cdots\cdots & \frac{\partial \varphi^n}{\partial x^n} & p_n \end{vmatrix} = 0.$$

From these equations it follows that

$$\frac{\partial \varphi^k}{\partial p_i} = \lambda^{ij} \frac{\partial \varphi^k}{\partial x^j}, \qquad \lambda^{ij} p_j = 0, \tag{2.9}$$

where the λ's are so defined. From the first of these sets of equations we have

$$\frac{\partial^2 \varphi^k}{\partial p_h \, \partial p_i} = \left(\frac{\partial \lambda^{ij}}{\partial p_h} + \lambda^{il} \frac{\partial \lambda^{hj}}{\partial x^l} \right) \frac{\partial \varphi^k}{\partial x^j} + \lambda^{ij} \lambda^{hr} \frac{\partial^2 \varphi^k}{\partial x^j \, \partial x^r}.$$

In consequence of (2.8) the conditions of integrability of the first set of equations (2.9) are

$$\frac{\partial \lambda^{ij}}{\partial p_h} - \frac{\partial \lambda^{hj}}{\partial p_i} + \lambda^{il} \frac{\partial \lambda^{hj}}{\partial x^l} - \lambda^{hl} \frac{\partial \lambda^{ij}}{\partial x^l} = 0.$$

Multiplying by p_j, summing for j and noting that from the second set of (2.9) we have

$$\frac{\partial \lambda^{ij}}{\partial x^l} p_j = 0, \qquad \frac{\partial \lambda^{ij}}{\partial p_h} p_j + \lambda^{ih} = 0,$$

we find that $\lambda^{ih} = \lambda^{hi}$, that is, λ^{ij} is symmetric in the indices. From (2.9) we have

$$\frac{\partial \varphi^k}{\partial p_i} \frac{\partial \varphi^l}{\partial x^i} = \lambda^{ij} \frac{\partial \varphi^k}{\partial x_j} \frac{\partial \varphi^l}{\partial x^i}.$$

Since λ^{ij} is symmetric in i and j, we have the identities

$$(\varphi^k, \varphi^l) \equiv \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^l}{\partial p_i} - \frac{\partial \varphi^k}{\partial p_i} \frac{\partial \varphi^l}{\partial x^i} = 0. \tag{2.10}$$

Conversely, if φ^i are any functions of x^i and p_i satisfying (2.7), (2.8) and (2.10) and ψ_i are functions defined by the first set of (2.3), then these functions are such that

$$\psi_j \frac{\partial \varphi^j}{\partial p_i} \frac{\partial \varphi^k}{\partial x^i} = \psi_j \frac{\partial \varphi^j}{\partial x^i} \frac{\partial \varphi^k}{\partial p_i} = p_i \frac{\partial \varphi^k}{\partial p_i} = 0.$$

Because of (2.8) it follows that these functions ψ_i satisfy the second set of (2.3). Hence we have:

A necessary and sufficient condition that a set of functions φ^i determine a contact transformation (2.1) *for which the ψ's are uniquely determined is that the functions φ^i be homogeneous of degree zero in the p's, that the*

jacobian of the φ's with respect to the x's be of rank n and that (2.10) *be satisfied.**

3. **Geometrical properties of contact transformations.** We consider now the geometrical significance of equations (2.4) and begin with the case when there is a single equation

$$F(\bar{x}^1, \cdots, \bar{x}^n;\ x^1, \cdots, x^n) = 0 \tag{3.1}$$

arising from the elimination of the p's from the first set of equations (2.1). In this case a point $P(x^i)$ is transformed into the points of the hypersurface $\bar{\Sigma}$, whose equation is given by (3.1) when the x's are given the values at P. Now equations (2.6) become

$$\bar{p}_i = \varrho \frac{\partial F}{\partial \bar{x}^i}, \qquad p_i = -\varrho \frac{\partial F}{\partial x^i}. \tag{3.2}$$

Each choice of p_i in (2.1) determines an elemental V_{n-1} at P, and for this choice there is a determined point $\bar{P}$ on the hypersurface $\bar{\Sigma}$. Moreover, from the first of (3.2) it is seen that the transform of the vector p_i is the covariant normal to the hypersurface $\bar{\Sigma}$ at $\bar{P}$. Also if the $\bar{x}$'s in (3.1) are given these values, this equation in the x's defines a hypersurface Σ, all of whose points transform into $\bar{P}$. In particular it follows from the second of (3.2) that the vector p_i chosen at P is the covariant normal to Σ at P.

Consider a hypersurface, S, whose equation is

$$f(x^1, \cdots, x^n) = 0. \tag{3.3}$$

To each of its points there corresponds a hypersurface $\bar{\Sigma}$ by means of (3.1). When in particular we put $p_i = \dfrac{\partial f}{\partial x^i}$ in the first of (2.1) these equations define a hypersurface $\bar{S}$, the $\bar{x}$'s being defined in terms of the x's as parameters, which are subject to the condition (3.3). Thus each point $P(x^i)$ of S is transformed into a definite point $\bar{P}(\bar{x}^i)$ of $\bar{S}$. A displacement on $\bar{S}$ is given by

$$d\bar{x}^i = \left(\frac{\partial \varphi^i}{\partial x^j} + \frac{\partial \varphi^i}{\partial p_k}\frac{\partial p_k}{\partial x^j}\right) dx^j$$

for each variation of the x's subject to the condition $\dfrac{\partial f}{\partial x^i} dx^i = 0$. Because of this consideration and (2.3) it follows that $\bar{p}_i$ given by (2.1) for this case satisfy the condition $\bar{p}_i d\bar{x}^i = 0$. Hence if $f(\bar{x}^1, \cdots, \bar{x}^n) = 0$

* This theorem was established in a different manner by Lie, l. c., p. 137.

is the equation of $\overline{S}$, resulting from the elimination of the x's from the first of (2.1) $\left(\text{in which } p_i = \frac{\partial f}{\partial x^i}\right)$ and (3.3), it follows that $\overline{p}_i = \lambda \frac{\partial \overline{f}}{\partial \overline{x}^i}$. If we take the hypersurface $\overline{\Sigma}$ corresponding to the point $P(x^i)$, the point on it determined by $p_i = \frac{\partial f}{\partial x^i}$ is the point $\overline{P}$ referred to above and $\overline{p}_i$ given by (2.1) for these p's are the components of the covariant normal to $\overline{\Sigma}$. Hence $\overline{\Sigma}$ and $\overline{S}$ are tangent at P.

We consider now the general case when $r > 1$ in (2.4). From (2.6) it follows that the jacobian matrix of the F's with respect to the x's is necessarily r, and likewise the matrix of the F's with respect to the $\overline{x}$'s. Geometrically this means that a point $P(x^i)$ is transformed into the points of a space of $n-r$ dimensions, say $\overline{\Sigma}_{n-r}$, and that a point $\overline{P}(\overline{x}_i)$ is the transform of the points of a space of $n-r$ dimensions, say Σ_{n-r}.* Each choice of p_i at a point P determines an elemental V_{n-1} at P, and for this choice there is a unique point $\overline{P}$ of $\overline{\Sigma}_{n-r}$ defined by the first set of equations (2.1). Any displacement in $\overline{\Sigma}_{n-r}$ at $\overline{P}$ satisfies the conditions $\frac{\partial \overline{F}_\alpha}{\partial \overline{x}^i} d\overline{x}^i = 0$ and consequently $\overline{p}_i d\overline{x}^i = 0$, as follows from (2.6). Hence $\overline{p}_i$ are the components of a covariant normal to $\overline{\Sigma}_{n-r}$, in the sense that a covariant vector λ_i in a V_n which satisfies the condition $\lambda_i dx^i = 0$ for all displacements at a point in a sub-space of V_n is called a covariant normal to the sub-space at the point. When p_i are chosen, $\overline{x}^i$ are determined as we have seen and then the corresponding Σ_{n-r}, and from the second set of (2.6) it is seen that p_i are the components of a covariant normal to this Σ_{n-r} at P.

As in the case when $r = 1$, each point of a hypersurface (3.3) is transformed into a $\overline{\Sigma}_{n-r}$ and each of these varieties is tangent to the hypersurface whose equation is obtained, when we replace p_i by $\frac{\partial f}{\partial x^i}$ in the first of (2.1) and eliminate the x's from these equations and (3.3).

We call the space $\overline{\Sigma}_{n-r}$ defined by (2.4) for a point the *fundamental variety* of the transformation for the point. Also we say that two sub-spaces, of orders p and q respectively $(p \geqq q)$, of a space V_n are tangent at a point P, if every covariant normal to the former at P is a covariant normal to the latter at P. Accordingly we have:

When the points of a hypersurface and all the elements at these points undergo a contact transformation, the fundamental varieties of the transformation for the points are tangent to the hypersurface which is the transform of the points for elements tangential to the given hypersurface; and the transformed elements are tangential to the hypersurface.

* These observations are consequences of the fundamental theorem on the existence of solutions of a set of equations involving implicit functions, cf. Fine, l. c.

As a corollary of this theorem we have that if two hypersurfaces are tangent at a point, that is, have the same covariant normal to the point, the envelopes of the two sets of varieties $\bar{\Sigma}_{n-r}$ for the points of the two given hypersurfaces will be tangent at the point which is the transform of the given point. The foregoing results justify the term contact transformations.

Consider a sub-space of $n-p$ dimensions, S_{n-p}, defined by the equations

$$f_\sigma(x^1, \cdots, x^n) = 0 \qquad (\sigma = 1, \cdots, p). \tag{3.4}$$

With each point $P(x^i)$ of S_{n-p} there is an associated $\bar{\Sigma}_{n-r}$. If in the first set of equations (2.1) we put

$$p_i = u^\sigma \frac{\partial f_\sigma}{\partial x^i}, \tag{3.5}$$

where the u's are parameters, we obtain the coördinates of a point $\bar{P}(\bar{x}^i)$ of $\bar{\Sigma}_{n-r}$. Since the functions φ^i are homogeneous of degree zero in the p's, when we eliminate the x's and u's from (2.1), (3.4) and (3.5), we get a single equation in the $\bar{x}$'s, that is, the locus of the points is a hypersurface $\bar{S}$. Its equations in parametric form are

$$\bar{x}^i = \varphi^i\left(x^1, \cdots, x^n; u^\sigma \frac{\partial f_\sigma}{\partial x^1}, \cdots, u^\sigma \frac{\partial f_\sigma}{\partial x^n}\right), \tag{3.6}$$

the parameters being the x's, subject to the conditions (3.4), and the ratio of the u's. Any displacement on $\bar{S}$ at $\bar{P}$ is given by

$$d\bar{x}^i = \frac{\partial \varphi^i}{\partial x^j} dx^j + \overline{\frac{\partial \varphi^i}{\partial p_k}} d\left(u^\sigma \frac{\partial f_\sigma}{\partial x^k}\right), \tag{3.7}$$

for any variations of the u's and for variations of the x's subject to the conditions $\frac{\partial f_\sigma}{\partial x^i} dx^i = 0$; in these equations $\overline{\frac{\partial \varphi^i}{\partial p_k}}$ indicates the expression which results when p_j in $\frac{\partial \varphi^i}{\partial p_k}$ are replaced by the expressions (3.5). Since (2.3) hold whatever be p_i, we have

$$\bar{p}_i d\bar{x}^i = p_j dx^j = u^\sigma \frac{\partial f_\sigma}{\partial x^j} dx^j = 0.$$

Since this equation holds for all displacements in $\bar{S}$, it follows that $\bar{p}_i$ are the components of the covariant normal to $\bar{S}$. When the coördinates of a point $P(x^i)$ of S_{n-p} are substituted in the right-hand member of (3.5),

the p's so defined for all values of the u's are the covariant normals to S_{n-p} at P. Hence we have:

When the points of a sub-space of dimensions $n-p$ and all the elements at these points undergo a contact transformation, the fundamental varieties of the transformation for the points are tangent to the hypersurface which is the transform of the points and the vectors which are the covariant normals to the sub-space; and the transformed elements are tangent to the hypersurface.

4. Definition of contact transformations by means of fundamental varieties. Suppose that in a V_n of coördinates x^i we have r equations of the form

$$F_\alpha(\overline{x}^1, \cdots, \overline{x}^n; x^1, \cdots, x^n) = 0 \qquad (\alpha = 1, \cdots, r), \tag{4.1}$$

where the $\overline{x}$'s are parameters. If, and only if, the jacobian matrix $\left\| \frac{\partial F_\alpha}{\partial x^i} \right\|$ is of rank r because of the equations (4.1), then for each set of values of the $\overline{x}$'s equations (4.1) define a variety of $n-r$ dimensions, say Σ_{n-r}. Any covariant normal to Σ_{n-r} has components of the form

$$p_i = u^\alpha \frac{\partial F_\alpha}{\partial x^i} \tag{4.2}$$

for suitable values of the parameters u^α; and any choice of the u's determines a covariant normal. A necessary and sufficient condition that the $n+r$ equations (4.1), (4.2) can be solved for the $\overline{x}$'s and u's uniquely is that the jacobian of these equations with respect to these quantities does not vanish because of (4.1) and (4.2); this follows from the general theorem concerning implicit functions.* This jacobian is

$$\begin{vmatrix} \frac{\partial F_1}{\partial \overline{x}^1} & \cdots\cdots & \frac{\partial F_1}{\partial \overline{x}^n} & 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_r}{\partial \overline{x}^1} & \cdots\cdots & \frac{\partial F_r}{\partial \overline{x}^n} & 0 & \cdots\cdots & 0 \\ u^\alpha \frac{\partial^2 F_\alpha}{\partial x^1 \partial \overline{x}^1} & \cdots\cdots & u^\alpha \frac{\partial^2 F_\alpha}{\partial x^1 \partial \overline{x}^n} & \frac{\partial F_1}{\partial x^1} & \cdots\cdots & \frac{\partial F_r}{\partial x^1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u^\alpha \frac{\partial^2 F_\alpha}{\partial x^n \partial \overline{x}^1} & \cdots\cdots & u^\alpha \frac{\partial^2 F_\alpha}{\partial x^n \partial \overline{x}^n} & \frac{\partial F_1}{\partial x^n} & \cdots\cdots & \frac{\partial F_r}{\partial x^n} \end{vmatrix}. \tag{4.3}$$

* It is assumed that the functions F_α and their first and second derivatives are continuous for the domain under consideration. Cf. Fine, l. c.

Since this determinant does not involve the p's, it must not vanish because of (4.1) alone. Since the latter do not involve the u's, it must not vanish for arbitrary values of the u's and as a result of (4.1). If we apply the rule of Laplace to this determinant, we see that the rank of the matrix $\left\| \frac{\partial F_\alpha}{\partial \bar{x}^i} \right\|$ must be r.

Since we are concerned only with the determination of the $\bar{x}$'s, we may arrive at our result by solving (4.1) and the equations resulting from (4.2) on the elimination of the u's. By a suitable numbering of the x's, we have that the determinant $\left| \frac{\partial F_\alpha}{\partial x^\beta} \right|$ for $\alpha, \beta = 1, \cdots, r$ is of rank r, in consequence of the assumption concerning the matrix $\left\| \frac{\partial F_\alpha}{\partial x^i} \right\|$. Then by the elimination of the u's from (4.2) we have $n - r$ independent equations

$$\begin{vmatrix} \frac{\partial F_1}{\partial x^1} & \cdots\cdots & \frac{\partial F_1}{\partial x^r} & \frac{\partial F_1}{\partial x^\sigma} \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \frac{\partial F_r}{\partial x^1} & \cdots\cdots & \frac{\partial F_r}{\partial x^r} & \frac{\partial F_r}{\partial x^\sigma} \\ p_1 & \cdots\cdots & p_r & p_\sigma \end{vmatrix} = 0 \qquad (\sigma = r+1, \cdots, n). \tag{4.4}$$

In order that these equations and (4.1) be solvable for the $\bar{x}$'s, it is seen that the matrix $\left\| \frac{\partial F_\alpha}{\partial \bar{x}^i} \right\|$ must be of rank r for the equations (4.1), as previously seen from the considerations concerning the determinant (4.3). It is evident from (4.4) that these solutions, say

$$\bar{x}^i = \varphi^i(x, p) \tag{4.5}$$

are such that the φ's are homogeneous of degree zero in the p's. When these expressions are substituted in (4.1), we have identities in the x's and p's. Differentiating these identities, we have

$$\frac{\partial F_\alpha}{\partial x^i} + \left(\frac{\partial F_\alpha}{\partial \bar{x}^j}\right) \frac{\partial \varphi^j}{\partial x^i} = 0, \qquad \left(\frac{\partial F_\alpha}{\partial \bar{x}^j}\right) \frac{\partial \varphi^j}{\partial p_i} = 0, \tag{4.6}$$

where $\left(\frac{\partial F_\alpha}{\partial \bar{x}^j}\right)$ indicates the result of replacing the $\bar{x}$'s by the φ's in $\frac{\partial F_\alpha}{\partial \bar{x}^j}$. If we denote by $\bar{u}^\alpha$ the functions obtained by solving (4.2) for the u's after the $\bar{x}$'s have been replaced by the φ's, and put

$$\psi_j = -\bar{u}^\alpha \left(\frac{\partial F_\alpha}{\partial \bar{x}^j}\right), \tag{4.7}$$

then from (4.2) and (4.6) it follows that ψ_i satisfy the conditions (2.3), and consequently the functions φ^i and ψ_i define a contact transformation. Hence we have:

If F_α are r functions of $2n$ variables, $\bar{x}^i$ and x^i, such that the determinant (4.3) *is of rank $n+r$ for the equations $F_\alpha = 0$ for all values of the parameters u^α, equations* (4.5) *and* (4.7) *define a contact transformation.**

If we consider a hypersurface S, defined by (3.3), and in place of (4.2) take

$$\frac{\partial f}{\partial x^i} + \lambda^\alpha \frac{\partial F_\alpha}{\partial x^i} = 0, \tag{4.8}$$

and then solve for the $\bar{x}$'s these equations, after the elimination of the λ's, and (4.1), we get the equations in parametric form, the x's being the parameters subject to the condition (3.3), of the hypersurface $\bar{S}$ defined in § 3. Eliminating the x's from these equations, we get the equation $\bar{f}(\bar{x}^1, \cdots, \bar{x}^n) = 0$ of $\bar{S}$, which would likewise be obtained by eliminating directly the x's and λ's from (4.1), (4.8) and (3.3), in accordance with the usual theory of envelopes.

When we have a contact transformation defined by (2.1) and obtain the corresponding set of equations (2.4), the above mentioned conditions upon the corresponding determinant (4.3) are necessarily satisfied. We observe furthermore, that in the form of (4.3) the variables x^i and $\bar{x}^i$ enter symmetrically. Consequently, if we interchange the roles of the x's and $\bar{x}$'s in the above process, we obtain a unique contact transformation

$$x^i = \bar{\varphi}^i(\bar{x}, \bar{p}), \qquad p_i = \bar{\psi}_i(\bar{x}, \bar{p}). \tag{4.9}$$

This transformation is the inverse of the former. For, if we take any point $P(x^i)$ and a point $\bar{P}(\bar{x}^i)$, such that (4.1) are satisfied, and in (4.5) replace p_i by the quantities (4.2), the resulting equations are satisfied identically for all values of u^α, and $\bar{p}_i$ are given by

$$\bar{p}_i = -u^\alpha \frac{\partial F_\alpha}{\partial \bar{x}_i}, \tag{4.10}$$

as follows from (2.6). A similar result is true, if we proceed with (4.9) by means of (4.10). Hence we have:

* Cf. Lie, l. c., p. 149.

Every contact transformation admits a unique inverse contact transformation.

If we have two contact transformations, namely (2.1) and

$$x_1^i = \overline{\varphi}^i(\overline{x}, \overline{p}), \qquad p_{1i} = \overline{\psi}_i(\overline{x}, \overline{p}),$$

we have

$$\overline{\psi}_i\, d\overline{\varphi}^i = \overline{p}_i\, d\overline{x}^i, \qquad \psi_i\, d\varphi^i = p_i\, dx^i.$$

Replacing $\overline{p}_i$ and $\overline{x}_i$ in the first by ψ_i and φ^i, we have $\overline{\psi}_i\, d\overline{\varphi}^i = p_i\, dx^i$. Consequently we have the theorem of Lie:*

The totality of contact transformations in $2n$ quantities x^i and p_i form an infinite group, which contains the inverse of each transformation.

For the inverse of the transformation (2.1) we have analogously to (2.3)

$$p_k \frac{\partial x^k}{\partial x'^i} = p'_i, \qquad p_k \frac{\partial x^k}{\partial p'_j} = 0.$$

Differentiating the first with respect to x'^j, we have from the resulting expression that

$$\{x'^i, x'^j\} \overset{\bullet}{\equiv} \frac{\partial x^k}{\partial x'^i}\frac{\partial p_k}{\partial x'^j} - \frac{\partial x^k}{\partial x'^j}\frac{\partial p_k}{\partial x'^i} = 0.$$

Differentiating the second with respect to p'_i, we have

$$\{p'_i, p'_j\} = 0.$$

If we differentiate the first with respect to p'_j and the second with respect to x'^i and subtract the resulting equations, we have

$$\{x'^i, p'_j\} = \delta^i_j$$

where

(4.11) $$\delta^i_j = 1 \text{ or } 0, \quad \text{as } i = j \text{ or } i \neq j.$$

Consider $2n$ independent functions u^α, for $\alpha = 1, \cdots, 2n$, of x^i and p_i. Since

$$\frac{\partial x^i}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^j} = \delta^i_j, \qquad \frac{\partial p_i}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial p_j} = \delta^j_i, \qquad \frac{\partial x^i}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial p_j} = \frac{\partial p_j}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^i} = 0,$$

we have

$$\sum_\alpha^{1,\cdots,2n} (u^\alpha, u^\beta)\{u^\alpha, u^\gamma\} = \delta^\beta_\gamma,$$

* L. c., p. 139.

where (u^α, u^β) is defined by (2.10). Applying this identity to the above case we have

$$(x'^i, x'^j) = 0, \qquad (x'^i, p'_j) = \delta^i_j, \qquad (p'_i, p'_j) = 0.$$

Hence we have the theorem:*

For a homogeneous contact transformation the functions φ^i and ψ_i satisfy the identities

$$\{\varphi^i, \varphi^j\} = \{\psi_i, \psi_j\} = 0, \qquad \{\varphi^i, \psi_j\} = \delta^i_j, \tag{4.12}$$

and

$$(\varphi^i, \varphi^j) = (\psi_i, \psi_j) = 0, \qquad (\varphi^i, \psi_j) = \delta^i_j. \tag{4.13}$$

5. **Infinitesimal contact transformations.** As we wish to consider one parameter continuous groups of contact transformations, we begin with a study of infinitesimal contact transformations. Such a transformation is defined by equations of the form

$$\bar{x}^i = x^i + \xi^i \delta t, \qquad \bar{p}_i = p_i + \eta_i \delta t, \tag{5.1}$$

where the ξ's and η's are functions of the x's and p's such that these equations must satisfy the conditions (2.3). This gives the conditions

$$\eta_i + p_j \frac{\partial \xi^j}{\partial x^i} = 0, \qquad p_j \frac{\partial \xi^j}{\partial p_i} = 0. \tag{5.2}$$

If we put

$$C = p_j \xi^j, \tag{5.3}$$

we have by differentiation and as a consequence of (5.2)

$$\frac{\partial C}{\partial x^i} = p_j \frac{\partial \xi^j}{\partial x^i} = -\eta_i, \qquad \frac{\partial C}{\partial p_i} = \xi^i + p_j \frac{\partial \xi^j}{\partial p_i} = \xi^i. \tag{5.4}$$

From the second set of these equations and (5.3) we have

$$C = p_j \frac{\partial C}{\partial p_j}. \tag{5.5}$$

Consequently C is homogeneous of first degree in the p's. Differentiating (5.5) with respect to x^i and p_i, we obtain respectively

$$\frac{\partial C}{\partial x^i} - p_j \frac{\partial^2 C}{\partial p_j \partial x^i} = 0, \qquad p_j \frac{\partial^2 C}{\partial p_i \partial p_j} = 0, \tag{5.6}$$

which are equivalent to (5.2) because of (5.4).

* Cf. Lie, p. 137; also Whittaker, Analytical Dynamics, p. 300, Cambridge, 1927.

Conversely if C is any function of the x's and p's homogeneous of first degree in the p's, and we take ξ^i and η_i as defined by (5.4), the conditions (5.2) are satisfied, because (5.6) are a consequence of (5.5). Accordingly we have:

Every infinitesimal contact transformation is defined by equations of the form

$$\bar{x}^i = x^i + \frac{\partial C}{\partial p_i}\delta t, \qquad \bar{p}_i = p_i - \frac{\partial C}{\partial x^i}\delta t, \tag{5.7}$$

where C is homogeneous of first degree in the p's; moreover, any such function C determines an infinitesimal contact transformation.

The function C is called the *characteristic function* of the transformation.

If we form the differentials of (5. 7), we obtain

$$\begin{aligned} d\bar{x}^i &= dx^i + \Big(\frac{\partial^2 C}{\partial p^i \partial x^j}\,dx^j + \frac{\partial^2 C}{\partial p_i \partial p_k}\,dp_k\Big)\delta t, \\ d\bar{p}_i &= dp_i - \Big(\frac{\partial^2 C}{\partial x^i \partial x^j}\,dx^j + \frac{\partial^2 C}{\partial x^i \partial p_k}\,dp_k\Big)\delta t. \end{aligned} \tag{5.8}$$

Equations (5.7) and (5.8) define the extended infinitesimal transformation of the elements x^i, p_i, dx^i and dp_i. For this extended transformation $p_i\,dx^i$ is invariant, as follows from the determination of (5.7). Hence in accordance with the general theory of continuous groups the quantity $p_i\,dx^i$ is invariant under the finite group G_1 generated by the extended infinitesimal transformation, and the finite equations* of G_1 are given by the integrals of the equations

$$\frac{\delta x^i}{\delta t} = \frac{\partial C}{\partial p^i}, \qquad \frac{\delta p_i}{\delta t} = -\frac{\partial C}{\partial x^i}, \tag{5.9}$$

say

$$\bar{x}^i = \varphi^i(x,\ p,\ t), \qquad \bar{p}_i = \psi_i(x,\ p,\ t), \tag{5.10}$$

and their differentials. Consequently (5.10) define a one-parameter group of contact transformations.

From the foregoing considerations we have:

The most general one-parameter group of contact transformations is given by the solution of equations (5.9) *in which C is an arbitrary analytic function of the x's and p's which is homogeneous of degree one in the latter.*

Equations (5.9) admit the integral $C = h$, where h is an arbitrary constant. This means that in the $2n$-dimensional space of coördinates x^i and p_i, each integral curve, or *trajectory*, lies in one of the hypersurfaces $C = h$. When in the right-hand members of (5.10), the quantities

* Cf. Lie, p. 212.

x^i and p_i are given particular values, these equations define such an integral curve, and the hypersurface is determined by the value of C for the particular point. Since C is homogeneous of degree one in the p's, φ^i are of degree zero and ψ_i of degree one. Hence if we replace p_i by p_i/h, we get a curve for which φ^i are unaltered and $\bar{p}_i = \psi_i/h$, and these satisfy the condition

$$C = 1. \tag{5.11}$$

From the standpoint of contact transformations this change is of no significance, so that except for the case $C = 0$, we may in all generality take (5.11) as holding.

When real initial values of the p's can be chosen so that $C = 0$, this equation holds along the trajectory; such a trajectory we call *singular*. In this case, as follows from (5.2) and $p_j \xi^j = 0$, the ξ's are determined only to within a factor. Consequently, if C is of such a form that $\dfrac{\partial C}{\partial p_i}$ and $\dfrac{\partial C}{\partial x^i}$ involve a factor which makes them vanish or become infinite when $C = 0$, we remove this factor. Consider for example the case when C is the square-root of a homogeneous quadratic form in the p's, say

$$C = \sqrt{g^{ij} p_i p_j}, \tag{5.12}$$

in which the g's are functions of the x's. For the non-singular trajectories we choose the p's so that $C = 1$, in which case equations (5.9) become

$$\frac{dx^i}{dt} = g^{ij} p_j, \qquad \frac{dp_i}{dt} = -\frac{1}{2} \frac{\partial g^{jk}}{\partial x^i} p_j p_k. \tag{5.13}$$

From (5.1) and (5.3) we have that when $C = 0$, $p_i \, dx^i = 0$. This condition is satisfied by the first set of (5.13) and accordingly we take (5.13) as the equations for the singular trajectories also, when there are such trajectories. We observe also that $g^{ij} p_i p_j = const.$ is a first integral of these equations.

If we transform equations (5.9) by means of a general contact transformation (2.1), written $x'^i = \varphi^i$, $p'_i = \psi_i$, we have

$$\frac{dx'^i}{dt} = \frac{\partial \varphi^i}{\partial x^j} \frac{\partial C}{\partial p_j} - \frac{\partial \varphi^i}{\partial p_j} \frac{\partial C}{\partial x^j}, \qquad \frac{dp'_i}{dt} = \frac{\partial \psi_i}{\partial x^j} \frac{\partial C}{\partial p_j} - \frac{\partial \psi_i}{\partial p_j} \frac{\partial C}{\partial x^j}.$$

If C' is the transform of C, we have

$$\begin{aligned} \frac{\partial \varphi^i}{\partial x^j} \frac{\partial C}{\partial p_j} - \frac{\partial \varphi^i}{\partial p_j} \frac{\partial C}{\partial x^j} = &\left(\frac{\partial \varphi^i}{\partial x^j} \frac{\partial \varphi^k}{\partial p_j} - \frac{\partial \varphi^i}{\partial p_j} \frac{\partial \varphi^k}{\partial x^j} \right) \frac{\partial C'}{\partial x'^k} \\ &+ \left(\frac{\partial \varphi^i}{\partial x^j} \frac{\partial \psi_k}{\partial p_j} - \frac{\partial \varphi^i}{\partial p_j} \frac{\partial \psi_k}{\partial x^j} \right) \frac{\partial C'}{\partial p'_k}. \end{aligned}$$

In consequence of (4.13) this reduces to $\frac{\partial C'}{\partial p_i'}$. In like manner the right-hand member of the second set of equations is equal to $-\frac{\partial C'}{\partial x'^i}$. Hence we have:

A group G_1 of contact transformations is transformed into another group G_1 by any contact transformation and the equations of the new group are integrals of the equations

$$\frac{dx'^i}{dt} = \frac{\partial C'}{\partial p_i'}, \qquad \frac{dp_i'}{dt} = -\frac{\partial C'}{\partial x'^i},$$

where C' is the transform of the characteristic function of the given group.

6. **Non-homogeneous contact transformations.** In his development of the theory of contact transformations Lie considers first transformations of the form

$$(6.1)\quad Z = \varphi(z, x^1, \cdots, x^n; p_1, \cdots, p_n), \quad X^i = \varphi^i(z, x; p), \quad P_i = \psi_i(z, x; p),$$

such that the equation

$$(6.2)\qquad dZ - P_i\, dX_i = 0$$

is satisfied whenever

$$(6.3)\qquad dz - p_i\, dx^i = 0$$

is. Consequently the analytical problem reduces to the determination of functions φ, φ^i, ψ_i and also a function ϱ of z, the x's and the p's, such that

$$(6.4)\qquad dZ - P_i\, dX^i = \varrho(dz - p_i\, dx^i)$$

for independent values of the differentials dz, dx^i and dp_i. When, in particular, we take $p_i = \frac{\partial f}{\partial x^i}$, where f is a function of the x's, the differentials in (6.3) correspond to displacements in the hypersurface $z = f(x^1, \cdots, x^n)$ and the elements are tangential to it. When these values are substituted in the first $n+1$ of equations (6.1), we have on the elimination of the x's, $Z = F(X^1, \cdots, X^n)$ and the P_i determine tangential elements at each point of this hypersurface in consequence of (6.2). We call a transformation (6.1) *non-homogeneous* and those of § 2 *homogeneous*.

For an n dimensional space of coördinates x^i, if we let x^1 play the role of z in the above equations, the analytical problem consists in determining transformations

$$(6.5)\quad \overline{x}^i = \varphi^i(x^1, \cdots, x^n; p_2, \cdots, p_n), \quad \overline{p}_i = \psi_i(x^1, \cdots, x^n; p_2, \cdots, p_n),$$

where the φ's and ψ's are such that the relation

$$(6.6)\qquad \overline{p}_i\, d\overline{x}^i = -dx^1 + p_\alpha\, dx^\alpha \qquad (\alpha = 2, \cdots, n)$$

holds for independent values of the differentials dx^i and dp_α. The resulting conditions are

$$(6.7)\quad \psi_j \frac{\partial \varphi^j}{\partial x^1} = -1, \quad \psi_j \frac{\partial \varphi^j}{\partial x^\alpha} = p_\alpha, \quad \psi_j \frac{\partial \varphi^j}{\partial p_\alpha} = 0 \quad (\alpha = 2, \cdots, n).$$

Comparing these equations with (2.3), we see that if we take any homogeneous transformation and in its equations put $p_1 = -1$, we obtain a non-homogeneous transformation.

Suppose conversely that we have a set of functions satisfying (6.7), and in the φ's we put

$$(6.8)\quad p_\alpha = -\frac{p'_\alpha}{p'_1},$$

and denote the resulting functions by φ'^i. Since

$$(6.9)\quad \frac{\partial \varphi'^j}{\partial p'_1} = \frac{\partial \varphi^j}{\partial p_\alpha}\frac{p'_\alpha}{p'^2_1} = -\frac{\partial \varphi^j}{\partial p_\alpha}\frac{p_\alpha}{p'_1}, \quad \frac{\partial \varphi'^j}{\partial p'_\alpha} = -\frac{\partial \varphi^j}{\partial p_\alpha}\frac{1}{p'_1} \quad (\alpha = 2, \cdots, n),$$

where α is summed in the first set of equations, we have from (6.7), (6.8) and (6.9)

$$\psi'_j \frac{\partial \varphi'^j}{\partial x^i} = p'_i, \qquad \psi'_j \frac{\partial \varphi'^j}{\partial p'_i} = 0,$$

where

$$(6.10)\quad \psi'_j = -p'_1 \psi_j\left(x^1, \cdots, x^n; -\frac{p'_2}{p'_1}, \cdots, -\frac{p'_n}{p'_1}\right).$$

Consequently the functions φ'^i and ψ'_i define a homogeneous contact transformation. In particular, we observe that, if the non-homogeneous transformation is derived from a homogeneous one as indicated in the preceding paragraph, the homogeneous transformation arising from the above process is the given one. It should be observed that a homogeneous transformation obtained in this manner does not apply to elements for which $p'_1 = 0$, unless the ψ's are homogeneous of degree one in the p's. From the foregoing considerations it follows that the results of § 3 apply to non-homogeneous transformations.

When we substitute the expressions (6.9) in equations of the form (2.10), we obtain

$$(6.11)\quad [\varphi^k, \varphi^l] \equiv \frac{\partial \varphi^l}{\partial p_\alpha}\left(\frac{\partial \varphi^k}{\partial x^\alpha} + p_\alpha \frac{\partial \varphi^k}{\partial x^1}\right) - \frac{\partial \varphi^k}{\partial p_\alpha}\left(\frac{\partial \varphi^l}{\partial x^\alpha} + p_\alpha \frac{\partial \varphi^l}{\partial x^1}\right) = 0.$$

Conversely, suppose we have a set of functions φ^i of $x^1, \cdots, x^n; p_2, \cdots, p_n$ satisfying these equations and such that the jacobian of the φ's with respect

to the x's is of rank n. A set of functions ψ_i is uniquely determined by the first n of equations (6.7). In fact, if equations (6.11) be multiplied by ψ_k and k be summed, we obtain

$$\psi_k \frac{\partial \varphi^k}{\partial p_\alpha}\left(\frac{\partial \varphi^l}{\partial x^\alpha} + p_\alpha \frac{\partial \varphi^l}{\partial x^1}\right) = 0. \tag{6.12}$$

These equations hold for $l = 1, \cdots, n$, since (6.11) are identically satisfied when $k = l$. Solving the first n of equations (6.7) for ψ_k, we have

$$\left|\frac{\partial \varphi^i}{\partial x^j}\right| \cdot \psi_k = (-1)^k \begin{vmatrix} 1 & \frac{\partial \varphi^1}{\partial x^1} & \cdots & \frac{\partial \varphi^{k-1}}{\partial x^1} & \frac{\partial \varphi^{k+1}}{\partial x^1} & \cdots & \frac{\partial \varphi^n}{\partial x^1} \\ -p_2 & \frac{\partial \varphi^1}{\partial x^2} & \cdots & & & & \frac{\partial \varphi^n}{\partial x^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -p_n & \frac{\partial \varphi^1}{\partial x^n} & \cdots & & & & \frac{\partial \varphi^n}{\partial x^n} \end{vmatrix}.$$

The determinant on the right is equal to the determinant of the quantities $\left(\frac{\partial \varphi^l}{\partial x^\alpha} + p_\alpha \frac{\partial \varphi^l}{\partial x^1}\right)$ for $\alpha = 2, \cdots, n$; $l = 1, \cdots, k-1, k+1, \cdots, n$.* Since one at least of the ψ's is not zero, it follows from (6.12) that $\psi_k \frac{\partial \varphi^k}{\partial p_\alpha} = 0$. Hence we have:

A necessary and sufficient condition that a set of functions φ^i of $x^1, \cdots, x^n$; $p_2, \cdots, p_n$ determine a non-homogeneous contact transformation is that the jacobian of the φ's with respect to the x's be of rank n and that $[\varphi^k, \varphi^l] = 0$; then the functions ψ_i of the transformation are uniquely determined.†

From (6.10) and (6.8) we have

$$\frac{\partial \psi_j'}{\partial x^i} = -p_1' \frac{\partial \psi_j}{\partial x^i}, \quad \frac{\partial \psi_k'}{\partial p_1'} = -\psi_k + \frac{\partial \psi_k}{\partial p_\alpha} p_\alpha, \quad \frac{\partial \psi_k'}{\partial p_\alpha'} = \frac{\partial \psi_k}{\partial p_\alpha}.$$

Hence from (4.13) we have, using the notation (6.11),

$$\begin{aligned} [\psi_j, \psi_k] &= \psi_k \frac{\partial \psi_j}{\partial x^1} - \psi_j \frac{\partial \psi_k}{\partial x^1}, \\ [\varphi^j, \psi_k] &= \delta_k^j + \psi_k \frac{\partial \varphi^j}{\partial x^1}. \end{aligned} \tag{6.13}$$

* Cf. Fine, College Algebra, p. 505.

† Cf. Lie, l. c., p. 124.

Equations (6.11) and (6.13) follow directly also, if in the equations (2.7) and $p_i \frac{\partial \psi_j}{\partial p_i} = \psi_j$ we solve for $\frac{\partial \varphi^j}{\partial p_1}$ and $\frac{\partial \psi^j}{\partial p_1}$ and put $p_1 = -1$, giving

$$\frac{\partial \varphi^j}{\partial p_1} = \left(p_\alpha \frac{\partial \varphi^j}{\partial p_\alpha}\right)_{p_1=-1}, \qquad \frac{\partial \psi^j}{\partial p_1} = \left(p_\alpha \frac{\partial \psi^j}{\partial p_\alpha} - \psi_j\right)_{p_1=-1},$$

and substitute in (4.13).

If the rank of the jacobian matrix $\left\| \frac{\partial \varphi^i}{\partial p_\alpha} \right\|$ for equations (6.5) is $n-r$, on the elimination of the p's from the first n of these equations we obtain equations

$$F_\sigma(\bar{x}^1, \cdots, \bar{x}^n; x^1, \cdots, x^n) = 0 \qquad (\sigma = 1, \cdots, r).$$

Making use of (6.6), we have analogously to (2.6)

$$\bar{p}_i = \varrho_\sigma \frac{\partial F_\sigma}{\partial \bar{x}^i}, \qquad 1 = \varrho_\sigma \frac{\partial F_\sigma}{\partial x^1}, \qquad p_\alpha = -\varrho_\sigma \frac{\partial F_\sigma}{\partial x^\alpha}.$$

Conversely, if we have a set of equations $F_\sigma = 0$, it can be shown, as in § 4, that these determine a non-homogeneous contact transformation, provided the determinant (4.3) is of rank $n+r$ for arbitrary values of the parameters ϱ_σ. Then the further determination of the functions φ^i of the transformation requires the solution for the $\bar{x}$'s of the equations $F_\sigma = 0$ and (4.4) in which $p_1 = -1$. In so doing it is understood that the variables are numbered in such a way that the determinant $\left| \frac{\partial F_\sigma}{\partial x^\tau} \right|$ for $\sigma, \tau = 1, \cdots, r$ is of rank r.*

For an infinitesimal transformation (5.1) in which $p_1 = -1$, the equations (6.7) become

$$\eta_i - \frac{\partial \xi^1}{\partial x^i} + p_\alpha \frac{\partial \xi^\alpha}{\partial x^i} = 0, \qquad \frac{\partial \xi^1}{\partial p_\alpha} - p_\beta \frac{\partial \xi^\beta}{\partial p_\alpha} = 0 \tag{6.14}$$

$$(\alpha, \beta = 2, \cdots, n).$$

If we put

$$W = p_\alpha \xi^\alpha - \xi^1,$$

these equations reduce to

$$\eta_i = -\frac{\partial W}{\partial x^i}, \qquad \xi^\alpha = \frac{\partial W}{\partial p_\alpha}, \qquad \xi^1 = p_\alpha \frac{\partial W}{\partial p_\alpha} - W.$$

Conversely, if W is any function of $x^1, \cdots, x^n$ and $p_2, \cdots, p_n$, these expressions satisfy (6.14). Hence we have:

The most general infinitesimal non-homogeneous contact transformation is defined by equations of the form

* Cf. Lie, l. c., p. 155.

$$(6.15)\quad \frac{\delta x^1}{\delta t} = p_\alpha \frac{\partial W}{\partial p_\alpha} - W, \quad \frac{\delta x^\alpha}{\delta t} = \frac{\partial W}{\partial p_\alpha}, \quad \frac{\delta p_i}{\delta t} = -\frac{\partial W}{\partial x^i},$$

*where W is an arbitrary function of $x^1, \cdots, x^n; p_2, \cdots, p_n$.**

7. Restricted non-homogeneous transformations. We consider now the particular case when the function φ^1 in equations (6.5) is of the form $x^1 + \varphi^1(x^2, \cdots, x^n; p_2, \cdots, p_n)$ and the functions φ^α for $\alpha = 2, \cdots, n$ do not involve x^1. Then equations (6.7) reduce to

$$(7.1)\quad \psi_1 = -1, \quad \psi_\beta \frac{\partial \varphi^\beta}{\partial x^\alpha} = p_\alpha + \frac{\partial \varphi^1}{\partial x^\alpha}, \quad \psi_\beta \frac{\partial \varphi^\beta}{\partial p_\alpha} = \frac{\partial \varphi^1}{\partial p_\alpha}.$$

Consequently the functions ψ_α do not involve x^1. Moreover equations (6.11) reduce in this case to

$$(7.2)\quad (\varphi^\beta, \varphi^\gamma) \equiv \frac{\partial \varphi^\beta}{\partial x^\alpha}\frac{\partial \varphi^\gamma}{\partial p_\alpha} - \frac{\partial \varphi^\beta}{\partial p_\alpha}\frac{\partial \varphi^\gamma}{\partial x^\alpha} = 0$$

and

$$(7.3)\quad (\varphi^1, \varphi^\beta) = -p_\alpha \frac{\partial \varphi^\beta}{\partial p_\alpha} \qquad (\alpha, \beta, \gamma = 2, \cdots, n).$$

In consequence of the first theorem of § 6 we have:

If φ^α are $n-1$ independent functions of $x^2, \cdots, x^n; p_2, \cdots, p_n$ satisfying (7.2) *and φ^1 is any function of these quantities satisfying* (7.3), *the equations*

$$(7.4)\quad \bar{x}^1 = x^1 + \varphi^1, \quad \bar{x}^\alpha = \varphi^\alpha, \quad \bar{p}_\alpha = \psi_\alpha \qquad (\alpha = 2, \cdots, n),$$

where the ψ's are uniquely determined by (7.1), *define a contact transformation for which*

$$(7.5)\quad d\bar{x}^1 - \bar{p}_\alpha\, d\bar{x}^\alpha = dx^i - p_\alpha\, dx^\alpha,$$

and consequently

$$(7.6)\quad \bar{p}_\alpha\, d\bar{x}^\alpha = p_\alpha\, dx^\alpha + d\varphi^1.\dagger$$

We call such transformations *restricted non-homogeneous.*

If the functions φ^α are homogeneous of degree zero in the p's, equations (7.3) reduce to $(\varphi^1, \varphi^\beta) = 0$. Looked upon as linear partial differential equations in φ^1, these equations form a complete system, as can be shown with the aid of the identity of Jacobi, namely

$$(f, (\varphi, \psi)) + (\varphi, (\psi, f)) + (\psi, (f, \varphi)) = 0.$$

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* If we write (1.12) with $p_1 = -1$ in the form $d\bar{x}_1 - P_\alpha\, d\bar{x}^\alpha = \varrho(dx^1 - p_\alpha\, dx^\alpha)$, then $P_\alpha = -\frac{\bar{p}_\alpha}{\bar{p}_1} = p_\alpha + (\eta_\alpha + p_\alpha \eta_1)\delta t$. Hence in place of the last set of (6.15) we have $\frac{\delta p_\alpha}{\delta t} = -\left(\frac{\partial W}{\partial x^\alpha} + p_\alpha \frac{\partial W}{\partial x^1}\right)$. This is the form of the theorem given by Lie, l. c., p. 252.

† Cf. Lie, l. c., p. 129, where it is shown that a solution of (7.3) is given by a quadrature.

Consequently these equations admit $n-1$ independent integrals. But φ^α are independent integrals in consequence of (7.2). Hence φ^1 is an arbitrary function of the φ^α, say $F(\varphi^2, \cdots, \varphi^n)$. In this case we have from (7.1)

$$\left(\psi_\beta - \frac{\partial F}{\partial \varphi^\beta}\right)\frac{\partial \varphi^\beta}{\partial x^\alpha} = p_\alpha, \qquad \left(\psi_\beta - \frac{\partial F}{\partial \varphi^\beta}\right)\frac{\partial \varphi^\beta}{\partial p_\alpha} = 0,$$

so that the transformation is essentially of the homogeneous type.

We remark also that for the restricted case equations (6.13) become

$$(7.7)\quad (\varphi^1, \psi_\alpha) = \psi_\alpha - p_\beta \frac{\partial \varphi_\alpha}{\partial p_\beta}, \quad (\varphi^\alpha, \psi_\beta) = \delta^\alpha_\beta, \quad (\psi_\alpha, \psi_\beta) = 0.$$

The jacobian of φ^α with respect to the p's is of rank $n-1-r$, where $r \geqq 0$. When the p's are eliminated from the first n of equations (7.4) we get

$$(7.8)\qquad \bar{x}^1 - x^1 = F(\bar{x}^2, \cdots, \bar{x}^n; x^2, \cdots, x^n)$$

and r equations

$$(7.9)\qquad F_\sigma(\bar{x}^2, \cdots, \bar{x}^n; x^2, \cdots, x^n) = 0 \qquad (\sigma = 1, \cdots, r).$$

In this case these two sets of equations take the place of equations (2.4) for the homogeneous case, and in place of (2.6) we have, on making use of (7.5),

$$(7.10)\qquad \bar{p}_\alpha = \frac{\partial F}{\partial \bar{x}^\alpha} + \varrho_\sigma \frac{\partial F_\sigma}{\partial \bar{x}^\alpha}, \qquad p_\alpha = -\frac{\partial F}{\partial x^\alpha} - \varrho_\sigma \frac{\partial F_\sigma}{\partial x^\alpha}.$$

Conversely, if a set of equations (7.9) and the second set of (7.10) can be solved for $\bar{x}^2, \cdots, \bar{x}^n; \varrho_1, \cdots, \varrho_r$ in terms of the x's and p's, when these are substituted in (7.8) and the first set of (7.10), the resulting equations define a restricted non-homogeneous transformation. The conditions to be satisfied by $F, F_1, \cdots, F_r$ can be determined in a manner similar to that followed in § 4.*

For an infinitesimal transformation of the restricted type the function W does not involve x^1, and consequently we have:

The most general infinitesimal restricted non-homogeneous transformation is defined by equations of the form

$$(7.11)\qquad \frac{\delta x^1}{\delta t} = p_\alpha \frac{\partial W}{\partial p_\alpha} - W, \quad \frac{\delta x^\alpha}{\delta t} = \frac{\partial W}{\partial p_\alpha}, \quad \frac{\delta p_\alpha}{\delta t} = -\frac{\partial W}{\partial x^\alpha},$$

where W is an arbitrary function of $x^2, \cdots, x^n; p_2, \cdots, p_n$.

* Cf. Lie, l. c., p. 156.

In this case we have in place of (7.6)

$$\bar{p}_\alpha \, d\bar{x}^\alpha = p_\alpha \, dx^\alpha + d\left(p_\alpha \frac{\partial W}{\partial p_\alpha} - W\right) \cdot \delta t,$$

to within terms of higher order in δt. Hence when W is homogeneous of degree one in the p's, the transformation is homogeneous (cf. § 5).

From the results of § 5 it follows that each infinitesimal transformation of this type generates a continuous group G_1 of restricted transformations, whose finite equations are the integrals of

$$(7.12) \qquad \frac{dx^1}{dt} = p_\alpha \frac{\partial W}{\partial p_\alpha} - W, \quad \frac{dx^\alpha}{dt} = \frac{\partial W}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = -\frac{\partial W}{\partial x^\alpha}.$$

From (7.6) it follows that the product of two restricted transformations is one of the same kind. Consequently the product of a restricted transformation and the transformations of a group G_1 of restricted transformations yields a new G_1 of restricted transformations, whose equations therefore are of the form

$$\frac{d\bar{x}^1}{dt} = \bar{p}_\alpha \frac{\partial \bar{W}}{\partial \bar{p}_\alpha} - \bar{W}, \quad \frac{d\bar{x}^\alpha}{dt} = \frac{\partial \bar{W}}{\partial \bar{p}_\alpha}, \quad \frac{d\bar{p}_\alpha}{dt} = -\frac{\partial \bar{W}}{\partial \bar{x}^\alpha}.$$

Because of (7.2), (7.3) and (7.7) we have $\bar{W} = W$, that is, $\bar{W}$ is the transform of W by means of the given restricted transformation.

Equations (7.12) with the exception of the first are the general form of the Hamiltonian equations for a conservative holonomic dynamical system for which the Hamiltonian function does not involve the time t. Consequently, if we apply to these equations any transformation of the form (7.4) where the φ's satisfy (7.2) and φ^1 equations (7.3), we have a new Hamiltonian system* and the Hamiltonian function is the transform of the original Hamiltonian function.

When the first of (7.12) is written

$$\frac{dx^1}{dt} = p_\alpha \frac{dx^\alpha}{dt} - W,$$

the right-hand member is the Lagrangian function of the dynamical system. Consequently, if we adjoin the first equation (7.12) to the Hamiltonian equations of a conservative holonomic system for which W does not involve the time, the difference in the values of x^1 at two points of a dynamical trajectory is the action for that segment.

* This result is due to Jacobi, Comptes Rendus, vol. 5 (1837), p. 67.

8. **Contact transformations of maximum rank.** From the second set of equations (5.6) it follows that the rank of hessian of C ($\neq 0$) with respect to p's, that is $\left|\frac{\partial^2 C}{\partial p_i \partial p_j}\right|$, is $n-1$ at most. If we put

$$H = \frac{1}{2}C^2, \tag{8.1}$$

we have

$$\frac{\partial H}{\partial p_i} = C\frac{\partial C}{\partial p_i}, \qquad \frac{\partial^2 H}{\partial p_i \partial p_j} = C\frac{\partial^2 C}{\partial p_i \partial p_j} + \frac{\partial C}{\partial p_i}\frac{\partial C}{\partial p_j}. \tag{8.2}$$

From the second set of these equations it follows that the hessian of H with respect to the p's is equal to*

$$C^{n-1}\begin{vmatrix} C & -\frac{\partial C}{\partial p_1} & \cdots\cdots & -\frac{\partial C}{\partial p_n} \\ \frac{\partial C}{\partial p_1} & \frac{\partial^2 C}{\partial p_1^2} & \cdots\cdots & \frac{\partial^2 C}{\partial p_1 \partial p_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial C}{\partial p_n} & \frac{\partial^2 C}{\partial p_1 \partial p_n} & \cdots\cdots & \frac{\partial^2 C}{\partial p_n^2} \end{vmatrix}. \tag{8.3}$$

If we multiply the last n rows of this determinant by $p_1, \cdots, p_n$ respectively and add the resulting terms of all but the last row to the corresponding terms of the last row, the terms of the latter become $C, 0 \cdots, 0$ in consequence of (5. 5) and (5. 6). Consequently a necessary condition that the hessian of H with respect to the p's be of rank n is that the determinant formed from (8. 3) by omitting the first column and last row be of rank n. If the columns of this determinant be multiplied by $p_1, \cdots, p_n$ respectively and the terms of the first $n-1$ columns be added to the last, the terms of the latter become $C, 0, \cdots, 0$. Hence if this determinant is to be of rank n it is necessary that the determinant $\left|\frac{\partial^2 C}{\partial p_\alpha \partial p_\beta}\right|$ for $\alpha, \beta = 1, \cdots, n-1$ be of rank $n-1$.

If we apply the above process to (8. 3) using any one of the last n rows as we used the last, we find that it is necessary that every minor of order $n-1$ of the hessian of C with respect to the p's be of rank $n-1$. But this is a consequence of one minor being of rank $n-1$. For if $\left|\frac{\partial^2 C}{\partial p_\alpha \partial p_\beta}\right|$ for $\alpha, \beta = 1, \cdots, n-1$ is of rank $n-1$, it follows from the

* Cf. Fine, College Algebra, p. 505.

equations $p_i \dfrac{\partial^2 C}{\partial p_i \partial p_\alpha} = 0$ $(i = 1, \cdots, n;\ \alpha = 1, \cdots, n-1)$ that each of the determinants of order $n-1$ of the matrix $\left\| \dfrac{\partial^2 C}{\partial p_i \partial p_\alpha} \right\|$ is of rank $n-1$. When then we apply the same reasoning to the equations $p_i \dfrac{\partial^2 C}{\partial p_i \partial p_\sigma} = 0$, for σ equal to n and any $n-2$ of the numbers $1, \cdots, n-1$, we are led to the conclusion that, if, the hessian of C with respect to the p's is of rank $n-1$, every minor of order $n-1$ is of rank $n-1$. Accordingly we have:

When, and only when, the rank of the hessian of C with respect to the p's is $n-1$, the rank of the hessian of H with respect to the p's is n.

Without loss of generality we take (§ 5) so that

$$C = 1. \tag{8.4}$$

Then equations (5.9) may be written, because of (8.2), in the form

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \tag{8.5}$$

If we assume that C satisfies the condition of the preceding theorem, the hessian of H with respect to the p's is of rank n and consequently the first set of equations (8.5) can be solved for the p's as functions of the x's and $\dot{x}$'s, where $\dot{x}^i = \dfrac{dx^i}{dt}$. Since H is homogeneous of degree two in the p's, it follows that the solutions p_i are homogeneous of first degree in the $\dot{x}$'s. When these expressions for the p's are substituted in H, we get a function $\bar{H}$ of the x's and $\dot{x}$'s, which is homogeneous of degree two in the $\dot{x}$'s. Thus we have under this transformation

$$\bar{H}(x; \dot{x}) = H(x; p). \tag{8.6}$$

Differentiating $\bar{H}$ with respect to $\dot{x}^i$, we have in consequence of (8.5)

$$\frac{\partial \bar{H}}{\partial \dot{x}^i} = \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial \dot{x}^i} = \dot{x}^j \frac{\partial p_j}{\partial \dot{x}^i}. \tag{8.7}$$

Hence by Euler's theorem

$$2\bar{H} = \dot{x}^i \frac{\partial \bar{H}}{\partial \dot{x}^i} = \dot{x}^j \dot{x}^i \frac{\partial p_j}{\partial \dot{x}^i} = \dot{x}^j p_j. \tag{8.8}$$

From this equation and (8.7) we have

$$\frac{\partial \bar{H}}{\partial \dot{x}^i} = \frac{1}{2}\left(p_i + \dot{x}^j \frac{\partial p_j}{\partial \dot{x}^i}\right) = \frac{1}{2} p_i + \frac{1}{2} \frac{\partial \bar{H}}{\partial \dot{x}^i},$$

and consequently

$$p_i = \frac{\partial \bar{H}}{\partial \dot{x}^i}. \tag{8.9}$$

From (8.6), (8.5) and (8.8) we have

$$\frac{\partial \bar{H}}{\partial x_i} = \frac{\partial H}{\partial x^i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial x^i} = \frac{\partial H}{\partial x^i} + \dot{x}^j \frac{\partial p_j}{\partial x^i} = \frac{\partial H}{\partial x^i} + 2\frac{\partial \bar{H}}{\partial x^i}.$$

Consequently

$$\frac{\partial \bar{H}}{\partial x^i} + \frac{\partial H}{\partial x^i} = 0. \tag{8.10}$$

If we denote by $\bar{C}$ the function of the x's and $\dot{x}$'s when the expressions for p_i from the solution of (8.5), that is (8.9), are substituted in C, we have

$$\bar{H} = \frac{1}{2}\bar{C}^2, \qquad \bar{C} = 1, \tag{8.11}$$

the latter being a consequence of (8.4). From (8.9) we have

$$p_i = \frac{\partial \bar{C}}{\partial \dot{x}^i} \tag{8.12}$$

and from (8.10)

$$\frac{\partial \bar{C}}{\partial x^i} + \frac{\partial C}{\partial x^i} = 0. \tag{8.13}$$

When now the expressions (8.12) are substituted in the second set of equations (5.9), we have, in consequence of (8.13),

$$\frac{d}{dt}\left(\frac{\partial \bar{C}}{\partial \dot{x}^i}\right) - \frac{\partial \bar{C}}{\partial x^i} = 0. \tag{8.14}$$

Thus in the quantities x^i and $\dot{x}^i$ equations (8.12) and (8.14) replace (5.9).

Conversely, if we have any function $\bar{C}$ of x's and $\dot{x}$'s, homogeneous of degree one in the $\dot{x}$'s, such that

$$\bar{C}(x; \dot{x}) = 1, \tag{8.15}$$

and the rank of the hessian of $\bar{C}$ with respect to the $\dot{x}$'s is $n-1$, equations (8.9) can be solved for the $\dot{x}$'s as homogeneous functions of degree one

in the p's, and by repeating the above processes we obtain the first set of equations (5.9), and the second set follow from (8.12) and (8.14).

If we take a particular point $P(x^i)$ of our space, the first set of equations (5.7) defines a near-by point, which varies for each choice of the p's subject to the condition (8.4). The equation of the locus of this point is obtained on the elimination of the p's from the first set of (5.7) and (8.4). If the hessian of C with respect to the p's is of rank $n-1$, this elimination can be accomplished by the process which led to (8.11). Consequently, if y^i are coördinates with P as origin so that $y^i = \bar{x}^i - x^i$, we have as the equation of the locus

$$\bar{C}(x; y) = \delta t, \tag{8.16}$$

since $\bar{C}$ is homogeneous of the first degree in the $\dot{x}$'s. Also from this property of $\bar{C}$ and from (8.12) it follows that

$$p_i = \frac{\partial \bar{C}(x; y)}{\partial y^i}. \tag{8.17}$$

Consequently p^i are the components of the covariant normal to the hypersurface (8.16), which we call the *elementary hypersurface* of the transformation for the point P.

If X^i are current coördinates, the tangential hyperplane at a point is defined by

$$p_i(X^i - y^i) = 0,$$

or

$$p_i X^i = w, \tag{8.18}$$

where in consequence of (5.7) and (5.5)

$$w = p_i \frac{\partial C}{\partial p_i} \delta t = C \delta t. \tag{8.19}$$

Consequently (8.18) with w given by (8.19) is the tangential equation of elementary hypersurface.*

9. **Geometrical properties of continuous groups of maximum rank. Waves.** Consider the finite equations

$$\bar{x}^i = \varphi^i(x, p, t), \qquad \bar{p}_i = \psi_i(x, p, t) \tag{9.1}$$

of a group G_1 of contact transformations for which the hessian of the characteristic function C with respect to the p's is of rank $n-1$. When

* Cf. Vessiot, l. c., p. 262; also Levi-Civita, l. c., pp. 452, 462.

the functions φ^i are written as power series in t, as solutions of (5.9), it is seen that the rank of the jacobian of the φ's with respect to the p's is $n-1$. Consequently when the p's are eliminated from the first set of equations (9.1), we get an equation

$$F(\bar{x}, x, t) = 0. \tag{9.2}$$

This equation for given values of the x's defines the hypersurface whose points are the transforms of the point $P(x)$ for the given value of t; when $t = 0$, the hypersurface reduces to the point P, as follows from φ^i expressed as power series in t. For other values of t equation (9.2) defines a family of hypersurfaces which may be interpreted as waves emanating from P, t being the time which has elapsed since the emanation.

When in the first set of equations (9.1) we give the x's and p's fixed values, these equations define a curve, the locus of corresponding points on the waves, and the second set of equations (9.1) defines the covariant normal to the waves at these points. We call them the *trajectories of the wave-motion.* When the initial values satisfy $C = 0$, we call the trajectory *singular.* The non-singular trajectories are integrals of equations (8.14). Along a singular trajectory we have $\bar{p}_i d\bar{x}^i = 0$ (§ 5). Consequently such a trajectory is tangential to a hypersurface (9.2) at its point of meeting with it. Substituting in (9.2) for the $\bar{x}$'s such a solution, we have on differentiating the resulting identity with respect to t

$$\frac{\partial F}{\partial \bar{x}^i}\frac{d\bar{x}^i}{dt} + \frac{\partial F}{\partial t} = 0.$$

Hence the singular trajectories meet (9.2) in points of the hypersurfaces

$$\frac{\partial F}{\partial t} = 0,$$

that is in the points of tangency of (9.2) with its envelope considering t as a parameter.

Excluding the points on the envelope of the hypersurfaces (9.2), we can solve this equation for t, thus

$$f(\bar{x}, x) - t = 0; \tag{9.3}$$

then $f(x, x) = 0$. Since the transformations form a group, f must be of such a form that

$$f\left(\bar{x} + \frac{\partial C}{\partial \bar{p}_i}\delta t, x\right) - (t + \delta t) = 0,$$

where

$$\bar{p}_i = \varrho \frac{\partial f}{\partial \bar{x}^i}. \tag{9.4}$$

Consequently

$$\frac{\partial f}{\partial \bar{x}^i} \frac{\partial C}{\partial \bar{p}_i} = 1. \tag{9.5}$$

Since C is homogeneous of degree one in the p's, it follows that f is a solution of the equation

$$C\left(\bar{x}, \frac{\partial f}{\partial \bar{x}}\right) = 1. \tag{9.6}$$

Moreover, it is a complete integral, since it involves n parameters x^i subject to the condition $f(x, x) = 0$.

We proceed now to the converse problem of deriving the trajectories from a complete integral of equation (9.6), say $\varphi(\bar{x}, a^1, \cdots, a^{n-1})$, where none of the a's is additive. Because of this requirement the jacobian of the functions φ and $\frac{\partial \varphi}{\partial a^\alpha}$, where $\alpha = 1, \cdots, n-1$, with respect to the $\bar{x}$'s is not zero. Hence the equations

$$\varphi(\bar{x}, a) + a^n - t = 0, \tag{9.7}$$

$$\frac{\partial \varphi}{\partial a^\alpha} = b_\alpha, \tag{9.8}$$

where a^n and the b's are arbitrary constants, admit a solution

$$\bar{x}^i = \varphi_1^i(a^1, \cdots, a^n, b_1, \cdots, b_{n-1}, t). \tag{9.9}$$

We denote by ψ_i the functions of the a's, b's, t and h which are obtained by the substitution of the φ_1's in the right-hand members of the equations

$$\bar{p}_i = h \frac{\partial \varphi}{\partial \bar{x}^i}, \tag{9.10}$$

where h is an arbitrary constant. We shall show that the φ_1's and ψ's constitute a solution of equations (5.9). In fact, if we substitute (9.9) in (9.7) and (9.8) and differentiate the resulting identities with respect to t, we obtain

$$\frac{\partial \varphi}{\partial \bar{x}^i} \frac{d\bar{x}^i}{dt} - 1 = 0, \qquad \frac{\partial^2 \varphi}{\partial a^\alpha \partial \bar{x}^i} \frac{d\bar{x}^i}{dt} = 0. \tag{9.11}$$

Also if we substitute the function φ in (9.6) and differentiate the resulting identity with respect to the a's and $\bar{x}$'s, we get, in consequence of (9.10),

$$(9.12)\qquad \frac{\partial C}{\partial \bar{p}_i}\frac{\partial^2 \varphi}{\partial \bar{x}^i \partial a^\alpha} = 0, \qquad \frac{\partial C}{\partial \bar{x}^i} + h\frac{\partial C}{\partial \bar{p}_j}\frac{\partial^2 \varphi}{\partial \bar{x}^j \partial \bar{x}^i} = 0.$$

From the first set of these equations, (9.11) and (9.5) we obtain

$$\frac{\partial \varphi}{\partial \bar{x}^i}\left(\frac{d\bar{x}^i}{dt} - \frac{\partial C}{\partial \bar{p}_i}\right) = 0, \qquad \frac{\partial^2 \varphi}{\partial \bar{x}^i \partial a^\alpha}\left(\frac{d\bar{x}^i}{dt} - \frac{\partial C}{\partial \bar{p}^i}\right) = 0,$$

which are equivalent to the first set of equations (5.9), since the determinant of these equations is a non-vanishing jacobian. From this result, (9.10) and the second set of equations (9.12) we have

$$\frac{d\bar{p}_i}{dt} = h\frac{\partial^2 \varphi}{\partial \bar{x}^i \partial \bar{x}^j}\frac{d\bar{x}^j}{dt} = h\frac{\partial^2 \varphi}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial C}{\partial \bar{p}_j} = -\frac{\partial C}{\partial \bar{x}^i},$$

as was to be proved.

If we denote by x^i and p_i the values of $\bar{x}^i$ and $\bar{p}_i$ when $t = 0$, we have from (9.7) and (9.8)

$$(9.13)\qquad a^n = -\varphi(x, a), \qquad b_\alpha = \frac{\partial}{\partial a^\alpha}\varphi(x, a^1, \cdots, a^{n-1}),$$

and from (9.10)

$$(9.14)\qquad p_i\frac{\partial}{\partial x^j}\varphi(x, a) - p_j\frac{\partial}{\partial x^i}\varphi(x, a) = 0.$$

These $n-1$ equations define the $n-1$ a's as functions of the x's and p's, homogeneous of degree zero in the p's. Then a^n and the b's are given by (9.13) as functions of these quantities and from (9.9) we have the first set of (9.1). For $t = 0$ we have from (9.6) and (9.10)

$$C\left(x, \frac{\partial \varphi}{\partial x}\right) = \frac{1}{h}C(x, p) = 1.$$

Consequently the second set of (9.1) is given by substituting the expressions for the $\bar{x}$'s, a's and b's in terms of the x's and p's in

$$(9.10')\qquad \bar{p}_i = C(x, p)\frac{\partial \varphi}{\partial \bar{x}^i},$$

which is homogeneous of degree one in the p's.

From (9.7) and (9.13) we have

$$(9.15)\qquad \varphi(\bar{x}, a) - \varphi(x, a) - t = 0.$$

For given values of the p's, the $\overline{p}$'s given by (9.10′) are the components of the covariant normal to the hypersurface (9.15) for a given value of t. From (9.4) these are the components of the covariant normal to the hypersurface (9.3). Hence the latter is the envelope of the hypersurfaces (9.15) for the same value of t, on treating the p's as parameters. Since the a's are independent functions of the ratios p_α/p_n, the envelope of the hypersurfaces considering these ratios as parameters is the same as the envelope considering the a's as parameters. If then we eliminate the a's from (9.15) and

$$\frac{\partial \varphi(\overline{x}, a)}{\partial a^\alpha} - \frac{\partial \varphi(x, a)}{\partial a^\alpha} = 0,$$

we obtain the equation (9.3).

Consider any hypersurface

$$F(x^1, \cdots, x^n) = 0, \tag{9.16}$$

F being irreducible. From the results of §§ 3, 4 it follows that the envelope of the waves emanating from each point of the hypersurface after a time t is defined by

$$\overline{x}^i = \varphi^i\left(x, \frac{\partial F}{\partial x}, t\right), \tag{9.17}$$

in which the x's are in the relation (9.16), and that

$$\overline{p}_i = \psi_i\left(x, \frac{\partial F}{\partial x}, t\right) \tag{9.18}$$

are the components of the covariant normal at each point. As t varies the envelopes constitute a series of wave-fronts whose character is determined by the *hypersurface of origin* (9.16). Eliminating the x's from (9.16) and (9.17), we get as the equation of the wave-fronts

$$\Phi(\overline{x}^1, \cdots, \overline{x}^n, t) = 0. \tag{9.19}$$

Excluding the points on the envelope of these hypersurfaces with t as parameter, we may replace (9.19) by

$$\psi(\overline{x}^1, \cdots, \overline{x}^n) - t = 0. \tag{9.20}$$

It follows from the considerations applied to (9.3) that ψ is a solution of equation (9.6). It is the general integral determined by the relation (9.16) between the parameters x^i in the complete integral $f(\overline{x}, x)$, since (9.20) is the envelope of the hypersurfaces (9.3) for the relation (9.16).

Thus the waves (9.3) and the wave-fronts (9.20) are related in accordance with Huygen's principle.

When the coördinates x^i of a point P on the hypersurface (9.16) are substituted in (9.17), the resulting equations define a curve, the locus of the points of contact with the wave-fronts of the waves emanating from P as t varies. The congruence of curves so defined are the trajectories of motion of the wave-fronts; evidently they are integral curves of equations (8.14).

The hypersurface (9.20) for a particular value of t is the envelope of the hypersurfaces (9.7) in which the a's are functions of the x's determined by the $n-1$ equations

$$\frac{\partial F}{\partial x^i}\frac{\partial \varphi(x,a)}{\partial x^j} - \frac{\partial F}{\partial x^j}\frac{\partial \varphi(x,a)}{\partial x^i} = 0, \tag{9.21}$$

the x's being subject to the relation $F=0$. If the x's are eliminated from (9.21), the first of (9.13) and $F=0$, we get a relation

$$\psi(a^1, \cdots, a^{n-1}) = a^n.$$

Consequently (9.20) is the envelope of (9.7) for the a's in this relation. Conversely, if we have any such relation, the envelope of

$$\varphi(x,a) + \psi(a) = 0$$

is the hypersurface of origin (9.16) of the waves (9.15).

In order to make a similar discussion of the singular trajectories of the group G_1, defined by a characteristic function C, we consider the differential equation

$$C\left(x, \frac{\partial \varphi}{\partial x}\right) = 0. \tag{9.22}$$

Suppose we have an integral of this equation involving $n-2$ arbitrary constants $a^1, \cdots, a^{n-2}$, none of which is additive, and consider the $n-1$ equations

$$\varphi(\overline{x}, a) = b, \quad \frac{\partial \varphi(\overline{x}, a)}{\partial a^\sigma} = b_\sigma \qquad (\sigma = 1, \cdots, n-2), \tag{9.23}$$

where the b's are arbitrary constants. For each set of values of the a's and b's these equations define a curve along which the differentials satisfy the conditions

$$\frac{\partial \varphi}{\partial \overline{x}^i} d\overline{x}^i = 0, \quad \frac{\partial^2 \varphi}{\partial a^\sigma \partial \overline{x}^i} d\overline{x}^i = 0.$$

If we define functions $\overline{p}_i$ by (9.10), it follows from (9.22) that

$$\frac{\partial \varphi}{\partial \overline{x}^i}\frac{\partial C}{\partial \overline{p}_i} = 0, \quad \frac{\partial^2 \varphi}{\partial a^\sigma \partial \overline{x}^i}\frac{\partial C}{\partial \overline{p}_i} = 0.$$

Since the matrix of $\frac{\partial \varphi}{\partial \overline{x}^i}$ and $\frac{\partial^2 \varphi}{\partial a^\sigma \partial \overline{x}^i}$ is of rank $n-1$, we have from the above equations

$$\frac{d\overline{x}^i}{\frac{\partial C}{\partial \overline{p}_i}} = \cdots\cdots = \frac{d\overline{x}^n}{\frac{\partial C}{\partial \overline{p}_n}} = \varrho,$$

where ϱ is the factor of proportionality. From (9.10) and the second set of (9.12) we have

$$d\overline{p}_i = h\frac{\partial^2 \varphi}{\partial \overline{x}^i \partial \overline{x}^j}d\overline{x}^j = -\varrho\frac{\partial C}{\partial \overline{x}^i}.$$

Consequently the curves (9.23) are integrals of (5.9).

In order to obtain the curves through a point $P(x)$, we replace (9.23) by

$$\varphi(\overline{x}, a) - \varphi(x, a) = 0, \quad \frac{\partial \varphi(\overline{x}, a)}{\partial a^\sigma} - \frac{\partial \varphi(x, a)}{\partial a^\sigma} = 0. \tag{9.24}$$

Since the p's in this case must satisfy the condition $C(x, p) = 0$, there are $n-2$ of equations (9.14) for the determation of the a's as functions of the x's and ratios of the p's. When these are substituted in (9.24), we have the equations of the singular trajectories, and the $\overline{p}$'s are given by (9.10) in which

$$h = p_i \Big/ \frac{\partial \varphi}{\partial x^i}.$$

10. **Application to geodesics of a Riemannian space.** Consider a Riemannian space whose metric is defined by

$$ds^2 = g_{ij}\,dx^i\,dx^j, \tag{10.1}$$

where the g's are functions of the x's such that $g_{ij} = g_{ji}$ and the determinant of the g's is not zero. We define functions g^{ij} by the equations

$$g^{ij}g_{jk} = \delta^i_k, \tag{10.2}$$

where

$$\delta^i_k = 1 \text{ or } 0 \text{ as } i = k \text{ or } i \neq k. \tag{10.3}$$

The equations of the non-minimal geodesics of the space are*

$$g_{ij}\frac{d^2x^j}{ds^2}+[jk,i]\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \tag{10.4}$$

where

$$[jk,i]=\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^i}\right). \tag{10.5}$$

Consider now the group G_1 of contact transformations for which the characteristic function is

$$C=\sqrt{g^{ij}p_ip_j}. \tag{10.6}$$

In this case we have equations (5.13). When the trajectories are non-singular it follows from (5.13) and (10.1) that $t=s$. Hence we have

$$\frac{dx^i}{ds}=g^{ij}p_j, \qquad \frac{dp_i}{ds}=-\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i}p_jp_k. \tag{10.7}$$

In consequence of (10.2) and the first set of (10.7) we have

$$p_i=g_{ij}\frac{dx^j}{ds}. \tag{10.8}$$

Substituting this expression in the second set of (10.7), we obtain equations (10.4).

If the form (10.1) is not definite, there are real values of the p's such that

$$g^{ij}p_ip_j=0. \tag{10.9}$$

As in the preceding case we obtain from (5.13) equations (10.4) with s replaced by t, and from the first set of (5.13) and (10.9) it follows that $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=0$, that is, the curves are minimal geodesics. Hence we have:

For the group G_1 with the characteristic function (10.6) *the non-singular trajectories are the non-minimal geodesics of the Riemannian space with the fundamental form* (10.1) *and $t=s$; and the singular trajectories are the minimal geodesics of the space.*

For the case of non-minimal geodesics equation (9.6) is

$$g^{ij}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}=1. \tag{10.10}$$

* Eisenhart, Riemannian geometry, p. 50.

From the results of § 9 it follows that if $\varphi(x, a^1, \cdots, a^{n-1})$ is a complete integral of (10.10), the corresponding equations (9.8) define $n-1$ hypersurfaces which intersect in the non-minimal geodesics. Moreover, from (9.11) and (10.7) it follows that for particular values of the a's these geodesics are the orthogonal trajectories of the hypersurfaces $\varphi = \text{const.}$* Also if (10.1) is not definite and $\varphi(x, a^1, \cdots, a^{n-2})$ is an integral of

$$g^{ij} \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = 0,$$

then equations (9.23) define the minimal geodesics.

When $\varphi(\bar{x}, a^1, \cdots, a^{n-1})$ is an integral of (10.10) and the a's are eliminated from the equations

$$(10.11)\quad \varphi(\bar{x}, a) - \varphi(x, a) - s = 0, \qquad \frac{\partial \varphi(\bar{x}, a)}{\partial a^\alpha} - \frac{\partial \varphi(x, a)}{\partial a^\alpha} = 0 \qquad (\alpha = 1, \cdots, n-1),$$

in which the x's have particular values, the resulting equation

$$(10.12)\qquad f(\bar{x}, x) - s = 0$$

defines the hypersurfaces orthogonal to the geodesics through the point $P(x)$. Each of these hypersurfaces is the locus of points at the same distance s from P, measured along the geodesics. Consequently we call them *geodesic hyperspheres* for the space. The function f is a solution of (10.10) and the geodesics are the integrals of the equations

$$\frac{d\bar{x}^1}{g^{1j} \frac{\partial f}{\partial \bar{x}^j}} = \cdots\cdots = \frac{d\bar{x}^n}{g^{nj} \frac{\partial f}{\partial \bar{x}^j}}.$$

If the integrals of these equations are $\psi^\alpha(\bar{x}, x) = b^\alpha$, $(\alpha = 1, \cdots, n-1)$, and we put

$$y^\alpha = \psi^\alpha, \qquad y^n = f(\bar{x}, x),$$

and denote by $a_{ij}\, dy^i\, dy^j$ the transform of (10.1), we have

$$a^{\alpha n} = g^{ij} \frac{\partial \psi}{\partial \bar{x}^i} \frac{\partial f}{\partial \bar{x}^j} = 0, \qquad a^{nn} = g^{ij} \frac{\partial f}{\partial \bar{x}^i} \frac{\partial f}{\partial \bar{x}^j} = 1.$$

Consequently the form (10.1) becomes

$$ds^2 = (dy^n)^2 + a_{\alpha\beta}\, dy^\alpha\, dy^\beta,$$

* Cf. Eisenhart, Riemannian geometry, p. 58.

the hypersurfaces $y^n = \text{const.}$ being the geodesic hyperspheres with $P(x)$ as center.

Suppose that the x's and p's, as defined by (10.8), are subjected to any homogeneous contact transformation (2.1) and that $\overline{C}$ is the transform of C. Then $\overline{C}$ is homogeneous of degree one in the $\overline{p}$'s. From the results of § 5 it follows that equations (10.7) are transformed into

$$\frac{d\overline{x}^i}{ds} = \frac{\partial \overline{C}}{\partial \overline{p}_i}, \qquad \frac{d\overline{p}_i}{ds} = -\frac{\partial \overline{C}}{\partial \overline{x}^i}. \tag{10.13}$$

If we define functions $\overline{g}^{ij}$ by

$$\overline{g}^{ij} = \frac{1}{2}\frac{\partial^2(\overline{C}^2)}{\partial \overline{p}_i \partial \overline{p}_j}, \tag{10.14}$$

then $\overline{g}^{ij}$ is homogeneous of degree zero in the $\overline{p}$'s and it follows from the second set of equations (5.6) that

$$\overline{g}^{ij}\overline{p}_i\overline{p}_j = \overline{C}^2. \tag{10.15}$$

When the $\overline{p}$'s are such that $\overline{C} \neq 0$ and they are chosen so that $\overline{C} = 1$, the above equations may be written

$$\frac{d\overline{x}^i}{ds} = \overline{g}^{ij}\overline{p}_j, \quad \frac{d\overline{p}_i}{ds} = -\frac{1}{2}\frac{\partial \overline{g}^{jk}}{\partial \overline{x}^i}\overline{p}_j\overline{p}_k, \tag{10.16}$$

since we have in consequence of (10.14) that

$$\frac{\partial \overline{g}^{jk}}{\partial \overline{p}_i}\overline{p}_j\overline{p}_k = \frac{1}{2}\frac{\partial^3 \overline{C}^2}{\partial \overline{p}_i \partial \overline{p}_j \partial \overline{p}_k}\overline{p}_j\overline{p}_k = 0.$$

If we drop the bars temporarily from these quantities, the first of (10.13) become, in consequence of (6.1),

$$\dot{x}^i \equiv \frac{dx^i}{ds} = \frac{\partial H}{\partial p_i}.$$

As in § 6, these equations can be solved for p_i as functions of the x's and $\dot{x}$'s, linear homogeneous in the latter; we denote by $\hat{H}$ the function resulting from the substitution in H of these expressions for p_i. Since $\hat{H}$ is homogeneous of second degree in the $\dot{x}$'s, we have from (8.12)

$$p_i = \frac{\partial \hat{H}}{\partial \dot{x}^i} \equiv \frac{1}{2}\frac{\partial \hat{C}^2}{\partial \dot{x}^i} = \frac{1}{2}\frac{\partial^2 \hat{C}^2}{\partial \dot{x}^i \partial \dot{x}^j}\dot{x}^j = \overline{g}_{ij}\dot{x}^j; \tag{10.17}$$

$$\overline{g}_{ij} \equiv \frac{1}{2}\frac{\partial^2 \hat{C}^2}{\partial \dot{x}^i \partial \dot{x}^j},$$

where $\bar{g}_{ij}$ as thus defined are functions of x^i and $\dot{x}^i$, homogeneous of degree zero in the latter; also from (10.16) we have $\hat{C}^2 = \bar{g}_{ij}\,\dot{x}^i\,\dot{x}^j$. In consequence of (8.13) the second set of (10.13) become, on resuming bars,

$$\bar{g}_{ij}\frac{d\bar{x}^j}{ds^2} + [jk, i]\frac{d\bar{x}^j}{ds}\frac{d\bar{x}^k}{ds} = 0,$$

where $[jk, i]$ are defined by (10.5) in terms of $\bar{g}_{ij}$. Thus the trajectories defined by (10.13) are the geodesics of the space with the generalized Riemannian metric $ds^2 = \bar{g}_{ij}\,d\bar{x}^i\,d\bar{x}^j$ into which the given Riemannian space has been transformed.*

11. **Application to dynamics.** Consider a conservative dynamical system for which neither the constraints nor the potential energy V involve the time t. Suppose that there are n independent variables x^i; then the kinetic energy is given by

$$T = \frac{1}{2}g_{ij}\,\dot{x}^i\,\dot{x}^j,$$

where the g's are functions of the x's, and involve the masses of the system, and the lagrangian function is

$$L = \frac{1}{2}g_{ij}\,\dot{x}^i\,\dot{x}^j - V.$$

Substituting in the equations of Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0,$$

we find that the resulting equations may be written

$$g_{ij}\,\ddot{x}^j + [jk, i]\,\dot{x}^j\,\dot{x}^k + \frac{\partial V}{\partial x^i} = 0, \tag{11.1}$$

in consequence of (10.5). These equations admit the first integral

$$\frac{1}{2}g_{ij}\,\dot{x}^i\,\dot{x}^j + V = E, \tag{11.2}$$

where E is the energy constant.

If we put

$$p_i = g_{ij}\,\dot{x}^j, \qquad \dot{x}^i = g^{ij}\,p_j, \tag{11.3}$$

we have from equation (11.2)

$$g^{ij}\,p_i\,p_j = 2(E - V).$$

* Spaces with a generalized Riemannian metric have been studied by Finsler, Dissertation, Göttingen (1918), and by Berwald, Jahresbericht der Deut. Math.-Verein. (1925), p. 213.

Hence if we take for the characteristic function of a group G_1 of transformations the quantity*

$$C = \sqrt{\frac{1}{2(E-V)}\, g^{ij}\, p_i\, p_j} = 1, \tag{11.4}$$

the equations of the transformation are

$$\frac{dx^i}{ds} = \frac{1}{2(E-V)}\, g^{ij}\, p_j, \qquad \frac{dp_i}{ds} = -\frac{1}{4}\,\frac{\partial}{\partial x^i}\left(\frac{1}{E-V}\, g^{jk}\right) p_j\, p_k, \tag{11.5}$$

where the parameter s is given by (cf. § 10)

$$ds^2 = 2(E-V)\, g_{ij}\, dx^i\, dx^j. \tag{11.6}$$

The Hamiltonian function of the dynamical system is given by

$$H = p_i\, \dot{x}^i - L = \frac{1}{2}\, g^{ij}\, p_i\, p_j + V,$$

and consequently the Hamiltonian equations are

$$\frac{dx^i}{dt} = g^{ij}\, p_j, \qquad \frac{dp_i}{dt} = -\left(\frac{1}{2}\,\frac{\partial g^{jk}}{\partial x^i}\, p_j\, p_k + \frac{\partial V}{\partial x^i}\right).$$

These equations are equivalent to (11.5) with the relation

$$\frac{ds}{dt} = 2(E-V) \tag{11.7}$$

holding along any trajectory. Thus we have the known result that the trajectories, of energy E, of the given system are in one-to-one correspondence with the geodesics of the space with the fundamental form (11.6).

From the results of § 10 it follows that, if φ is any solution of the equation

$$g^{ij}\,\frac{\partial \varphi}{\partial x^i}\,\frac{\partial \varphi}{\partial x^j} = 2(E-V), \tag{11.8}$$

the orthogonal trajectories of the hypersurfaces $\varphi = \text{const.}$ are geodesics. If $\varphi(x, a^1, \cdots, a^{n-1})$ is a complete integral of this equation, on eliminating the a's from the corresponding equations (10.11), the resulting equation (10.12) defines geodesic hyperspheres with $P(x)$ as center.

* This problem for the motion of a particle was discussed by Lie, Die infinitesimalen Berührungstransformationen der Mechanik, Leipz. Ber., vol. 41 (1889), pp. 145–156; and the more general problem by Vessiot, l. c., pp. 266–268.

Consider the case of a single particle of mass m and write (11.2) in the form

$$m\left(\frac{ds_0}{dt}\right)^2 = 2(E-V), \tag{11.9}$$

where ds_0 is the element of length of the space of the particle. Then from (11.6) we have

$$ds^2 = 2m(E-V)\,ds_0^2, \tag{11.10}$$

and consequently

$$s = \int \sqrt{2m(E-V)}\,ds_0 = \int mv\,ds_0.$$

Hence s is the action and the dynamical trajectories are the extremals of the integral of action.

From the view-point of optics in accordance with Fermat's principle the trajectories are the same as the paths of light through an isotropic non-homogeneous medium of refractive index equal to $\sqrt{2m(E-V)}$. In accordance with this principle $s = \varkappa\tau$, where τ is the time and $\varkappa$ a factor of proportionality. Then the velocity of light is

$$u = \frac{ds_0}{d\tau} = \frac{\varkappa}{\sqrt{2m(E-V)}}. \tag{11.11}$$

Comparing this with (11.9) we have

$$u = \frac{\varkappa}{mv}. \tag{11.12}$$

From (11.10) it follows that the spaces with metric (11.6) and ds_0 are conformal. Consequently in the latter metric the trajectories through a point $P(x)$ are orthogonal to the hypersurfaces (10.12). If we take $\varkappa = E$, the equation of these hypersurfaces is

$$f(\overline{x}, x) - E\tau = 0. \tag{11.13}$$

From (11.7) it follows that along any trajectory the relation between the time t of the particle and τ of the light are given by

$$\frac{d\tau}{dt} = \frac{2(E-V)}{E} \tag{11.14}$$

which is in conformity with (11.12).

PRINCETON UNIVERSITY.

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS OF INFINITE ORDER, WITH CONSTANT COEFFICIENTS.*

By I. M. Sheffer.

1. **Introduction.** A linear functional equation in the field of analytic functions (to limit ourselves) gives rise formally to a system of infinitely many linear equations in infinitely many unknowns. We have but to assume a power series expansion for the unknown function, substitute into the functional equation, and equate coefficients.

Conversely there is a general linear functional equation to which every system of linear equations in infinitely many unknowns can be formally led, namely the linear differential equation of infinite order; i. e., where the derivatives of the unknown function appear to all orders.

More precisely: Given the system of equations

$$\sum_{j=0}^{\infty} a_{ij}\, y_j = c_i, \qquad i = 0, 1, \cdots, \tag{1}$$

where a_{ij}, c_i are wholly arbitrary. Then there exists a linear differential equation

$$P_0(x)\, y(x) + P_1(x)\, y'(x) + \cdots + P_n(x)\, y^{(n)}(x) + \cdots = C(x), \tag{2}$$

where the functions $C(x) = \sum_0^{\infty} c_k x^k$, $P_i(x) = \sum_0^{\infty} p_{i,k} x^k$ are formal power series, such that on setting $y(x) = \sum_0^{\infty} y_k x^k$ and substituting into (2), the resulting system of equations is precisely (1). The formulas which permit one to go from one to the other are readily found to be

$$\begin{cases} a_{nk} = k!\left[\dfrac{p_{0,n-k}}{k!} + \dfrac{p_{1,n-k+1}}{(k-1)!} + \cdots + \dfrac{p_{k,n}}{0!}\right], \\ p_{kn} = \dfrac{1}{k!}\left[a_{n,k} - \binom{k}{1} a_{n-1,k-1} + \binom{k}{2} a_{n-2,k-2} - \cdots + (-1)^k \binom{k}{k} a_{n-k,0}\right]. \end{cases} \tag{3}$$

It is interesting to note that when we consider a system of equations such as

$$a_{ii}\, y_i + a_{i,i+1}\, y_{i+1} + a_{i,i+2}\, y_{i+2} + \cdots = c_i, \quad i = 0, 1, \cdots,$$

the resulting formal power series $P_k(x)$ of (2) are *polynomials*, $P_k(x)$ being of degree $\leqq k$.

* Received May 24, 1928; presented to the American Mathematical Society, December 28, 1927.

This formal equivalence between the linear differential equation of infinite order and the system of linear equations in infinitely many unknowns suggests that it is of interest to study the former. The present paper is devoted to the particular, but important, case where the coefficients in the differential equation are constants:

$$a_0 y(x) + a_1 y'(x) + \cdots + a_n y^{(n)}(x) + \cdots = g(x), \tag{4}$$

and to the corresponding case of *systems* of such equations.

Putting suitable restrictions on $g(x)$ and the a_n's, we obtain the existence theorems for equation (4), in § 2. This equation appears to have been treated first by Schürer.* The results given in § 2 duplicate for the most part the results obtained by him;† we give them here both for completeness and for a basis for the succeeding work. The method is an operational one, and it enables us to treat the general case of a system of n equations in n unknowns, as is done in § 3. So far as the writer is aware, systems have never before been treated. § 3 forms a complete discussion.

In another paper we hope to treat in some detail the case of one equation in one unknown in which the coefficients are polynomials of degree not exceeding one.

2. One equation in one unknown, with constant coefficients.

Let $A(t) = \sum_0^\infty a_n t^n$ be convergent, $|t| < R$.

Notation: $A[y(x)] \equiv a_0 y(x) + a_1 y'(x) + \cdots + a_n y^{(n)}(x) + \cdots$.

Definition: If $g(x)$ is an entire function with the property $\overline{\lim_{n=\infty}} \sqrt[n]{|g^{(n)}_{(0)}|} = q$, we shall say that $g(x)$ is of *exponential value* q; or for short, $g(x)$ is of exp. val. q.‡

Remark: If $g(x)$ is of exp. val. q, then $\overline{\lim_{n=\infty}} \sqrt[n]{|g^{(n)}_{(x)}|} = q$ for every x.

LEMMA 1. *If $A(t)$ is analytic, $|t| < R$, and $g(x)$ is of exp. val. $q < R$, then $A[g]$ is entire and is of exp. val. $\leqq q$.*

Proof: From the remark made above we have $g^{(n)}(x) \ll M q'^n e^{q'x}$, $n = 0, 1, \cdots$, where $q' = q + \varepsilon$, $\varepsilon > 0$ arbitrary. Choose ε so that $q' < R$. Then

$$A[g] \ll M \sum_0^\infty |a_n| q'^n e^{q'x} = M e^{q'x} \sum_0^\infty |a_n| q'^n.$$

* F. Schürer, Leipz. Ber., vol. 70 (1918), pp. 185–240.

† We add that the methods here used were in the possession of the writer before he was aware of Schürer's paper or of any other paper dealing with equations of infinite order.

‡ Unless the contrary is stated, it will be assumed that q is finite. Most writers on differential equations of infinite order restrict themselves to the same class of functions.

The series on the right converges, since $q' < R$. Hence $A[g]$ is entire, and clearly is of exp. val. $\leqq q'$. Since ϵ is arbitrary, it follows that $A[g]$ is of exp. val. $\leqq q$.

Let $A(t)$ be analytic, $|t| < R$, and $g(x)$ be of exp. val. $q < R$. We wish to solve the differential equation:

$$a_0 y(x) + a_1 y'(x) + \cdots + a_n y^{(n)}(x) + \cdots = g(x).$$

Let us introduce a parameter λ as follows:

$$a_0 y(x) + \lambda a_1 y'(x) + \cdots + \lambda^n a_n y^{(n)}(x) + \cdots = g(x),$$

and assume a solution in the form

$$y(x) = Y_0(x) + \lambda Y_1(x) + \cdots + \lambda^n Y_n(x) + \cdots.$$

On substituting in and equating coefficients, we find equations for $Y_0, Y_1, \cdots$ which yield (on assuming $a_0 \neq 0$)

$$Y_0 = \frac{1}{a_0} g(x), \quad Y_1 = \frac{-a_1}{a_0^2} g'(x), \quad Y_2 = \left(\frac{a_1^2 - a_0 a_2}{a_0^3}\right) g''(x), \ \cdots.$$

Now the coefficients of $g, g', g'', \cdots$ in these expressions are precisely the Taylor coefficients for $1/A(t)$. This suggests then, on taking $\lambda = 1$, that the original equation has a formal solution of the form

$$y(x) = b_0 g(x) + b_1 g'(x) + \cdots + b_n g^{(n)}(x) + \cdots,$$

where $1/A(t) = \sum_0^\infty b_n t^n$; that is

$$y(x) = \frac{1}{A}[g].$$

We use this as a guide. Our actual proofs will be carried through by an operational method.

LEMMA 2. *If* $A_1(t) = \sum_0^\infty a_n' t^n$, $A_2(t) = \sum_0^\infty a_n'' t^n$ *are analytic,* $|t| < R$, *and* $g(x)$ *is of exp. val.* $q < R$, *then*

$$A_1[A_2[g]] = A_2[A_1[g]] = A_1 A_2[g].$$

Proof: By Lemma 1, $A_1[g]$, $A_2[g]$, $A_1 A_2[g]$ are of exp. val. $\leqq q$, and therefore the same is true of the expressions of this lemma. From the proof of Lemma 1,

$$\frac{d^n}{dx^n}\{A_2[g]\} \ll l M q'^n e^{q'x}. \qquad q < q' < R,$$

where $l = \sum_0^\infty |a_k''| q'^k$. Consequently, the following double series for $A_1[A_2[g]]$ converges absolutely and may be summed in any manner:

$$A_1[A_2[g]] = a_0'(a_0'' g + a_1'' g' + \cdots + a_n'' g^{(n)} + \cdots) + a_1'(a_0'' g' + \cdots\cdots\cdots + a_{n-1}'' g^{(n)} + \cdots) + \cdots.$$

On summing by columns we obtain

$$A_1[A_2[g]] = (a_0' a_0'') g + (a_0' a_1'' + a_1' a_0'') g' + \cdots,$$

which is precisely $A_1 A_2[g]$. The lemma is then established, since we can interchange the roles of A_1, A_2.

COROLLARY: *If $A(t)$, $B(t)$ are analytic, $|t| < R$, and if $f(x)$, $g(x)$, $h(x)$, of exp. val. $< R$, satisfy the two relations*

$$(1)\ A[g] = f(x), \qquad (2)\ B[h] = g(x),$$

then they also satisfy

$$(3)\ AB[h] = f(x);$$

conversely, if $B(t) \not\equiv 0$, $|t| < R$, and if

$$(1')\ B[h] = g(x), \qquad (2')\ AB[h] = f(x),$$

then also

$$(3')\ A[g] = f(x).$$

THEOREM 1: *If $A(t)$ is analytic, $|t| < R$, with $A(t) \not\equiv 0$ in $|t| < R$, and if $g(x)$ is of exp. val. $q < R$, then equation*

$$(1)\ A[y] = g(x)$$

*has an entire function solution $y(x)$ of exp. val. q, given by**

$$(2)\ y(x) = \frac{1}{A}[g],$$

and (2) is the only solution of exp. val. $< R$; and the homogeneous equation

$$(3)\ A[y] = 0$$

has no solution† of exp. val. $< R$.

Proof: From Lemma 1, $y(x)$ is of exp. val. $\leqq q$. Since A, $1/A$ are analytic, $|t| < R$, we may apply Lemma 2. This gives $A[y] = A \cdot \frac{1}{A}[g] = 1[g] = g(x)$. That is, (2) *does* give a solution of (1). We already

* Since $A(t) \not\equiv 0$, $|t| < R$, then $1/A(t)$ is analytic and $\not\equiv 0$, $|t| < R$.

† We always disregard the trivial solution $y \equiv 0$.

know that $y(x)$ is of exp. val. $\leqq q$. If exp. val. of $y(x) < q$, then by Lemma 1, $g(x) = A[y]$ is of exp. val. $< q$, which contradicts our assumption on $g(x)$. Therefore $y(x)$ is of exp. val. $= q$.

If (1) has a second solution of exp. val. $< R$, then the homogeneous equation (3) has a solution $Y(x) \not\equiv 0 : A[Y] = 0$. Also the equation $\frac{1}{A}[z] = Y(x)$ has, by what has just been proved, a solution $z(x) = B[Y]$, where $B(t)$ is the reciprocal of $1/A(t)$. That is, $z(x) = A[Y] \equiv 0$. But if $z(x) \equiv 0$, then $Y(x) = \frac{1}{A}[z] = 0$, which is a contradiction. The theorem is thus proved.

Let us now consider the case where $A(t)$ has a finite number of zeros. We have

THEOREM 2: *If $A(t)$ is analytic, $|t| < R$, and in $|t| < R$ has a finite number, m, of zeros,* then the homogeneous equation*

$$A[y] = 0 \tag{1}$$

has precisely m linearly independent solutions of exp. val. $< R$. Furthermore, if $\xi_1, \xi_2, \cdots, \xi_s$, of order $l_1, l_2, \cdots, l_s$ respectively, are the distinct zeros of $A(t)$ in $|t| < R$, then these m linearly independent solutions can so be chosen that precisely l_i of them involve† only the zero ξ_i, $i = 1, 2, \cdots, s$.

Proof: We can write $A(t) = B(t)\,P(t)$, where $P(t) = (t-\xi_1)^{l_1} \cdots (t-\xi^s)^{l_s}$, and where $B(t) \not\equiv 0$, $|t| < R$. (1) can then be written (1′) $B[P[y]] = 0$. If y is any function of exp. val. $< R$, the same is true of $P[y]$. Consequently, since $B(t) \not\equiv 0$, $|t| < R$, we must have (1″) $P[y] = 0$. That is, (1) and (1″) are equivalent. Now (1″) is a *finite* linear differential equation with constant coefficients, so that there exist precisely m linearly independent solutions, which can be chosen as follows:

$$y_1 = e^{\xi_1 x},\ y_2 = x\,e^{\xi_1 x},\ \cdots,\ y_{l_1} = x^{l_1-1}\,e^{\xi_1 x};\quad y_{l_1+1} = e^{\xi_2 x},\ \cdots,\ y_m = x^{l_s-1}\,e^{\xi_s x}.$$

We observe that precisely l_i of these solutions involve the zero ξ_i. This establishes the theorem.

Turning to the non-homogeneous equation, we have

THEOREM 3: *If $A(t)$ is analytic, $|t| < R$, and in $|t| < R$ has a finite number, m, of zeros, and if $g(x)$ is of exp. val. $q < R$, then the equation (2) $A[y] = g(x)$ has a solution $y_0(x)$ of exp. val. $= q$; and the most general solution of exp. val. $< R$ is y_0 plus the most general solution of the corresponding homogeneous equation (1).*

* A multiple zero of order k is to be counted as k zeros.

† The meaning of this will be clear when we proceed to solve (1).

Proof: Let $B(t)$, $P(t)$ have the same significance as in Theorem 2. The equation $(2')$ $B[z] = g(x)$ has a unique solution $z(x)$, and $z(x)$ is of exp. val. $= q$. Consider now $(2'')$ $P[y] = z(x)$. Every solution $y(x)$ of $(2'')$ is a solution of (2) through $(2')$. The entire theorem will follow, then, if we show that $(2'')$ has a solution $y(x)$ of exp. val. $=$ q.

Let $\lambda_1, \cdots, \lambda_r$ be those zeros of $P(t)$ which lie in $|t| \leqq q$, where this time each zero appears (with different subscript) as many times as its order, and choose q' in $q < q' < R$ such that $\lambda_1, \cdots, \lambda_r$ are the only zeros of $P(t)$ in $|t| < q'$. Set $Q(t) = (t-\lambda_1) \cdots (t-\lambda_r)$, so that $P(t) = Q(t)\,M(t)$, $M(t) \neq 0$ in $|t| < q'$. We may then operate on $(2'')$ with $1/M$, obtaining $(2''')$ $Q[y] = \frac{1}{M}[z] = w(x)$, $w(x)$ of exp. val. $= q$. If y, of exp. val. $< q'$, satisfies $(2''')$, it also satisfies $(2'')$. We may then consider $(2''')$.

Let $Q_1(t) = t-\lambda_1$. The equation $Q_1[y] = w(x)$, which is the equation $y' - \lambda_1 y = w(x)$, has a solution $y(x) = \sum_0^\infty y_n x^n$ with $y_0 = 0$, $|y_n| \leqq \frac{M'}{n!} q''^n$, where $|\lambda_1| \leqq q < q''$, $q'' > q$ arbitrary. Therefore $y(x) \ll M' e^{q''x}$, so that $y(x)$ is of exp. val. $\leqq q''$. Since q'' can be taken arbitrary close to q, it follows that $y(x)$ is of exp. val. $\leqq q$. Call this solution $y(x) = w_1(x)$.

If now we consider $Q_2[y] = w_1(x)$, where $Q_2(t) = t-\lambda_2$, we arrive at a solution $y(x) = w_2(x)$ of exp. val. $\leqq q$. And so on, through $Q_r[y] = w_{r-1}(x)$, where $Q_r(t) = t-\lambda_r$, determining a solution $y(x) = w_r(x)$ of exp. val. $\leqq q$. Then $y = w_r(x)$ is a solution of $Q_1[Q_2[\cdots[Q_{r-1}[Q_r[y]]]\cdots]] = w(x)$; i. e., of $Q[y] = w(x)$. $y = w_r(x)$ is therefore a solution of (2), of exp. val. $\leqq q$. There remains only to show that $y_0(x) = w_r(x)$ is of exp. val. precisely q. Suppose the inequality sign holds: exp. val. $y_0(x) < q$. Then (Lemma 1) $A[y_0]$ is of exp. val. $< q$. But $A[y_0] \equiv g(x)$ which is of exp. val. $= q$; a contradiction. This completes the theorem.

To complete our discussion we set down

THEOREM 4. *If $A(t)$ is analytic, $|t| < R$, and in $|t| < R$ has infinitely many zeros, then the homogeneous equation* (1) *has infinitely many linearly independent solutions, while the non-homogeneous equation* (2) *has a solution $y_0(x)$ of exp. val. q, to which may be added any solution of the homogeneous equation. In particular, the most general solution of* (2) *of exp. val. $\leqq q$ is $y_0(x)$ plus those linearly independent solutions of* (1) *which involve zeros of $A(t)$ which lie in $|t| \leqq q$.*

Proof: Choose R' in $q < R' < R$. Then in $|t| < R'$, Theorems 2 und 3 apply. As we let R' increase toward R, we introduce more and more zeros of $A(t)$, and consequently more and more (linearly independent) solutions of the homogeneous equation, the number increasing beyond limit.

3. System of equations in n unknowns, with constant coefficients. We now turn to the general systems of n equations in n unknown functions, homogeneous and non-homogeneous:

$$\text{(I)}\qquad \begin{aligned} A_{11}[y_1]+A_{12}[y_2]+\cdots+A_{1n}[y_n] &= 0 \qquad (1)\\ &\cdots\cdots\cdots\cdots\\ A_{n1}[y_1]+A_{n2}[y_2]+\cdots+A_{nn}[y_n] &= 0 \qquad (n)\end{aligned}$$

and

$$\text{(II)}\qquad \begin{aligned} A_{11}[y_1]+A_{12}[y_2]+\cdots+A_{1n}[y_n] &= g_1(x) \qquad (1),\\ &\cdots\cdots\cdots\cdots\\ A_{n1}[y_1]+A_{n2}[y_2]+\cdots+A_{nn}[y_n] &= g_n(x) \qquad (n).\end{aligned}$$

Set

$$\text{(3)}\qquad \Delta(t) = \begin{vmatrix} A_{11} & \cdots & A_{1n}\\ \cdots & \cdots & \cdots\\ A_{n1} & \cdots & A_{nn}\end{vmatrix},$$

and let $M_{ij}(t)$ denote the cofactor* of $A_{ij}(t)$.

We shall prove the following theorems:

THEOREM 5: *If the $A_{ij}(t)$ are analytic, $|t|<R$, and if $\Delta(t)$ has m zeros in $|t|<R$, then the homogeneous system* (I) *has precisely m linearly independent solutions $(y_1,\cdots,y_n)$ of exp. val. $<R$. Furthermore, if $\xi_1,\cdots,\xi_r$ are the distinct zeros of $\Delta(t)$ in $|t|<R$, and these are of order $p_1,\cdots,p_r$ respectively, then the m linearly independent solutions can be so chosen that precisely p_i of them correspond to the zero ξ_i, $i=1,\cdots,r$.*

THEOREM 6: *Under the same hypotheses on $A_{ij}(t)$, $\Delta(t)$, if $g_1(x),\cdots,g_n(x)$ are of exp. val. $\leqq q<R$, then there exists a solution $(y_1,\cdots,y_n)$ of system* (II) *of exp. val. $\leqq q$; and the general solution of exp. val. $<R$ is obtained by adding in the general solution of system* (I).

We require

LEMMA 3: *If $E(t)$ is analytic, $|t|<R$, and if $|\xi|<R$, then*

$$E[x^s e^{\xi x}] = e^{\xi x}\left\{E(\xi)x^s+\binom{s}{1}E'(\xi)x^{s-1}+\cdots+\binom{s}{s}E^{(s)}(\xi)x^0\right\}.$$

Proof: We have only to substitute the identity

$$(x^s e^{\xi x})^{(n)} = e^{\xi x}\left[\xi^n x^s+\binom{n}{1}\xi^{n-1}sx^{s-1}+\binom{n}{2}\xi^{n-2}s(s-1)x^{s-2}+\cdots\right.$$
$$\left.\cdots+\binom{n}{n-s}\xi^{n-s}s(s-1)\cdots 1\,x^0\right]$$

into $E[x^s e^{\xi x}]\equiv\sum_0^\infty e_n(x^s e^{\xi x})^{(n)}$ and collect common terms.

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* Then $\Delta(t)=\sum_{i=1}^{n}A_{ij}(t)M_{ij}(t)$, $j=1,2,\cdots,n$.

COROLLARY: *If* $E(\xi) \neq 0$, *then the equation*

$$E[y] = (c_k x^k + c_{k-1} x^{k-1} + \cdots + c_0 x^0) e^{\xi x}, \ (c_k \neq 0),$$

has a solution $y(x)$ *of the form*

$$y(x) = (d_k x^k + d_{k-1} x^{k-1} + \cdots + d_0 x^0) e^{\xi x}, \ d_k \neq 0;$$

and there are no two solutions of the form $y(x) = e^{\xi x} Q(x)$, $Q(x)$ *a polynomial. In particular, if the righthand member is identically* 0, *the only solution of this form is* $y(x) \equiv 0$.

Proof: Assume $y(x) = e^{\xi x}(d_n x^n + \cdots + d_0 x^0)$ and substitute into the equation. Using Lemma 3 we see that, since $E(\xi) \neq 0$, n must be equal to k. The successive coefficients d_i are seen to be uniquely determined, and this completes the proof.

We proceed to the proofs. Theorems 5 and 6 have been proved for $n = 1$. We assume them true up to $n-1$, and shall prove them true for n. Operate on the n equations of (I) by $M_{1i}, \cdots, M_{ni}$, respectively, and add. Clearly every column drops out except the ith, leaving

$$\Delta[y_i] = 0, \qquad i = 1, 2, \cdots, n. \tag{4}$$

Hence every solution of (I) must satisfy equations (4).

Consider a particular zero,* say ξ, of $\Delta(t)$. Let its order be p.

Case (i). Not all the functions $A_{ij}(t)$ vanish at $t = \xi$.

Then at $t = \xi$ the determinant Δ will be of rank k, $0 < k < n$. We may suppose subscripts rearranged so that the principal minor of order k: $\Delta_k(t)$, is $\neq 0$ at $t = \xi$.

Let $(y_{k+1}, \cdots, y_n)$ be any solution (involving only ξ) of the system of equations

$$\Delta[y_i] = 0, \qquad i = k+1, \cdots, n. \tag{a}$$

Since $\Delta_k(\xi) \neq 0$, we see (Lemma 3 and Corollary) that the first k equations of (I) then determine $y_1, \cdots, y_k$ *uniquely* (to involve only the zero ξ). Define

$$\Delta_{rs} = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1k} & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2k} & A_{2r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{k1} & A_{k2} & \cdots & A_{kk} & A_{kr} \\ A_{s1} & A_{s2} & \cdots & A_{sk} & A_{sr} \end{vmatrix}, \tag{5}$$

* If we have a solution $y_1, \cdots, y_n$ involving more than one zero ξ of $\Delta(t)$, say $\xi_1, \xi_2, \cdots, \xi_r$, then it is clear that the components $y_{1i}, \cdots, y_{ni}$ of $y_1, \cdots, y_n$ which involve only ξ_i themselves form a solution, $i = 1, \cdots, r$, and the r solutions so obtained are linearly independent. It follows that we may consider solutions corresponding to only one zero at a time.

and let*

$$L_i^{(s)} = \text{cofactor of } A_{ir} \text{ in } \Delta_{rs}. \tag{6}$$

Operate on equations (I, 1), $\cdots$, (I, k) by $L_1^{(s)}, \cdots, L_k^{(s)}$, respectively, and add. We obtain

$$0 = \sum_{i=1}^{n} (A_{1i} L_1^{(s)} + \cdots + A_{ki} L_k^{(s)})[y_i] = \sum_{i=1}^{n} (\Delta_{is} - A_{si} \Delta_k)[y_i]$$
$$= -\sum_{i=1}^{k} A_{si} \Delta_k [y_i] + \sum_{i=k+1}^{n} (\Delta_{is} - A_{si} \Delta_k)[y_i],$$

or

$$\Delta_k [A_{s1}[y_1] + \cdots + A_{sn}[y_n]] = \Delta_{k+1,s}[y_{k+1}] + \Delta_{k+2,s}[y_{k+2}] + \cdots + \Delta_{ns}[y_n], \quad s = 1, \cdots, n. \tag{7}$$

That is, equations (I) imply equations (I, 1), $\cdots$, (I, k) and equations (a), and these in turn imply equations (7).

Now for any solution $(y_1, \cdots, y_n)$ of (I), we have $A_{s1}[y_1] + \cdots + A_{sn}[y_n] = 0$, $s = 1, 2, \cdots, n$, so that from (7) the following equations are satisfied:

$$\Delta_{k+1,s}[y_{k+1}] + \Delta_{k+2,s}[y_{k+2}] + \cdots + \Delta_{ns}[y_n] = 0, \quad s\dagger = k+1, \cdots, n. \tag{8}$$

Conversely, let $(y_{k+1}, \cdots, y_n)$ be any solution of (8) involving only the zero ξ. If such a set $(y_{k+1}, \cdots, y_n)$ also satisfies equations (a), then as before we determine $(y_1, \cdots, y_k)$ uniquely (involving only ξ) to satisfy (I, 1), $\cdots$, (I, k), and therefore also equations (7). But from (7) and (8),

$$\Delta_k [A_{s1}[y_1] + \cdots + A_{sn}[y_n]] = 0, \quad s = k+1, \cdots, n; \tag{7'}$$

and since $A_{s1}[y_1] + \cdots + A_{sn}[y_n]$ involves only the zero ξ, and $\Delta_k(\xi) \neq 0$, it follows (Corollary, Lemma 3) that

$$A_{s1}[y_1] + \cdots + A_{sn}[y_n] = 0, \qquad s = k+1, \cdots, n.$$

That is, *all* of the equations (I) are satisfied.

Hence the number of linearly independent solutions $(y_1, \cdots, y_n)$ of (I) involving only the zero ξ is the same as the number of linearly independent solutions $(y_{k+1}, \cdots, y_n)$ involving only ξ which satisfy *both* (a) *and* (8).

We now examine the determinant $\delta(t)$ of (8):

$$\delta(t) = \begin{vmatrix} \Delta_{k+1,k+1} & \Delta_{k+2,k+1} & \cdots & \Delta_{n,k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta_{k+1,n} & \Delta_{k+2,n} & \cdots & \Delta_{nn} \end{vmatrix}. \tag{9}$$

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* It is clear that $L_i^{(s)}$ is independent of r. Observe that $L_s^{(s)} \equiv \Delta_k$.

† For $s = 1, 2, \cdots, k$, $\Delta_{k+1,s} = \Delta_{k+2,s} = \cdots = \Delta_{ns} \equiv 0$, so that in (8) we need only consider $s = k+1, \cdots, n$.

Let

$$\text{(i)}\quad \Delta = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}, \qquad \text{(ii)}\ \Delta_k = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \cdot & \cdots & \cdot \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}\ (1 \leqq k < n),$$

$$\text{(iii)}\quad \Delta_{sr} = \begin{vmatrix} a_{11} & \cdots & a_{1k} & a_{1r} \\ \cdot & \cdots & \cdot & \cdot \\ a_{k1} & \cdots & a_{kk} & a_{kr} \\ a_{s1} & \cdots & a_{sk} & a_{sr} \end{vmatrix}, \qquad \text{(iv)}\quad \delta = \begin{vmatrix} \Delta_{k+1,k+1} & \cdots & \Delta_{n,k+1} \\ \cdot & \cdots & \cdot \\ \Delta_{k+1,n} & \cdots & \Delta_{nn} \end{vmatrix}.$$

Then we have the following theorem on determinants:

LEMMA 4: *δ as defined by* (iv) *satisfies the identity*

$$\text{(v)}\quad \delta = \Delta_k^{n-k-1} \cdot \Delta.$$

Proof:* We may suppose that $\Delta \neq 0$, $\Delta_k \neq 0$, since if it is then true, it will remain true for $\Delta = 0$ or $\Delta_k = 0$ (or both) by continuity. For the same reason we can assume that no one of the principal minors $\Delta_1, \Delta_2, \cdots, \Delta_{k-1}$ of Δ_k vanishes. We proceed to form a new determinant from Δ, in which the elements† (1,2) and (2,1) are zero. We make $(1,2) = 0$ by adding $-a_{12}/a_{11}$ times the first column to the second column; and make $(2,1) = 0$ by then adding $-a_{21}/a_{11}$ times the first row to the second row. In the new determinant, the principal minor of order two has the non-zero elements: $(1,1) = \Delta_1$, $(2,2) = \Delta_2/\Delta_1$. We form in the same manner still a new determinant in which the following additional elements are zero: (1,3), (2,3), (3,1), (3,2). In this new determinant the principal minor of order three has its non-zero elements: $(1,1) = \Delta_1$, $(2,2) = \Delta_2/\Delta_1$, $(3,3) = \Delta_3/\Delta_2$.

And so on. After the kth reduction we arrive at a determinant Δ' which has as its principal minor of order k the determinant Δ_k':

$$\Delta_k' = \begin{vmatrix} \Delta_1 & & & \\ & \Delta_2/\Delta_1 & & \mathrm{O} \\ & & \Delta_3/\Delta_2 & \\ & & & \ddots \\ \mathrm{O} & & & \Delta_k/\Delta_{k-1} \end{vmatrix}, \qquad \Delta' = \begin{vmatrix} \Delta_k' & \mathrm{O} \\ \mathrm{O} & \begin{matrix} c_{k+1,k+1} & \cdots & c_{k+1,n} \\ \cdot & \cdots & \cdot \\ c_{n,k+1} & \cdots & c_{nn} \end{matrix} \end{vmatrix},$$

where the c_{ij}'s are given in terms of the a_{ij}'s.‡ The explicit values of the c_{ij}'s are not needed.

* Professor G. Y. Rainich very kindly suggested this method of proof to me.

† By (1,2) is meant the element in the first row and second column.

‡ The method we have given shows how we may get rid of the non-diagonal terms in the principal minor of order k. After doing this, we can by a further application of

The transformations we have made obviously have not changed the values of the determinants Δ, Δ_k, Δ_{sr}, and therefore of δ. That is, $\Delta' = \Delta$, $\Delta'_k = \Delta_k$, $\Delta'_{sr} = \Delta_{sr}$, $\delta' = \delta$. But $\Delta'_{sr} = c_{sr}\,\Delta_k$, so that

$$\delta = \delta' = \Delta_k^{n-k} \cdot \begin{vmatrix} c_{k+1,k+1} & \cdots & c_{n,k+1} \\ \cdot & \cdots & \cdot \\ c_{k+1,n} & \cdots & c_{nn} \end{vmatrix}, \quad \Delta = \Delta' = \Delta_k \cdot \begin{vmatrix} c_{k+1,k+1} & \cdots & c_{k+1,n} \\ \cdot & \cdots & \cdot \\ c_{n,k+1} & \cdots & c_{nn} \end{vmatrix}.$$

Since the two c-determinants differ only in that the rows of one are the columns of the other, it follows that $\delta = \Delta_k^{n-k-1} \cdot \Delta$.

Let us return to the proof of Theorem 5. By Lemma 4, $\delta(t) = \Delta_k^{n-k-1}(t)\,\Delta(t)$. Since $\Delta_k(\xi) \neq 0$, $\delta(t)$ has a p-fold zero at $t = \xi$. By our assumption that Theorems 5 und 6 are true up to $n-1$, then, the number of linearly independent solutions of (8) involving only the zero ξ is precisely p. Let $(y_{k+1}, \cdots, y_n)$ be one such solution. These functions $y_i(x)$ satisfy (as we know) the relations

$$\delta[y_i] = 0, \qquad i = k+1, \cdots, n.$$

That is,
$$\Delta_k^{n-k-1}\,\Delta[y_i] = 0, \qquad i = k+1, \cdots, n.$$

But since $\Delta_k(\xi) \neq 0$, we have (Corollary, Lemma 3)

$$\Delta[y_i] = 0, \qquad i = k+1, \cdots, n.$$

This is the set of equations (a). That is, every solution $(y_{k+1}, \cdots, y_n)$ of (8) is a solution of (a). Consequently there are precisely p linearly independent solutions $(y_1, \cdots, y_n)$ of (I) involving only the zero ξ.

Case (ii): All the functions $A_{ij}(t)$ vanish at $t = \xi$.

Let k be the largest order (of the zero ξ) common to all the functions $A_{ij}(t)$. Then on setting $B_{ij}(t)\,P(t) = A_{ij}(t)$, $P(t) = (t-\xi)^k$, at least one function $B_{ij}(t)$ is $\neq 0$ at $t = \xi$. Equations (I) can then be written

(I′) $$B_{i1}[u_1] + \cdots + B_{in}[u_n] = 0, \qquad i = 1, \cdots, n,$$

where

(1″) $$P[y_i] = u_i.$$

Let

(3′) $$\omega(t) = \begin{vmatrix} B_{11} & \cdots & B_{1n} \\ \cdot & \cdots & \cdot \\ B_{n1} & \cdots & B_{nn} \end{vmatrix}.$$

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the method, using only the first k rows and first k columns, get rid of the elements (i, j) in the upper right- and lower left-hand corners:

$$(i, j) = 0, \quad \begin{cases} i = 1, 2, \cdots, k \\ j = k+1, \cdots, n \end{cases} \text{ and } \begin{cases} i = k+1, \cdots, n \\ j = 1, 2, \cdots, k \end{cases}.$$

Then $\omega(t)\,P^n(t) = \Delta(t)$, so that at $t = \xi$, $\omega(t)$ has a zero of order $(p - kn)$.

By Case (i), (I′) has precisely $(p - kn)$ linearly independent solutions $(u_1, \cdots, u_n)$ of exp. val. $< R$, corresponding to $t = \xi$. Let $(v_1, \cdots, v_n)$ be one of these solutions, and let $(z_1, \cdots, z_n)$ be any solution of $P[z_i] = v^i$, $i = 1, \cdots, n$. If we do this for each of the $(p - kn)$ solutions $(u_1, \cdots, u_n)$, we obtain $(p - kn)$ solutions $(y_1, \cdots, y_n)$, which must be linearly independent since the corresponding u-solutions are linearly independent.

Now for every solution $(v_1, \cdots, v_n)$, $(z_1, \cdots, z_n)$ is not unique, since we can add in to each z_i an arbitrary solution of the homogeneous equation $P[y] = 0$. There is therefore the possibility of further linearly independent solutions. Since for a given solution $(v_1, \cdots, v_n)$, any two solutions $(y_1, \cdots, y_n)$ differ *only* by solutions of $P[y] = 0$, we can be certain that we have *all* the linearly independent solutions $(y_1, \cdots, y_n)$ if we can show that every solution $y_1, \cdots, y_n$ of $P[y] = 0$ is a solution of equations (I). But this is clearly true. There are kn such solutions and these are given by*

$$y_1 = 0,\ y_2 = 0, \cdots, y_i = x^r e^{\xi x},\ y^{i+1} = 0, \cdots, y_n = 0, \begin{cases} r = 0, 1, \cdots, k-1 \\ i = 1, 2, \cdots, n. \end{cases}$$

Then, these kn solutions together with the $(p - kn)$ previously found are all the linearly independent solutions involving only ξ; and their total number is p.†

Hence in both cases, (i), (ii), corresponding to $t = \xi$ we have precisely p linearly independent solutions of exp. val. $< R$. If we sum up the number of solutions for all the zeros of $\Delta(t)$, we have the conclusion of Theorem 5.

And now we take up the proof of Theorem 6. We shall want

LEMMA 5: *If the functions $A_{ij}(t)$ are analytic, $|t| < R$, with $\Delta(t) \not\equiv 0$, and if ξ is any number for which $|\xi| < R$, then the system*

$$\text{(A)} \qquad A_{i1}[y_1] + \cdots + A_{in}[y_n] = P_i(x)e^{\xi x}, \qquad i = 1, \cdots, n,$$

where the $P_i(x)$ are polynomials, has a solution $(y_1, \cdots, y_n)$ of the form $y_i(x) = Q_i(x)\,e^{\xi x}$, the $Q_i(x)$ being polynomials.

Proof: The lemma is easily verified for $n = 1$. We assume it true up to $n - 1$, and proceed to establish it for n.

In (A) we may assume that the A_{ij}'s in any column have no common

* They correspond to the solution $(u_1 = 0, \cdots, u_n = 0)$ of (I′).

† It is easy to see that the above mentioned p solutions *are* linearly independent.

zero.* Then at least one function A_{ij} is $\neq 0$ at $t = \xi$; and we may clearly assume it to be A_{11}. Then there is a principal minor $\Delta_k (k < n)$ which does not vanish† at $t = \xi$.

Consider now the system

$$\text{(B)} \qquad \begin{aligned} &\Delta_{k+1,s}[y_{k+1}] + \cdots + \Delta_{ns}[y_n] \\ &= \{L_1^{(s)}[P_1 e^{\xi x}] + \cdots + L_k^{(s)}[P_k e^{\xi x}]\} + \Delta_k [P_s e^{\xi x}], \qquad s = k+1, \cdots, n, \end{aligned}$$

where the notation is that used in the proof of Theorem 5. The right hand members of (B) are of the form $S_i(x) e^{\xi x}$, $i = k+1, \cdots, n$. Also, the determinant of (B) is, as we know, $\delta(t) = \Delta_k^{n-k-1}(t) \Delta(t)$, and is therefore $\not\equiv 0$. By our assumption that the lemma is true up to $n-1$, system (B) has then a solution $(y_{k+1}, \cdots, y_n)$ of the form $y_i(x) = Q_i(x) e^{\xi x}$, $i = k+1, \cdots, n$. Substitute these values into the first k equations of (A):

$$\text{(A')} \qquad A_{i1}[y_1] + \cdots + A_{ik}[y_k] = P_k e^{\xi x} - \{A_{i,k+1}[y_{k+1}] + \cdots + A_{in}[y_n], \qquad i = 1, \cdots, k.$$

This system satisfies the conditions of the lemma, so that, k being $\leqq n-1$, there is a solution $(y_1, \cdots, y_k)$ of (A') of the form $y_i = Q_i e^{\xi x}$, $i = 1, \cdots, k$. That is, $(y_1, \cdots, y_n)$ of the form $y_i = Q_i e^{\xi x}$ exists satisfying the first k equations of (A).

Now operate on equations (A, 1), $\cdots$, (A, k) by $L_1^{(s)}, \cdots, L_k^{(s)}$, respectively, $s > k$, and add. This gives

$$\text{(a)} \qquad \begin{aligned} &\Delta_k [A_{s1}[y_1] + \cdots A_{sn}[y_n]] \\ &= \{\Delta_{k+1,s}[y_{k+1}] + \cdots + \Delta_{ns}[y_n]\} - \{L_1^{(s)}[P_1 e^{\xi x}] + \cdots + L_k^{(s)}[P_k e^{\xi x}]\}, \\ &\qquad\qquad s = k+1, \cdots, n. \end{aligned}$$

On using (B) this becomes

$$\text{(b)} \qquad \Delta_k [A_{s1}[y_1] \cdots + A_{sn}[y_n] - P_s e^{\xi x}] = 0, \qquad s = k+1, \cdots, n.$$

Now $\Delta_k(\xi) \neq 0$, while $A_{s1}[y_1] + \cdots + A_{sn}[y_n] - P_s e^{\xi x}$ is of the form $Q(x) e^{\xi x}$; whence from Corollary, Lemma 3, we must have

* For if $S_j(t)$ is the polynomial with all the zeros common to the A_{ij}'s in the jth column, $j = 1, \cdots, n$, we consider the system

(B) $\quad B_{i1}[u_1] + \cdots + B_{in}[u_n] = P_i(x) e^{\xi x}$, where $S_j(t) B_{ij}(t) = A_{ij}(t)$, $u_j = S_j[y_j]$.

(B) satisfies the conditions of the lemma. If the lemma is true for (B), then u_i exist of form $Q_i(x) e^{\xi x}$, and therefore the y_i exist of the same form, defined by $S_j[y_j] = u_j$, and the y_i satisfy (A).

† We can take $k = 1$, since $\Delta_1 = A_{11} \neq 0$ at t_1. However, we prefer to keep k in order to use the notation introduced in proving Theorem 5 and Lemma 4.

$$A_{s1}[y_1]+\cdots+A_{sn}[y_n]-P_s e^{\xi x} = 0, \quad s = k+1, \cdots, n.$$

These are the last $n-k$ equations of system (A), so the lemma is proved.

We are assuming that Theorem 6 is true up to $n-1$, and wish to prove it then true for n. Since $\Delta(t) \not\equiv 0$, at least one minor of order $n-1$ is $\not\equiv 0$, and by a rearrangement of rows and columns we can place it in the upper left hand corner. That is, we may suppose that $\Delta_{n-1}(t) \not\equiv 0$. If in system

(II) $$A_{i1}[y_1]+\cdots+A_{in}[y_n] = g_i(x), \qquad i = 1, \cdots, n,$$

we operate on equations (II, 1), $\cdots$, (II, n) respectively by $M_{1i}, \cdots, M_{ni}$, and add, we find necessary conditions for a solution:

(A) $$\Delta[y_i] = M_{1i}[g_1]+\cdots+M_{ni}[g_n], \qquad i = 1, \cdots, n.$$

Choose $y_n(x)$ as any* solution of (A) (for $i = n$):

(a) $$\Delta[y_n] = M_{1n}[g_1]+\cdots+M_{nn}[g_n].$$

The first $n-1$ equations of (II) then form a system

(II′) $$A_{i1}[y_1]+\cdots+A_{i,n-1}[y_{n-1}] = g_i(x)-A_{in}[y_n], \quad i = 1, \cdots, n-1,$$

which satisfies all the conditions of the theorem, so that we can by our assumption find a solution $(y_1, \cdots, y_{n-1})$ of (II′) of exp. val. $< R'$. That is, a solution $(y_1, \cdots, y_n)$ of exp. val. $< R'$ exists of the first $n-1$ equations of (II). If we operate on (II,1), $\cdots$, (II, $n-1$) by $M_{1n}, \cdots, M_{n-1,n}$, respectively, and add, we obtain, on using (a),

$$M_{nn}[A_{n1}[y_1]+\cdots+A_{nn}[y_n]-g_n] = 0;$$

or, since $M_{nn} \equiv \Delta_{n-1}$,

(b) $$\Delta_{n-1}[A_{n1}[y_1]+\cdots+A_{nn}[y_n]-g_n] = 0.$$

Therefore

(c) $$A_{n1}[y_1]+\cdots+A_{nn}[y_n]-g_n = \sum_{i=1}^{s}(c_{i,l_i-1}x^{l_i-1}+\cdots+c_{i,0}x^0)e^{\xi_i x},$$

where $\xi_1, \cdots, \xi_s$, of order $l_1, \cdots, l_s$, respectively, are the zeros of $\Delta_{n-1}(t)$. If all the c's are zero, then (c) is equation (II, n), and we have a solution

* To establish Theorem 6, it suffices to prove the existence of *one* solution $(y_1, \cdots, y_n)$ of exp. val. $\leqq q$. Let us then choose R' in $q < R' < R$, as close to q as we please (such that Δ and Δ_{n-1} do not vanish in $q < |t| \leqq R'$), and consider only the region $|t| < R'$. In particular, then, the solution y_n of (a) is chosen so that its exp. val. is $< R'$.

of system (II). Suppose not all the c's are zero. For each zero ξ_i in the set $\xi_1, \cdots, \xi_s$, we can find a solution $(y_1^{(i)}, \cdots, y_n^{(i)})$ of

$$A_{j1}[y_1] + \cdots + A_{jn}[y_n] = \begin{cases} 0, & j \neq n; \\ -(c_{i,l_i-1}\, x^{l_i-1} + \cdots + c_{i,0}\, x^0)\, e^{\xi_i x}, & j = n. \end{cases}$$

This follows from Lemma 5. On adding these s sets to the set $(y_1, \cdots, y_n)$ already found, we annihilate the right hand member of (c), and so obtain a solution of (II).

Now the solution we obtain is of exp. val. $< R'$, and since R' can be chosen as near to q as we please, there is a solution of (II) of exp. val. $\leqq q$. This proves Theorem 6.

Remark. In the preceding work we have dealt only with functions of finite exp. val. If we wish to widen the class of functions to be considered, we must restrict the functions $A_{ij}(t)$. One important restriction is that $A_{ij}(t)$ be entire functions of finite order. We do not in this paper consider the theory of equations under these new conditions.

HARVARD UNIVERSITY.

SOLUTION TO A PROBLEM IN DIFFUSION EMPLOYING A NON-ORTHOGONAL SINE SERIES.*

By R. L. Peek jr.

1. A problem arising in the study of diffusion phenomena involves boundary conditions which can only be satisfied by the use of a sine series in which the terms are non-orthogonal, but by a novel modification of Fourier's method it is possible to determine the coefficients of the terms directly. A cosine series having terms of the same form as those of this sine series is orthogonal. A similar sine series obtained by Fourier for a special case of heat conduction is orthogonal, while the corresponding cosine series is not. It can be shown that these two types of series include all cases of orthogonality for sine and cosine series, and that the ordinary Fourier series may be regarded as a special case of either type.

The problem referred to arises in the use of a convenient experimental method for the study of the rate of diffusion of moisture through a solid such as rubber.† In this method, the sample of the solid, in the form of a thin sheet, separates two closed chambers, in one of which a constant pressure of water vapor is maintained, while the other, initially evacuated, is provided with a manometer by which may be read the pressure of the water vapor which has diffused through the membrane into the second chamber. The flow will be considered unilinear, and Fick's law held to apply. Under the conditions of the experimental method the diffusion in the measuring chamber is so rapid compared with that in the membrane that the concentration throughout the measuring chamber may be taken as uniform. Then the time rate of change of concentration at the boundary $x = a$ of the membrane next to the measuring chamber will equal the rate of flow into this chamber divided by the volume (V) of the latter, or,

$$\frac{\partial \theta}{\partial t} = -\frac{Ah^2}{V}\frac{\partial \theta}{\partial x}, \qquad (x = a), \tag{1}$$

in which A is the area of the surface of the membrane exposed to the chamber, and h^2 the diffusivity constant. The other conditions in the practical problem are that the initial concentration be constant, and that the concentration at the boundary $x = 0$ be constant for all positive

* Received April 14, 1928.

† Schumacher and Ferguson, Journ. Amer. Chem. Soc., 49 (1926), 427.

values of t. A solution satisfying these conditions may be deduced from that for a more general problem, where,

$$\theta = 0, \qquad (x = 0), \tag{2}$$

$$\theta = f(x), \quad (t = 0). \tag{3}$$

The basic differential equation for diffusion, commonly known as Fick's law, is, for unilinear flow,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}. \tag{4}$$

The problem is then to obtain a solution of (4) satisfying (1), (2), and (3).*

2. A series may be formed by adding terms each of which is a well known particular solution of (4). Such a series is

$$\theta = \sum A_n \sin(\lambda_n x)\, e^{-h^2 \lambda_n^2 t}. \tag{5}$$

This series satisfies (2) and (4). A_n and λ_n are entirely unrestricted constants. By substitution, (1) will be satisfied, if

$$\lambda \sin a\lambda - p \cos a\lambda = 0, \tag{6}$$

where $p = \frac{A}{V}$ (a positive quantity). This condition restricts λ to being a root of (6), there being an infinite number of such roots.

If $t = 0$, from (5),

$$\theta = \sum A_n \sin(\lambda_n x). \tag{7}$$

Thus, if $f(x)$ can be expanded within the limits $x = 0$ to $x = a$ in a series of the form (7), in which λ is a root of (6), all the conditions are satisfied, and (5) is a solution to the problem. The limits of the expansion being as stated, $f(x)$ may be taken as an odd function and only positive roots of (6) considered.

Following Fourier's method, multiply through by $\sin(\lambda_m x) \cdot dx$ and integrate from $x = 0$ to $x = a$. Any term on the right-hand side of the resulting equality can (employing (6)), be reduced to

$$-\frac{A_n \sin(a\lambda_m) \sin(a\lambda_n)}{p}. \tag{8}$$

* A similar problem, the cooling of a sphere in a well stirred mass of liquid, has been solved by Peddie (Edinb. Math. Soc. Proc., 19 (1901), 34. In this solution, the coefficients of a non-orthogonal sine series are obtained by another modification of Fourier's method, but could have been equally well determined by the method described in this paper. This reference was kindly called to the writer's attention by Professor H. W. March of the University of Wisconsin.

The series is therefore not orthogonal. But by substituting $x = a$ in (7) and multiplying through by $-\frac{\sin(a\lambda_m)}{p}$, any term on the right-hand side of the resulting equality is precisely (8). Hence

$$\int_0^a f(x)\sin(\lambda_n x)\cdot dx + \frac{\sin(a\lambda_n)}{p} f(a) = A_n\left(\frac{\sin^2(a\lambda_n)}{p} + \int_0^a \sin^2(\lambda_n x)\cdot dx\right). \tag{9}$$

From (9), by means of (6), the series may be written as

$$f(x) = \sum \frac{2(p^2+\lambda_n^2)}{a\lambda_n^2 + ap^2 + p}\left(\frac{\sin(a\lambda_n)}{p} f(a) + \int_0^a f(\mu)\sin(\lambda_n\mu)\cdot d\mu\right)\sin\lambda_n x. \tag{10}$$

The values of λ, being positive roots of (6), may be obtained by a method of successive approximations. If not too high an order of accuracy is required, the roots may be readily obtained graphically from the intersections of

$$y = \tan(a\lambda) \quad \text{and} \quad y = \frac{p}{\lambda}.$$

3. The corresponding cosine series

$$f(x) = \sum B_n \cos(\lambda_n x), \tag{11}$$

in which λ is a root of (6), is orthogonal. Multiplying through by $\cos(\lambda_m x)\cdot dx$ and integrating from $x = 0$ to $x = a$, all terms in the series but that in A_m disappear, and, employing (6), the coefficients are so determined that

$$f(x) = \sum \frac{2(p^2+\lambda_n^2)}{a\lambda_n^2 + ap^2 + p}\cos(\lambda_n x)\int_0^a f(\mu)\cos(\lambda_n\mu)\cdot d\mu. \tag{12}$$

From (12) may be drawn an interesting corollary with respect to (10). Expand $f(x) = 1/p$ by means of (12), and differentiate with respect to x. Then,

$$0 = \sum \frac{2(p^2+\lambda_n^2)}{a\lambda_n^2 + ap^2 + p}\sin(\lambda_n x)\frac{\sin a\lambda_n}{p}. \tag{13}$$

From which it follows that the first of the two terms in brackets in (10) may be omitted without affecting the result. In the practical problem referred to, however, in which the initial concentration is constant ($f(x) = \theta_0$), a simpler expression is obtained for the coefficients by using (10) as given than by adopting the modification permitted by (13).

4. The convergence of the two series developed above may be very elegantly demonstrated by employing Cauchy's Theorem of Residues in the manner developed by Picard.* In the text referred to, II, pp. 179 et seq., it is shown by means of the theorem just mentioned, that a function $f(x)$ obeying Dirichlet's conditions, may be expanded in a series of the form

$$f(x) = -\sum \frac{\psi(\varrho)}{\pi'(\varrho)} e^{\varrho x} \int_{x_0}^{x_1} e^{-\varrho\mu} f(\mu)\, d\mu, \tag{14}$$

the summation being extended to the roots (ϱ) of $\pi(z)$, z being a complex variable. Certain conditions of convergence must be satisfied by $\psi(z)$ and $\pi(z)$. The developments discussed above are similar to that obtained by Fourier for the conduction of heat in a sphere radiating heat into an atmosphere of constant temperature. Paraphrasing the convergence proof for this development given in the text referred to (p. 190), the assumed functions may be taken as

$$\pi(z) = z(e^{az} - e^{-az}) + p(e^{az} + e^{-az}), \tag{15}$$

$$\psi(z) = e^{-az}(p - z). \tag{16}$$

That the conditions of convergence are satisfied, provided $x_1 - 2a < x < x_0 + 2a$ may be shown for (15) and (16) in the same way as for the functions considered in the text. If therefore, x_1 is taken as a and x_0 as $-a$, the series (14) will be convergent for values of x within the limits of integration. Also the roots of $\pi(z)$ may be shown in the same manner to be pure imaginaries, $i\lambda$, where λ is a root of (6), provided p is real and positive.

Substituting (15) and (16) in (14), writing $\varrho = i\lambda$, and using the values assigned x_0 and x_1 to satisfy the conditions of convergence, there is obtained

$$f(x) = \sum \frac{(ip + \lambda)}{2((1 + ap)\sin(a\lambda) + a\lambda\cos(a\lambda))} \int_{-a}^{a} e^{i\lambda(x-\mu-a)} f(\mu)\, d\mu. \tag{17}$$

This development is, from the conditions of convergence, valid within the limits of integration. Zero is not a root of (6) and for each positive root there is an equal and opposite negative root. Pairing the terms involving equal and opposite roots, and using (6) to simplify the coefficients, the development becomes

$$f(x) = \sum \frac{p^2 + \lambda^2}{a\lambda^2 + ap^2 + p} \int_{-a}^{+a} \cos(\lambda(x - \mu)) f(\mu) \cdot d\mu. \tag{18}$$

* E. Picard. Traité d'Analyse, 2nd Ed., Paris, 1905.

Assuming $f(x)$ to be an odd function, (18) reduces to (10), if consideration of (13) is included. If $f(x)$ is taken as an even function, (18) reduces to (12). Hence both (10) and (12) are convergent within their limits provided $f(x)$ fulfills Dirichlet's conditions.

5. The similar series developed by Fourier, to which reference was made above, is of the form of (7) in which λ_n is a root of

$$\lambda \cos(a\lambda) + p \sin(a\lambda) = 0. \tag{19}$$

This sine series is orthogonal; the determination of the coefficients by Fourier's method is given in Byerly,* p. 117. The corresponding cosine series is not orthogonal but the coefficients may be determined by the method employed above to determine the coefficients of (10). Either the sine or cosine series may be deduced from the equation developed from Cauchy's Theorem of Residues by Picard (loc. cit.).

It is noted in Byerly (loc. cit.) that the condition for the orthogonality of a sine series such as (7) reduces to the condition (19). Similarly, the condition for the orthogonality of a cosine series such as (11) reduces to the condition (6). Hence the two series here discussed include all cases of orthogonality for sine and cosine series, and Fourier's series may be considered as a special case for which p is zero in (6) or infinite in (19).

Acknowledgment is due Dr. T. C. Fry, of these Laboratories, for much aid and advice kindly contributed to the work reported in this paper.

* W. E. Byerly. Fourier's Series and Spherical Harmonics, New York.

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NOTE ON THE PRECEDING PAPER BY MR. PEEK.

By Einar Hille.

Mr. Peek has obtained the null-series

$$\sum_{n=0}^{\infty} \frac{\lambda_n^2 + p^2}{a\lambda_n^2 + ap^2 + p} \sin(\lambda_n a) \sin(\lambda_n x) = 0, \tag{1}$$

valid for $-a < x < +a$. The series remains convergent for $x = \pm a$, but its sum is clearly not zero. In deriving this expansion Mr. Peek uses term-by-term differentiation. The object of the present note which is written at the suggestion of the Editors of these Annals, is to show that this procedure is legitimate.

In order to verify formula (1) for $-a+\varepsilon \leqq x \leqq a-\varepsilon$ it is sufficient to prove that the series converges uniformly on this interval, since (1) is obtained by formal differentiation of a cosine series known to represent a constant. This is proved as follows. The λ_n are the positive roots of the equation

$$\lambda \sin(\lambda a) - p \cos(\lambda a) = 0, \tag{2}$$

where a and p are positive constants. All the roots are real; there is one and only one, λ_n, in each of the intervals $\frac{n\pi}{a} < \lambda < \frac{(n+1)\pi}{a}$, $n = 0, 1, 2, \cdots$. In virtue of (2)

$$\lambda_n = \frac{n\pi}{a} + \frac{p}{n\pi} + O\left(\frac{1}{n^2}\right). \tag{3}$$

It follows that

$$\sin(\lambda_n a) = (-1)^n \frac{p^a}{n\pi}\left[1 + O\left(\frac{1}{n^2}\right)\right],$$

$$\sin(\lambda_n x) = \sin\left(\frac{n\pi}{a} x\right)\left[1 + O\left(\frac{1}{n^2}\right)\right] + \cos\left(\frac{n\pi}{a} x\right) \frac{px}{n\pi}\left[1 + O\left(\frac{1}{n^2}\right)\right],$$

and the nth term of the series in (1) equals

$$(-1)^n \frac{p}{n\pi} \sin\left(\frac{n\pi}{a} x\right) + O\left(\frac{1}{n^2}\right). \tag{4}$$

This expression shows that the series converges uniformly on the interval $-a+\varepsilon \leqq x \leqq a-\varepsilon$.

The boundary value problem encountered by Mr. Peek,

$$(5) \qquad y'' + \lambda^2 y = 0, \qquad y(0) = 0, \qquad y'(a) - \lambda^2 y(a) = 0,$$

was considered first by Poisson* in discussing the motion of a heavy particle suspended at the end of an elastic thread. It seems to be one of the simplest of all boundary value problems which lead to null expansions. It has been pointed out by Tamarkin that null series always arise when the boundary conditions involve the parameter in a particulur way. Indeed, system (5) belongs to the class of boundary problems discussed by Tamarkin from this point of view.†

* Journal de l'École Polytechnique, Cah. 18 (1820), pp. 417–489 (476–489).

† J. D. Tamarkin, Petrograd thesis (1917), xiv + 308 pp. (279–283). An English translation in abbreviated and generalized form has appeared in Math. Zeitschrift, vol. 27 (1927), pp. 1–54 (see p. 53). The earliest reference to the null series seems to occur in the Théorie de l'élasticité des corps solides de Clebsch, traduit par B. de Saint-Venant et Flamant, Paris, 1885, p. 500, in connection with Poisson's problem.

PRINCETON UNIVERSITY.

SUR LA FORME QUADRATIQUE FONDAMENTALE D'UN ESPACE DE RIEMANN EN COORDONNÉES NORMALES.*

PAR ROGER PIEDVACHE.

1. Soit $ds^2 = g_{ij}\,dy^i\,dy^j$ la forme quadratique fondamentale d'un espace de Riemann en coordonnées normales d'origine $y^i = 0$. H. Vermeil† a indiqué que ce ds^2 peut se mettre sous la forme:

$$ds^2 = d\sigma^2 + P_{ijhk}\,p^{ij}\,p^{hk}, \tag{1}$$

où $d\sigma^2$ est la forme euclidienne:

$$d\sigma^2 = g^0_{ij}\,dy^i\,dy^j$$

et où:

$$p^{ij} = y^j\,dy^i - y^i\,dy^j,$$

la fonction P_{ijhk} vérifiant les identités suivantes:

$$\begin{aligned} P_{ijhk} &= -P_{jihk} = -P_{ijkh} = P_{hkij}, \\ P_{ijhk} &+ P_{ihkj} + P_{ikjh} = 0, \\ P_{ijhk,l} &+ P_{ijkl,h} + P_{ijlh,k} = 0, \end{aligned} \tag{2}$$

où:

$$P_{ijhk,l} = \frac{\partial P_{ijhk}}{\partial y^l}.$$

O. Veblen‡ en a donné une démonstration, en utilisant les tenseurs normaux et les identités qu'ils vérifient. Dans une lettre au Prof. Veblen, mademoiselle E. Noether indique une démonstration basée sur les identités $g_{ij}\,y^i = g^0_{ij}\,y^i$, qui caractérisent les coordonnées normales, et sur la formule de Clebsch-Gordan.§ Je me propose de développer en détail la méthode de mademoiselle E. Noether. Auparavant je vais démontrer dans un cas particulier la formule de Clebsch-Gordan.

2. **Cas particulier de la formule de Clebsch-Gordan.** Soit $\varphi(xy)$ une fonction de deux séries de n variables

$$x:\ x^1\,x^2\cdots x^n, \qquad y:\ y^1\,y^2\cdots y^n.$$

* Received April, 1928.

† Vermeil, Math. Ann., vol. 79 (1918), p. 289. Dans Webers Anmerkungen in Riemanns Werken et dans le mémoire de Schur, auxquels renvoie Vermeil, on ne trouve pas la démonstration en question.

‡ Veblen, Invariants of quadratic differential forms. Chap. 6.

§ Clebsch, Theorie der binären Formen. Leipzig.

Posons:

$$D\varphi = x^i \frac{\partial \varphi}{\partial y^i}, \qquad \Delta \varphi = y^i \frac{\partial \varphi}{\partial x^i},$$

$$\Omega_{ij}\,\varphi = \frac{\partial^2 \varphi}{\partial x^i\,\partial y^j} - \frac{\partial^2 \varphi}{\partial x^j\,\partial y^i}.$$

On a:

$$\Omega_{ij}\,\varphi = -\,\Omega_{ji}\,\varphi$$

et on vérifie facilement que les opérations D et Ω_{ij} sont permutables:

(3) $$D\,\Omega_{ij}\,\varphi = \Omega_{ij}\,D\varphi.$$

Soit, pour simplifier:

$$(ij;\,hk) = \frac{\partial^4 \varphi}{\partial x^i\,\partial x^j\,\partial y^h\,\partial y^k},$$

on a:

$$\Omega_{hk}\,\Omega_{ij}\,\varphi = (ih;\,jk) + (jk;\,ih) - (ik;\,jh) - (jh;\,ik).$$

Posons encore:

(3′) $$\Omega_{ijhk}\,\varphi = \Omega_{hk}\,\Omega_{ij}\,\varphi = \Omega_{ij}\,\Omega_{hk}\,\varphi.$$

On vérifiera que:

$$\Omega_{ijhk} = -\,\Omega_{jihk} = -\,\Omega_{ijkh} = \Omega_{hkij},$$

(4) $$\Omega_{ijhk} + \Omega_{ihkj} + \Omega_{ikjh} = 0,$$

$$\Omega_{ijhk,l} + \Omega_{ijkl,h} + \Omega_{ijlh,k} = 0,$$

où:

$$\Omega_{ijhk,l} = \frac{\partial\,\Omega_{ijhk}}{\partial\,x^l}.$$

Nous allons maintenant faire sur φ les hypothèses suivantes:

(5) 1°) φ est homogène de degré p en x, de degré q en y,
2°) φ vérifie l'identité: $D\varphi \equiv 0$.

On a donc:

$$x^k \frac{\partial \varphi}{\partial y^k} = 0.$$

Différentions par rapport à x^i:

$$\frac{\partial \varphi}{\partial y^i} + x^j \frac{\partial^2 \varphi}{\partial x^i\,\partial y^j} = 0.$$

A cause de l'homogénéité par rapport à y, on a:

$$q\varphi\,(x;\,y) = y^i \frac{\partial \varphi}{\partial y^i},$$

donc:

$$q\varphi = -y^i x^j \frac{\partial^2 \varphi}{\partial x^i \partial y^j}.$$

Mais on a aussi à cause de la double homogénéité en x et en y:

$$pq\varphi = x^i y^j \frac{\partial^2 \varphi}{\partial x^i \partial y^j}.$$

On déduit de ces deux égalités:

$$q(p+1)\varphi = (x^i y^j - x^j y^i) \frac{\partial^2 \varphi}{\partial x^i \partial y^j}.$$

Posons encore:

$$a^{ij} = x^i y^j - x^j y^i.$$

Puisque $\Omega_{ij}\varphi = -\Omega_{ji}\varphi$, on a:

$$\varphi = \frac{1}{2q(p+1)} a^{ij} \Omega_{ij} \varphi \tag{6}$$

pour toute fonction $\varphi(x, y)$ satisfaisant aux conditions 5.

Telle est la formule fondamentale à laquelle je voulais arriver.

Je vais, par itération, en déduire une autre formule. Pour cela remarquons que la fonction $\Omega_{ij}\varphi$ est homogène de degré $(p-1)$ en x, $q-1$ en y; puisque $D\Omega_{ij}\varphi = \Omega_{ij} D\varphi$, on a aussi;

$$D\Omega_{ij}\varphi = 0;$$

cette fonction satisfait donc à (5), et la formule (6) lui est applicable:

$$\Omega_{ij}\varphi = \frac{1}{2(q-1)p} a^{hk} \Omega_{hk} \Omega_{ij} \varphi.$$

Portons cette expression dans (6); il vient:

$$\varphi = \frac{1}{2^2 q(q-1)p(p+1)} a^{ij} a^{hk} \Omega_{ijhk} \varphi \tag{6'}$$

à condition que $q \neq 0$ et $\neq 1$, $p \neq 0$, pour toute fonction φ satisfaisant aux conditions (5).

3. Application aux formes quadratiques. Revenons aux coordonnées normales:

$$ds^2 = g_{ij}\, dy^i\, dy^j = \varphi(y; dy).$$

Nous supposons les g_{ij} développables en série par rapport aux y.

Posons:

$$g_{ij,ab\cdots c} = \left(\frac{\partial^p g_{ij}}{\partial y^a\, \partial y^b \cdots \partial y^c}\right)_{y=0}$$

nous avons, puisque $\left(\frac{\partial g_{ij}}{\partial y^a}\right)_0 = 0$:

$$g_{ij} = g^0_{ij} + \sum_{p=2}^{\infty} \frac{1}{p!}\, g_{ij,ab\cdots c}\, y^a y^b \cdots y^c,$$

$a, b, \cdots, c$ étant au nombre de p. Soit

$$\varphi^p = \frac{1}{p!}\, g_{ijab\cdots c}\, y^a y^b y^c\, dy^i\, dy^j.$$

On a:

$$\varphi = \varphi^0 + \varphi^2 + \cdots + \varphi^p + \cdots.$$

φ^p étant homogène de degré p en y et $q = 2$ en dy, et $\varphi^1 = 0$. D'autre part, on a $g_{ij}\, y^i = g^0_{ij}\, y^i$* c'est à dire:

$$\frac{1}{p!}\, g_{ijab\cdots c}\, y^i\, y^a\, y^b \cdots y^c = 0,$$

ou, comme on le vérifie facilement:

$$D\varphi^p = y^k \frac{\partial \varphi^p}{\partial (dy^k)} = 0.$$

La fonction φ^p satisfait donc aux conditions (5), et nous pouvons lui appliquer la formule (6'), ce qui donne:

$$\varphi^p = \frac{1}{8p(p+1)}\, a^{ij}\, a^{hk}\, \Omega_{ijhk}\, \varphi^p \qquad (p \neq 0)$$

et ici $a^{ij} = y^i\, dy^j - y^j\, dy^i$. Mais la fonction $\Omega_{ijhk}\, \varphi^p$ est de degré 0 en dy, et φ^p est algébrique en dy; donc $\Omega_{ijhk}\, \varphi^p$ n'est pas fonction de dy.

Posons:

$$P_{ijhk} = \frac{3}{2} \sum_{p=2}^{\infty} \frac{1}{p(p+1)}\, \Omega_{ijhk}\, \varphi^p.$$

P_{ijhk} n'est fonction que des y, et on a:

$$\varphi = \varphi_0 + \frac{1}{12}\, P_{ijhk}\, p^{ij}\, p^{hk}$$

avec:

$$p^{ij} = -a^{ij} = y^j\, dy^i - y^i\, dy^j,$$

* O. Veblen, loc. cit., p. 96.

ou:

$$ds^2 = d\sigma^2 + \frac{1}{12} P_{ijhk}\, p^{ij}\, p^{hk} \tag{7}$$

la fonction P_{ijhk} vérifiant les identités (2), puisque Ω_{ijhk} vérifie les identités (4).

La formule (1) est donc démontrée.

4. Les tenseurs de O. Veblen. Calculons maintenant la fonction $\Omega_{ijhk}\,\varphi^p$. Puisque:

$$\varphi^p = \frac{1}{p!}\, g_{ijab\cdots c}\, y^a\, y^b \cdots y^c\, dy^i\, dy^j;$$

on a:

$$\frac{\partial^2 \varphi^p}{\partial(dy^h)\,\partial(dy^k)} = \frac{2}{p!}\, g_{hkab\cdots c}\, y^a\, y^b \cdots y^c$$

et:

$$\frac{\partial^4 \varphi^p}{\partial y^i\,\partial y^j\,\partial(dy^h)\,\partial(dy^k)} = \frac{2p(p-1)}{p!}\, g_{hkijab\cdots c}\, y^a\, y^b \cdots y^c;$$

les indices $a, b, \cdots c$ étant au nombre de $p-2$. Donc:

$$\Omega_{ijhk}\,\varphi^p = \frac{2p(p-1)}{p!}\,[g_{ihjkab\cdots c} + g_{jkihab\cdots c} - g_{ikjhab\cdots c} - g_{jhikab\cdots c}]\, y^a\, y^b \cdots y^c.$$

Posons:

$$p(p+1)\, K_{ijhkab\cdots c} = 3\,[g_{ihjkab\cdots c} + g_{jkihab\cdots c} - g_{ikjhab\cdots c} - g_{jhikab\cdots c}];$$

les indices $a, b, \cdots, c$ étant au nombre de $p-2$. En particulier, pour $p=2$, on a:

$$K_{ijhk} = \frac{1}{2}\,[g_{ihjk} + g_{jkih} - g_{ikjh} - g_{jhik}],$$

ou:

$$K_{ijhk} = (R_{ijhk})_{y=0},$$

R_{ijhk} étant le tenseur de Riemann-Christoffel. On a alors:

$$\frac{1}{p(p+1)}\, \Omega_{ijhk}\, \varphi^p = \frac{2p(p-1)}{3p!}\, K_{ijhkab\cdots c}\, y^a\, y^b \cdots y^c$$

et par suite:

$$P_{ijhk} = \sum_{p=2}^{\infty} \frac{p(p-1)}{p!}\, K_{ijhkab\cdots c}\, y^a\, y^b \cdots y^c;$$

les indices $a, b, \cdots, c$ étant au nombre de $p-2$. Changeons p en $p+2$; il vient:

$$P_{ijhk} = \sum_{p=0}^{\infty} \frac{(p+1)(p+2)}{(p+2)!}\, K_{ijhkab\cdots c}\, y^a\, y^b \cdots y^c.$$

Le nombre des indices $a, b, \cdots, c$ étant cette fois égal à p. On peut encore écrire:

$$P_{ijhk} = K_{ijhk} + \sum_{p=1}^{\infty} \frac{1}{p!} K_{ijhkab\cdots c}\, y^a\, y^b \cdots y^c,$$

qui montre que les quantités $K_{ijhkab\cdots c}$ sont les valeurs à l'origine des dérivées pièmes de P_{ijhk}. Ces quantités sont identiques, comme on le vérifiera facilement, aux quantités $K_{ijhkab\cdots c}$ introduites par M. O. Veblen,* et permettent, ainsi qu'il l'a montré, de définir une suite de tenseurs.

5. **Théorème réciproque.**† Je vais montrer, que, ainsi que l'indique Vermeil, la fonction P_{ijhk} est caractérisée par les identités (2). Pour cela je vais d'abord revenir au cas d'une fonction $\varphi(x, y)$ homogène en x et en y.

Supposons que l'on ait trouvé des fonctions $P_{ij}(xy) = -P_{ji}(xy)$ homogènes de degrés $p-1$ en x et $q-1$ en y, telles que:

$$\varphi(x, y) = \frac{1}{2} a^{ij} P_{ij}(x, y), \tag{8}$$

(la fonction φ étant homogène de degrés p et q) et que:

$$\begin{aligned} \frac{\partial P_{ij}}{\partial x^k} + \frac{\partial P_{jk}}{\partial x^i} + \frac{\partial P_{ki}}{\partial x^j} &= 0, \\ \frac{\partial P_{ij}}{\partial y^k} + \frac{\partial P_{jk}}{\partial y^i} + \frac{\partial P_{ki}}{\partial y^j} &= 0, \\ DP_{ij} &= 0. \end{aligned} \tag{9}$$

Je vais montrer, qu'on a nécessairement:

$$P_{ij} = \frac{1}{q(p+1)} \Omega_{ij}\, \varphi,$$

autrement dit, la fonction $P_{ij}(x, y)$ est entièrement déterminée par l'équation (8) et les conditions (9).

On tire de l'équation (8):

$$\Omega_{hk}\, \varphi = 2 P_{hk} + y^j \left(\frac{\partial P_{hj}}{\partial y^k} - \frac{\partial P_{kj}}{\partial y^h} \right) + x^i \left(\frac{\partial P_{ik}}{\partial x^h} - \frac{\partial P_{ih}}{\partial x^k} \right) + x^i\, y^j \left(\frac{\partial^2 P_{ij}}{\partial x^h\, \partial y^k} - \frac{\partial^2 P_{ij}}{\partial x^k\, \partial y^h} \right),$$

ou, en utilisant (9):

$$\Omega_{hk}\, \varphi = 2 P_{hk} + y^j \frac{\partial P_{hk}}{\partial y^j} + x^i \frac{\partial P_{hk}}{\partial x^i} + x^i\, y^j \left(\frac{\partial^2 P_{ij}}{\partial x^h\, \partial y^k} - \frac{\partial^2 P_{ij}}{\partial x^k\, \partial y^h} \right). \tag{10}$$

* O. Veblen, loc. cit., p. 100, formule (17.11).

† H. Vermeil, loc. cit.

D'autre part, on déduit de la seconde équation (9):

$$\frac{\partial^2 P_{ij}}{\partial x^h \partial y^k} + \frac{\partial^2 P_{jk}}{\partial x^h \partial y^i} + \frac{\partial^2 P_{ki}}{\partial x^h \partial y^j} = 0$$

et de la première équation (9):

$$\frac{\partial^2 P_{jk}}{\partial x^h \partial y^i} + \frac{\partial^2 P_{kh}}{\partial x^j \partial y^i} + \frac{\partial^2 P_{hj}}{\partial x^k \partial y^i} = 0,$$

d'où:

$$\frac{\partial^2 P_{ij}}{\partial x^h \partial y^k} - \frac{\partial^2 P_{kh}}{\partial x^j \partial y^i} = \frac{\partial^2 P_{hj}}{\partial x^k \partial y^i} - \frac{\partial^2 P_{ki}}{\partial x^h \partial y^j}.$$

Permutons i et j; h et k. Le second membre change de signe, d'où:

$$\frac{\partial^2 P_{ij}}{\partial x^h \partial y^k} - \frac{\partial^2 P_{kh}}{\partial x^j \partial y^i} + \frac{\partial^2 P_{ji}}{\partial x^k \partial y^h} - \frac{\partial^2 P_{kh}}{\partial x^i \partial y^j} = 0,$$

ou:

$$\frac{\partial^2 P_{ij}}{\partial x^h \partial y^k} - \frac{\partial^2 P_{ij}}{\partial x^k \partial y^h} = \frac{\partial^2 P_{hk}}{\partial x^i \partial y^j} - \frac{\partial^2 P_{hk}}{\partial x^j \partial y^i}.$$

(10) devient alors:

$$\Omega_{hk}\,\varphi = 2\,P_{hk} + y^j \frac{\partial P_{hk}}{\partial y^j} + x^i \frac{\partial P_{hk}}{\partial x^i} + x^i y^j \left(\frac{\partial^2 P_{hk}}{\partial x^i \partial y^j} - \frac{\partial^2 P_{hk}}{\partial x^j \partial y^i} \right),$$

ou en tenant compte de ce que $P_{hk}(x, y)$ *est homogène de degrés* $p-1$ *et* $q-1$:

$$\Omega_{hk}\,\varphi = (p\,q+1)\,P_{hk} - x^i y^j \frac{\partial^2 P_{hk}}{\partial x^j \partial y^i}.$$

Mais on a

$$x^i y^j \frac{\partial^2 P_{hk}}{\partial x^j \partial y^i} = y^j \frac{\partial}{\partial x^j} \left(x^i \frac{\partial P_{hk}}{\partial y^i} \right) - y^j \frac{\partial P_{hk}}{\partial y^j},$$

ou:

$$x^i y^j \frac{\partial^2 P_{hk}}{\partial x^j \partial y^i} = -(q-1)\,P_{hk}, \quad \text{puisque } x^i \frac{\partial P_{hk}}{\partial y^i} \equiv 0,$$

d'où:

$$\Omega_{hk}\,\varphi = P_{hk} \cdot (p\,q+1+q-1) = q(p+1)\,P_{hk},$$

ou:

$$P_{hk}(x, y) = \frac{1}{q(p+1)}\,\Omega_{hk}\,\varphi;$$

ce que je voulais démontrer.

Supposons de même que l'on ait une égalité de la forme:

$$\varphi(x, y) = \frac{1}{4}\,a^{ij}\,a^{hk}\,P_{ijhk}(x, y) \tag{11}$$

avec:

$$P_{ijhk} = P_{hkij}; \qquad P_{ijhk} + P_{jhik} + P_{hijk} = 0,$$

$$\frac{\partial P_{ijhk}}{\partial x^r} + \frac{\partial P_{jrhk}}{\partial x^i} + \frac{\partial P_{rihk}}{\partial x^j} = 0, \tag{12}$$

$$\frac{\partial P_{ijhk}}{\partial y^r} + \frac{\partial P_{jrhk}}{\partial y^i} + \frac{\partial P_{rihk}}{\partial y^j} = 0,$$

$$D P_{ijhk} = 0,$$

P_{ijhk} étant homogène de degrés $p-2$ en x et $q-2$ en y.

Posons:

$$Q_{ij} = \frac{1}{2} a^{hk} P_{ijhk}.$$

$Q_{ij}(x, y)$ est homogène de degrés $p-1$ en x, $q-1$ en y.

On a:

$$\varphi(x, y) = \frac{1}{2} a^{ij} Q_{ij}(x, y)$$

$$\frac{\partial Q_{ij}}{\partial x^r} = y^k P_{ijrk} + x^h y^k \frac{\partial P_{ijhk}}{\partial x^r}$$

d'où:

$$\frac{\partial Q_{ij}}{\partial x^r} + \frac{\partial Q_{jr}}{\partial x^i} + \frac{\partial Q_{ri}}{\partial x^j} = 0$$

à cause de (12), et de même:

$$\frac{\partial Q_{ij}}{\partial y^r} + \frac{\partial Q_{jr}}{\partial y^i} + \frac{\partial Q_{ri}}{\partial y^j} = 0$$

$$D Q_{ij} = 0.$$

La fonction $Q_{ij}(x, y)$ satisfait aux conditions (9). Elle est donc unique.

De l'égalité:

$$Q_{ij} = \frac{1}{2} a^{hk} P_{ijhk}(x, y);$$

on déduit de la même manière que la fonction $P_{ijhk}(x, y)$ est unique.

Supposons que $\varphi(x, y)$ soit un polynome du second degré en y; P_{ijhk} n'est pas fonction des y et les deux dernières conditions (12) sont satisfaites d'elles-mêmes, et on peut énoncer le théorème suivant:

Soit $\varphi(x, y)$ un polynome homogène du second degré par rapport aux y, dont les coéfficients sont des fonctions homogènes de degrés $p \geqq 2$ des variables x, satisfaisant a la condition:

$$D\varphi = 0.$$

Il existe un seul système de fonctions $P_{ijhk}(x)$ telles que:

$$\varphi(x, y) = \frac{1}{4} a^{ij} a^{hk} P_{ijhk}(x),$$

$$(2) \qquad \begin{gathered} P_{ijhk} = -P_{jihk} = -P_{ijkh} = P_{hkij}, \\ P_{ijhk} + P_{ihkj} + P_{ikjh} = 0, \\ \frac{\partial P_{ijhk}}{\partial x^r} + \frac{\partial P_{jrhk}}{\partial x^i} + \frac{\partial P_{rihk}}{\partial x^j} = 0. \end{gathered}$$

Supposons maintenant que l'on ait une forme quadratique:

$$ds^2 = g_{ij}(y)\, dy^i\, dy^j$$

mise sous la forme:

$$ds^2 = d\sigma^2 + P_{ijhk}\, p^{ij}\, p^{hk},$$

P_{ijhk} vérifiant les identités (2). Développons P_{ijhk} en série:

$$P_{ijhk} = P^0_{ijhk} + \cdots + P^q_{ijhk} + \cdots,$$

$P^q_{ijhk}(y)$ étant homogène de degré q en y. La fonction $P^q_{ijhk}(y)$ satisfait aussi à (2); elle est donc unique, d'après le théorème précédent; la fonction P_{ijhk} est donc elle-même unique, c. q. f. d.

PRINCETON,
Février, 1928.

SUR LE DÉVELOPPEMENT DE CLEBSCH-GORDAN.*

PAR ROGER PIEDVACHE.

1. Je veux donner ici une démonstration de la formule générale de Clebsch-Gordan† pour le cas d'une fonction homogène de deux séries de n variables x^i, y^i.

Dans un précédent article‡ j'avais donné la démonstration d'un cas particulier de cette formule; je renvoie à cet article pour la définition des notations employées ici.

2. **Formule fondamentale.** Soit $\varphi(x, y)$ la fonction en question homogène de degré p en x, de degré q en y.

Calculons $\Delta D\varphi(x, y)$; on a à cause de l'homogénéité

$$\Delta D\varphi = q\varphi(x, y) + x^i y^j \frac{\partial^2 \varphi}{\partial x^j \partial y^i}.$$

Mais pour la même raison:

$$pq\varphi(x, y) = x^i y^j \frac{\partial^2 \varphi}{\partial x^i \partial y^j},$$

d'où:

$$\Delta D\varphi = q(p+1)\varphi - x^i y^j \Omega_{ij}\varphi,$$

ou, en posant:

$$a^{ij} = x^i y^j - x^j y^i, \tag{1}$$

il vient:

$$q(p+1)\varphi = \Delta D\varphi + \frac{1}{2} a^{ij} \Omega_{ij} \varphi(x, y).$$

Posons:

$$\alpha_k = \frac{1}{p+k+1} \tag{2}$$

la formule s'écrit:

$$q\varphi = \alpha_0 \Delta D\varphi + \frac{1}{2} \alpha_0 a^{ij} \Omega_{ij} \varphi. \tag{3}$$

3. **Formule générale.** Nous allons, par itération, en déduire une nouvelle formule: Appliquons (3) à la fonction $D\varphi$ qui est homogène de degrés $(p+1)$ en x, $(q-1)$ en y. Il vient

$$(q-1) D\varphi = \alpha_1 \Delta D^2 \varphi + \frac{1}{2} \alpha_1 a^{ij} \Omega_{ij} D\varphi,$$

* Received April 1, 1928.

† Clebsch, Théorie der binären Formen. Leipzig.

‡ Sur la forme quadratique fondamentale d'un espace de Riemann en coordonnées normales, Ann. of Math., vol. 30 (1929), p. 272.

d'où on déduit:

$$(q-1)\,\Delta D\varphi = \alpha_1\,\Delta^2 D^2\varphi + \frac{1}{2}\,\alpha_1\,\Delta(a^{ij}\,\Omega_{ij}\,D\varphi),$$

où encore:

$$(q-1)\,\Delta D\varphi = \alpha_1\,\Delta^2 D^2\varphi + \frac{1}{2}\,\alpha_1\,a^{ij}\,\Delta D\,\Omega_{ij}\,\varphi;$$

cela suppose $q-1 \neq 0$ ou $q>1$.*

Appliquons de même (3) à $\Omega_{ij}\varphi$, de degrés $p-1$ et $q-1$:

$$(q-1)\,\Omega_{ij}\varphi = \alpha_{-1}\,\Delta D\,\Omega_{ij}\varphi + \frac{1}{2}\,\alpha_{-1}\,a^{hk}\,\Omega_{ijhk}\varphi,$$

multiplions maintenant les deux membres de (3) par $(q-1)$ et remplaçons dans le second membre $\Delta D\varphi$ et $\Omega_{ij}\varphi$ par leurs expressions; il vient:

$$q(q-1)\varphi = \alpha_0\,\alpha_1\,\Delta^2 D^2\varphi + \frac{1}{2}\,\alpha_0\,\alpha_1\,a^{ij}\,\Delta D\,\Omega_{ij}\varphi + \frac{1}{2}\,\alpha_0\,a^{ij}\left[\alpha_{-1}\,\Delta D\,\Omega_{ij}\varphi + \frac{1}{2}\,\alpha_{-1}\,a^{hk}\,\Omega_{ijhk}\varphi\right],$$

ou:

$$q(q-1)\varphi = \alpha_0\,\alpha_1\,\Delta^2 D^2\varphi + \frac{1}{2}\,\alpha_0\,(\alpha_1+\alpha_{-1})\,a^{ij}\,\Delta D\,\Omega_{ij}\varphi + \frac{1}{2^2}\,\alpha_{-1}\,\alpha_0\,a^{ij}\,a^{hk}\,\Omega_{ijhk}\varphi.$$

Mais on voit facilement que:

$$\alpha_{-1}+\alpha_1 = \frac{2(p+1)}{p(p+2)} = \frac{2\,\alpha_{-1}\cdot\alpha_1}{\alpha_0}$$

et on a:

$$q(q-1)\varphi = \alpha_0\,\alpha_1\,\Delta^2 D^2\varphi + \frac{2}{2}\,\frac{\alpha_{-1}\,\alpha_0\,\alpha_1}{\alpha_0}\,a^{ij}\,\Delta D\,\Omega_{ij}\varphi + \frac{1}{2^2}\,\alpha_{-1}\,\alpha_0\,a^{ij}\,a^{hk}\,\Omega_{ijhk}\varphi. \tag{4}$$

Le même procédé d'itération peut s'appliquer à la formule (4), et si $q>2$, on obtient une formule qui peut se mettre sous la forme:

$$\begin{aligned} q(q-1)(q-2)\varphi = {} & \frac{1}{1}\,\frac{\alpha_0\,\alpha_1\,\alpha_2\,\alpha_3}{\alpha_3}\,\Delta^3 D^3\varphi \\ & + \frac{3}{2}\,\frac{\alpha_{-1}\,\alpha_0\,\alpha_1\,\alpha_2}{\alpha_1}\,a^{ij}\,\Delta^2 D^2\,\Omega_{ij}\varphi \\ & + \frac{3}{2^2}\,\frac{\alpha_{-2}\,\alpha_{-1}\,\alpha_0\,\alpha_1}{\alpha_{-1}}\,a^{ij}\,a^{hk}\,\Delta D\,\Omega_{ijhk}\varphi \\ & + \frac{1}{2^3}\,\frac{\alpha_{-3}\,\alpha_{-2}\,\alpha_{-1}\,\alpha_0}{\alpha_{-3}}\,a^{ij}\,a^{hk}\,a^{rs}\,\Omega_{ijhkrs}\varphi. \end{aligned} \tag{5}$$

* Car on a: $D(a^{ij}\varphi) = a^{ij}\,D\varphi$; $\Delta(a^{ij}\varphi) = a^{ij}\Delta\varphi$.

On voit apparaître, sur cette formule, la loi de formation des coéfficients. Au bout de k itérations, on obtiendra une formule de la forme:

$$(6)\qquad q(q-1)\cdots(q-k+1)\varphi = A_0^k \Delta^k D^k \varphi + A_1^k a^{ij} \Delta^{k-1} D^{k-1} \Omega_{ij}\varphi + \cdots + A_h^k a^{ij}\cdots a^{rs} \Delta^{k-h} D^{k-h} \Omega_{ij\ldots rs}\varphi + \cdots.$$

Je dis que l'on a:

$$A_h^k = \frac{C_h^k}{2^h} \frac{\alpha_{-h}\cdot\alpha_{-h+1}\cdots\alpha_{-h+k}}{\alpha_{-2h+k}}, \qquad C_h^k = \frac{k!}{h!\,(k-h)!}.$$

Supposons en effet la formule vraie pour la valeur k:

$$q(q-1)\cdots(q-k+1)\varphi = \sum_{h=0}^{k} A_h^k \underbrace{a^{ij}\cdots a^{rs}}_{h} \Delta^{k-h} D^{k-h} \Omega_{ij\ldots rs}\varphi.$$

Appliquons-la à $D\varphi$, de degrés $p+1$ et $q-1$, sans oublier que les α sont des fonctions de p: $\alpha_k = 1/(p+k+1)$. Il vient:

$$q(q-1)\cdots(q-k)\,D\varphi = \sum_{h=0}^{k} \frac{1}{2^h} C_h^k \frac{\alpha_{-h+1}\cdots\alpha_{-h+k+1}}{\alpha_{-2h+k+1}} a^{ij}\cdots a^{rs} \Delta^{k-h} D^{k-h+1} \Omega_{ij\ldots rs}\varphi,$$

d'où:

$$q(q-1)\cdots(q-k)\Delta D\varphi = \sum_{h=0}^{k} \frac{C_h^k}{2^h} \frac{\alpha_{-h+1}\cdot\alpha_{-h+2}\cdots\alpha_{-h+k+1}}{\alpha_{-2h+k+1}} a^{ij}\cdots a^{rs} \Delta^{k-h+1} D^{k-h+1} \Omega_{ij\ldots rs}\varphi.$$

Appliquons de même à $\Omega_{ij}\varphi$; de degrés $p-1$ et $q-1$:

$$q(q-1)\cdots(q-k)\,\Omega_{ij}\varphi = \sum_{h=0}^{k} \frac{C_h^k}{2^h} \frac{\alpha_{-h-1}\cdots\alpha_{-h+k-1}}{\alpha_{-2h+k-1}} a^{rs}\cdots a^{pq} \Delta^{k-h} D^{k-h} \Omega_{rs\ldots pqij}\varphi.$$

Multiplions par a^{ij} et changeons h en $h-1$:

$$q(q-1)\cdots(q-k)\,a^{ij}\,\Omega_{ij}\varphi = \sum_{h=1}^{k+1} \frac{C_{h-1}^k}{2^{h-1}} \frac{\alpha_{-h}\cdots\alpha_{-h+k}}{\alpha_{-2h+k+1}} a^{ij}\cdots a^{rs} \Delta^{k-h+1} D^{k-h+1} \Omega_{ij\ldots rs}\varphi.$$

Portons ces expressions dans (3), il vient:

$$q(q-1)\cdots(q-k)\varphi = \alpha_0 \sum_{h=0}^{k} \frac{C_h^k}{2^h} \frac{\alpha_{-h+1}\cdots\alpha_{-h+k+1}}{\alpha_{-2h+k+1}} a^{ij}\cdots \Delta^{k-h+1} D^{k-h+1} \Omega_{ij\ldots}\varphi + \frac{1}{2}\alpha_0 \sum_{h=1}^{k+1} \frac{C_{h-1}^k}{2^{h-1}} \frac{\alpha_{-h}\cdots\alpha_{-h+k}}{\alpha_{-2h+k+1}} a^{ij}\cdots \Delta^{k-h+1} D^{k-h+1} \Omega_{ij\ldots}\varphi.$$

Ou, en groupant les termes de même nature:

$$q(q-1)\cdots(q-k)\varphi = \frac{\alpha_0\,\alpha_1\cdots\alpha_{k+1}}{\alpha_{k+1}}\Delta^{k+1}D^{k+1}\varphi + \sum_{h=1}^{k} B_h \cdot \underbrace{a^{ij}\cdots}_{h}\,\Delta^{k-h+1}D^{k-h+1}\,\underbrace{\Omega_{ij}\cdots}_{h}\varphi + \frac{C_k^k}{2^{k+1}}\,\frac{\alpha_{-k-1}\cdots\alpha_1}{\alpha_{-k-1}}\,\underbrace{a^{ij}\cdots}_{k+1}\,\underbrace{\Omega_{ij}\cdots}_{k+1}\varphi.$$

Avec:

$$B_h = \frac{1}{2^h}\,\frac{\alpha_0}{\alpha^{-2h+k+1}}\,[C_h^k\,\alpha_{-h+1}\cdots\alpha_{-h+k+1} + C_{h-1}^k\,\alpha_{-h}\cdots\alpha_{-h+k}],$$

mais on a:

$$C_{h-1}^k = C_h^k\cdot\frac{h}{k-h+1}$$

et:

$$B_h = \frac{1}{2^h}\,\frac{\alpha_0}{\alpha_{-2h+k+1}}\,C_h^k\,\alpha_{-h+1}\cdots\alpha_{-h+k}\left[\alpha_{-h+k+1} + \frac{h}{k-h+1}\,\alpha_{-h}\right],$$

ou:

$$B_h = \frac{1}{2^h}\,\frac{1}{\alpha_{-2h+k+1}}\,C_h^k\,\alpha_{-h+1}\cdots\alpha_{-h+k}\cdot\frac{k+1}{k-h+1},$$

ou:

$$B_h = \frac{1}{2^h}\,C_h^{k+1}\,\frac{\alpha_{-h}\cdot\alpha_{-h+1}\cdots\alpha_{-h+k+1}}{\alpha_{-2h+k+1}} = A_h^{k+1};$$

on a donc:

$$q(q-1)\cdots(q-k)\varphi = A_0^{k+1}\,\Delta^{k+1}D^{k+1}\varphi + \cdots + A_h^{k+1}\,a^{ij}\cdots a^{rs}\,\Delta^{k+1-h}\,D^{k+1-h}\,\Omega_{ij\ldots rs}\varphi,$$

qui se déduit bien de (6) en remplaçant k par $(k+1)$. Donc si la formule (6) est vraie pour la valeur k, elle l'est aussi pour la valeur $(k+1)$. Elle a été démontrée pour $k = 2$; elle est donc générale.

PRINCETON,
Mars, 1928.

RIQUIER'S EXISTENCE THEOREMS.*

By Joseph Miller Thomas.

1. **Introduction.** The purpose of this paper is to expound the principal results obtained by Riquier† in his researches concerning existence theorems for systems of partial differential equation. An exposition of this subject by Janet‡ is already available and has been of great assistance to me in understanding Riquier's work and in preparing this paper. Janet's article, however, contains much about sets of monomials and forms which is not essential in the discussion of the existence theorems. Only the few properties of monomials necessary to the development of the existence theorems are treated in the present paper. These properties follow almost immediately from the consideration of Taylor's series for a function of several variables.

2. **Sets of monomials.** Let the variables x_i be n in number and let them be arranged in the order $x_1, x_2, \cdots, x_n$. By a monomial in x is meant an expression $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ in which the exponents are positive integers or zero. The subscript on each x, being the same as that on the corresponding i, will usually be omitted, and the above monomial will be written $x^{i_1} x^{i_2} \cdots x^{i_n}$. It will also be referred to as the monomial $i_1\, i_2 \cdots i_n$, or still more briefly as the monomial i.

By the product of two monomials is meant their algebraic product.

A monomial i is a multiple of a second j if the algebraic quotient of i by j is a monomial, that is, if none of the differences

$$i_1 - j_1,\ i_2 - j_2,\ \cdots,\ i_n - j_n \tag{1}$$

is negative.

The totality of differential operators

$$\frac{\partial^{i_1+i_2+\cdots+i_n}}{\partial x^{i_1}\, \partial x^{i_2} \cdots \partial x^{i_n}} \tag{2}$$

is in one-to-one correspondence with the monomials $i_1\, i_2 \cdots i_n$ in such a way that to the product of two differential operators corresponds the

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† Les systèmes d'équations aux dérivées partielles, Paris, 1910. References to his original articles will be found in this work.

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product of the corresponding monomials. The above differential operator will frequently be abbreviated as D_i.

The degree of the monomial $i_1 i_2 \cdots i_n$ is $i_1 + i_2 + \cdots + i_n$.

A monomial i is said to be of higher or lower rank than a different monomial j according as the first non-zero difference of the sequence (1) is positive or negative. If i is of higher rank than j, and j is of higher rank than k, it readily follows from this definition that i is of higher rank than k.

All the monomials in n variables can be arranged in a *canonical order* as follows. Arrange them first according to increasing degree. Then arrange the monomials of the same degree according to increasing rank. This is, essentially, the canonical order of Tresse and Delassus. It is also fundamental in the ordering of Riquier. An essential feature of this ordering is: if y precedes y' and z is any monomial, yz precedes $y'z$.

THEOREM 1. *Every sequence of monomials in which no monomial is a multiple of any of those preceding it is necessarily finite.*

The theorem is almost obvious in the case of a single variable. The sequence contains at most $a+1$ monomials x^a, x^{a-1}, $\cdots$, x, 1.

In the case of n variables, let $x^{i_1} x^{i_2} \cdots x^{i_n}$ be the first monomial of the sequence. Denote by E_{jk} the set of all monomials of the sequence in which x_j has the exponent k. If for a given j k ranges from 0 to i_j, every monomial of the sequence will be contained in at least one set E_{jk}. The monomials in each E_{jk} can be thought of as arranged in the same relative order as in the original sequence. Except for a common factor x_j^k, each E_{jk} is a sequence of monomials in $n-1$ variables such that no one of them is a multiple of any of the preceding. If the theorem is assumed for $n-1$ variables, it follows that there is but a finite number of monomials in each E_{jk}. The number of sets E_{jk} being finite, the theorem also holds for n variables, and the induction is complete.

Maclaurin's series for a function of n variables is

$$u = \sum_0^\infty \frac{a_{i_1 i_2 \cdots i_n}}{i_1!\, i_2! \cdots i_n!} x^{i_1} x^{i_2} \cdots x^{i_n}, \tag{3}$$

where $a_{i_1 i_2 \cdots i_n}$ is $D_i u$ evaluated for $x = 0$. Any finite set of monomials E separates the terms of such an expansion into non-overlapping classes, as will now be shown. It is assumed that no monomial of E is a multiple of any other. Such multiples, if present, are to be omitted. Let a be the greatest exponent of x_1 in the set E. The terms in (3) can be grouped as follows,

$$u = f_0 + x_1 f_1 + x_1^2 f_2 + \cdots + x_1^{a-1} f_{a-1} + x_1^a f_a, \tag{4}$$

where f_a is a function of all the variables and the other f's are functions of all the variables except x_1. The above process separates the monomials of the expansion in $a+1$ sets such that no two sets have a monomial in common.

Denote by E_λ, $\lambda = 0, 1, \cdots, a$, the monomials of E in which the exponent of x_1 is λ; and by e_λ the set of monomials in $n-1$ variables obtained by making $x_1 = 1$ in the monomials of E_λ. Treat each f_λ in the same way that the original u was treated, arranging the expansion of f_λ according to powers of x_2 and using as the set of monomials $e_0+e_1+\cdots+e_\lambda$. Again, any monomial which is a multiple of another is to be omitted from the set.

Repeat for each variable in order. After a finite number of steps the expansion is separated in the form

$$u = \sum x^{i_1} x^{i_2} \cdots x^{i_n} f_{i_1 i_2 \cdots i_n}, \tag{5}$$

where the summation extends over a finite number of terms. The variables in each f in (5) are called the *multipliers* of the corresponding monomial $i_1 i_2 \cdots i_n$, and the infinite series $x^{i_1} x^{i_2} \cdots x^{i_n} f_{i_1 i_2 \cdots i_n}$ contains all the products of the monomial $i_1 i_2 \cdots i_n$ by monomials in its multipliers. Any variable not in an f is called a *non-multiplier* for the corresponding monomial.

Consider a monomial $i_1 i_2 \cdots i_n$ in (5). Perform the corresponding differentiation on (5) and subsequently place the non-multipliers of i equal to zero. Any term not containing i as a factor will disappear on differentiation. Any term containing in addition to i one of its non-multipliers will vanish on evaluation. Hence the only terms giving a result different from zero are the products of i by its multipliers, that is, the results of performing the differentiation $i_1 i_2 \cdots i_n$ and evaluation for the corresponding non-multipliers equal to zero on u and on $x^{i_1} x^{i_2} \cdots x^{i_n} f_{i_1 i_2 \cdots i_n}$ are identical. The assignment of a function f will therefore determine a certain derivative of u evaluated for the non-multipliers equal to zero. Conversely, the result of evaluating the derivative in question being given, by integrating it i_1 times with respect to x_1 between the limits 0 and x_1, i_2 times with respect to x_2 between 0 and x_2, $\cdots$, i_n times with respect to x_n between 0 and x_n, and finally dividing by the monomial $i_1 i_2 \cdots i_n$, the corresponding $f_{i_1 i_2 \cdots i_n}$ is obtained.

The monomials i which multiply the f's in (5) are of two distinct types: multiples of the given monomials E, and non-multiples. Consider any monomial j of the set E, and any term in the expansion of u which is a multiple of j. Such a term is contained in one and only one of the

sub-expansions (5). Suppose it is contained in $x^{i_1}x^{i_2}\cdots x^{i_n}f_{i_1 i_2 \cdots i_n}$. From equation (4), j can be a divisor only of terms in the expansions

$$x_1^{j_1}f_{j_1}, \quad x_1^{j_1+1}f_{j_1+1}, \quad \cdots, \quad x_1^{a}f_{a}.$$

Hence it follows that $i_1 \geqq j_1$. In the same way, $i_2 \geqq j_2, \cdots, i_n \geqq j_n$. Consequently, all the multiples of the monomials E are given by the expansions $x^{i_1}x^{i_2}\cdots x^{i_n}f_{i_1 i_2 \cdots i_n}$ of (5) in which $i_1 i_2 \cdots i_n$ represents a multiple of a monomial of E. Because of this, the corresponding monomials i are called the *complete set* arising from and containing E. The monomials i in (5) which are non-multiples of E form the *set complementary to E*.

In accordance with the above, the product of any monomial i of the complete set by one of its non-multipliers is the product of another monomial j of the complete set by certain of the multipliers of the latter. Actually, it can be shown that j is of higher rank than i. Let x_k be the non-multiplier of i. In order that $i \cdot x_k$ be a multiple of j,

$$i_1 \geqq j_1, \; i_2 \geqq j_2, \; \cdots, \; i_{k-1} \geqq j_{k-1}, \; i_k + 1 \geqq j_k.$$

If j_1 were less than i_1, x_1 would not be a multiplier for j because in the process of separation (4) x_1 is retained as a multiplier only for those monomials containing the highest power of x_1. The degree of j in x_1 could not in that case be raised to i_1 by multiplication by multipliers of j. Hence $i_1 = j_1$. In exactly the same way it follows that

$$i_2 = j_2, \; \cdots, \; i_{k-1} = j_{k-1}, \; i_k + 1 = j_k.$$

The monomial j is therefore of higher rank than i. This result can be stated in

THEOREM 2. *The product of any monomial of a complete set by one of its non-multipliers is equal to the product of a monomial of the set of higher rank by multipliers of the latter.*

As an example, let u be a function of the three independent variables x, y, z, and let the set of monomials be

$$y^2z^2, \quad xz, \quad xy^2, \quad x^2.$$

The first step in the separation into form (5) is

$$u = f_0(y, z) + xf_1(y, z) + x^2 f_2(x, y, z).$$

The second step is to treat each of the f's in the same way, the set of monomials being that indicated in each case below and the variable y now assuming the rôle played by x above:

$$\begin{aligned} &f_0(y, z), && y^2 z^2; \\ &f_1(y, z), && z, y^2; \\ &f_2(x, y, z), && 1. \end{aligned}$$

Hence

$$\begin{aligned} f_0(y, z) &= f_{00}(z) + y f_{01}(z) + y^2 f_{02}(y, z), \\ f_1(y, z) &= f_{10}(z) + y f_{11}(z) + y^2 f_{12}(y, z), \\ f_2(x, y, z) &= f_{20}(x, y, z). \end{aligned}$$

The final step is

$$\begin{aligned} f_{00}(z) &= f_{000}(z), && \\ f_{01}(z) &= f_{010}(z), && \\ f_{02}(y, z) &= f_{020}(y) + z f_{021}(y) + z^2 f_{022}(y, z), && z^2 \\ f_{10}(z) &= f_{100} + z f_{101}(z), && z \\ f_{11}(z) &= f_{110} + z f_{111}(z), && z \\ f_{12}(y, z) &= f_{120}(y, z), && 1 \\ f_{20}(x, y, z) &= f_{200}(x, y, z), && 1 \end{aligned}$$

the sets of monomials being indicated in each case to the right.

The separation (5) is therefore

$$\begin{aligned} u = {} & f_{000}(z) + y f_{010}(z) + y^2 f_{020}(y) + y^2 z f_{021}(y) \\ & + y^2 z^2 f_{022}(y, z) + x f_{100} + x z f_{101}(z) + x y f_{110} \\ & + x y z f_{111}(z) + x y^2 f_{120}(y, z) + x^2 f_{200}(x, y, z). \end{aligned}$$

The complete set of monomials, together with the corresponding multipliers written to the right, is

$$\begin{aligned} &y^2 z^2 && y, z \\ &x z && z \\ &x y z && z \\ &x y^2 && y, z \\ &x^2 && x, y, z. \end{aligned}$$

The complementary set is

$$1, y, y^2, y^2 z, x, xy.$$

The assignment of the f's corresponding to the complementary set is equivalent to giving the values of

$$\begin{aligned} &u, && \frac{\partial u}{\partial y} && \text{for } x = y = 0, \\ &\frac{\partial^2 u}{\partial y^2}, && \frac{\partial^2 u}{\partial y\, \partial z} && \text{for } x = z = 0, \\ &\frac{\partial u}{\partial x}, && \frac{\partial^2 u}{\partial x\, \partial y} && \text{for } x = y = z = 0. \end{aligned}$$

In order to deal simultaneously with several expansions $u_1, u_2, \cdots, u_r$ like (3), consider symbols ξ_i and ζ_α corresponding to x_i and u_α respectively. Then $\xi^{i_1}\xi^{i_2}\cdots\xi^{i_n}\zeta_\alpha$ corresponds quite naturally the derivative $D_i u_\alpha$. Put

$$\xi_i = \theta_1\theta_2^{c_{2i}}\cdots\theta_s^{c_{si}}, \qquad \zeta_\alpha = \theta_1^{\gamma_{1\alpha}}\theta_2^{\gamma_{2\alpha}}\cdots\theta_s^{\gamma_{s\alpha}}, \tag{6}$$

where the c's and γ's are positive integers or zero, and s is an arbitrary positive integer. In this way, the derivative $D_i u_\alpha$ is made to correspond to a monomial in the s variables θ. The exponent of θ_λ in this monomial is called the *λth cote* of the derivative; it is given by

$$\begin{aligned} &\gamma_{1\alpha} + i_1 \quad + i_2 \quad + \cdots + i_n, && \lambda = 1, \\ &\gamma_{\lambda\alpha} + c_{\lambda 1} i_1 + c_{\lambda 2} i_2 + \cdots + c_{\lambda n} i_n, && \lambda = 2, 3, \cdots, s. \end{aligned} \tag{7}$$

Likewise, $c_{\lambda i}$ and $\gamma_{\lambda\alpha}$ are called the λth cote of x_i and u_α, respectively, the first cote of each of the independent variables being unity.

The assignment of values to the cotes establishes an order among the derivatives if the following definition is made. *A derivative precedes or follows another according as the monomial in θ corresponding to the first is of lower or of higher rank than that corresponding to the second.*

Since two monomials in θ may be identical for different derivatives, the ordering will not be complete for all ways of assigning the cotes. A necessary condition for complete ordering is that the unknowns themselves be in definite order, i. e., that the s equations

$$\gamma_{\lambda\alpha} = \gamma_{\lambda\beta}, \qquad \lambda = 1, 2, \cdots, s \tag{8}$$

imply $\alpha = \beta$. If this condition is not satisfied, additional cotes can clearly be assigned so that the new set of cotes completely orders the unknowns. The order actually established by the first s cotes is not affected by doing this, for the $(s+1)$th and following cotes play a rôle in determining order only when the two series of s cotes are identical.

Suppose, therefore, that the unknowns are completely ordered by the first s cotes. Let n additional cotes be assigned according to the following table:

	x_1	x_2	$\cdots$	x_n	u_1	u_2	$\cdots$	u_r,
$s+1$	1	0	$\cdots$	0	0	0	$\cdots$	0
$s+2$	0	1	$\cdots$	0	0	0	$\cdots$	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$s+n$	0	0	$\cdots$	1	0	0	$\cdots$	0.

If two derivatives $D_i u_\alpha$ and $D_j u_\beta$ have exactly the same series of $s+n$ cotes, from the equality of the last n cotes it follows that $i_1 = j_1$,

$i_2 = j_2, \cdots, i_n = j_n$, and then from the equality of the first s that equations (8) hold. Hence the two derivatives are identical. It will therefore be assumed in what follows that the set of cotes completely orders the derivatives.

THEOREM 3. *It is possible to assign to the variables a set of values, all greater than unity, for which the ratio of any monomial in a given finite set to any monomial of lower rank in the set is greater than a given positive number.*

In the case of a single variable, a monomial of lower rank is one of lower degree. If the variable θ is chosen greater than unity plus the given positive number ε, the truth of the theorem becomes apparent.

In the case of s variables, let e be the set of monomials in $s-1$ variables obtained by making $\theta_1 = 1$ in the given set. The theorem, assumed true for $s-1$ variables, states the existence of values of θ_2, $\theta_3, \cdots, \theta_s$ such that the ratio of any monomial of the set e to a monomial of lower rank in e is greater than ε. For such values of $\theta_2, \theta_3, \cdots, \theta_s$, the assertion of the theorem is seen to be true as applied to two monomials of the original set in which the exponents of θ_1 are the same. It remains to choose θ_1 so that the ratio of any monomial $\theta_1^{\lambda} \cdot f$ to any monomial $\theta_1^{\mu} \cdot g$, where $\lambda > \mu$ and f, g do not involve θ_1, is greater than ε. The conditions are all of the form

$$\theta_1^{\lambda-\mu} > \frac{g\varepsilon}{f},$$

and a value of θ_1 greater than unity can clearly be found to satisfy all of them. The theorem is therefore proved in general.

As an application of the foregoing, let the set of monomials referred to in Theorem 3 be the monomials in θ corresponding by means of (6) to certain derivatives of the u's which have been ordered by the assignment of cotes. For the values of θ whose existence is stated by Theorem 3, formulas (6) define constants ξ_i and ζ_α such that $\xi_i > 1$ and the ratio

$$\frac{\xi^{i_1} \cdots \xi^{i_n} \zeta_\alpha}{\xi^{j_1} \cdots \xi^{j_n} \zeta_\beta}$$

is greater than the given positive number ε provided $D_i u_\alpha$ follows $D_j u_\beta$.

3. **Two algebraic theorems.** It is convenient to prove next two theorems which will be of use in the convergence proof.

THEOREM 4. *If the system of equations*

$$X_i = \sum_{j=1}^{N} A_{ij} X_j + B_i, \qquad i = 1, 2, \cdots, N, \tag{9}$$

in which the A's are positive or zero and the B's positive, is satisfied by a set of positive values of the X's, the system

$$x_i = \sum_{j=1}^{N} a_{ij}\, x_j + b_i, \qquad i = 1, 2, \cdots, N, \tag{10}$$

in which $|a_{ij}| \leqq A_{ij}$, $|b_i| \leqq B_i$, *has a unique solution*; *and that solution satisfies the relations* $|x_i| \leqq X_i$.

The above theorem will be proved by the method of successive approximations. The recurrence formulas

$$X_i^1 = B_i, \quad X_i^n = \sum A_{ij}\, X_j^{n-1} + B_i, \qquad n > 1, \tag{11}$$

for each value of i define an infinite sequence. From (11) it follows that

$$X_i^2 - X_i^1 = \sum A_{ij}\, B_j \geqq 0,$$

and that

$$X_i^n - X_i^{n-1} = \sum A_{ij}\,(X_j^{n-1} - X_j^{n-2}), \qquad n > 2. \tag{12}$$

An induction based on these equations shows that the differences $X_i^n - X_i^{n-1}$ are positive or zero for all values of $n > 1$, that is, the sequences X_i^n are monotone increasing. Since X_i satisfy (9), the relations

$$X_i - X_i^1 = \sum A_{ij}\, X_j, \qquad X_i - X_i^n = \sum A_{ij}\,(X_j - X_j^{n-1})$$

hold. From them it follows by induction that X_i^n never exceeds the corresponding X_i. Each sequence X_i^n therefore approaches a limit $Y_i \leqq X_i$. From (11) it is seen that the Y's satisfy (9).

For the sequences defined by

$$x_i^1 = b_i, \qquad x_i^n = \sum a_{ij}\, x_j^{n-1} + b_i, \qquad n > 1, \tag{13}$$

an induction based on the relations

$$\begin{aligned} |x_i^2 - x_i^1| = |\sum a_{ij}\, b_j| &\leqq \sum |a_{ij}| \cdot |b_j| \\ &\leqq \sum A_{ij}\, B_j = X_i^2 - X_i^1, \end{aligned}$$

$$\begin{aligned} |x_i^n - x_i^{n-1}| = |\sum a_{ij}\,(x_j^{n-1} - x_j^{n-2})| &\leqq \sum |a_{ij}| \cdot |x_j^{n-1} - x_j^{n-2}| \\ &\leqq \sum A_{ij} |x_j^{n-1} - x_j^{n-2}|, \end{aligned}$$

taken in conjunction with (12) shows the truth of the inequalities

$$|x_i^n - x_i^{n-1}| \leqq X_i^n - X_i^{n-1}, \qquad n > 1.$$

Hence each infinite series whose nth term is $x_i^n - x_i^{n-1}$ is convergent. Their sums to n terms, namely, x_i^n approach limits x_i which do not exceed in absolute value the limits approached by X_i^n, that is, $|x_i| \leqq Y_i \leqq X_i$.

From elementary algebra, system (10) has a unique solution if and only if its determinant is different from zero. If the determinant is zero, the b's must satisfy certain linear relations in order that the system have any solution at all. Any one of these relations can be violated by choosing b's which satisfy $|b_i| \leqq B_i$. As a solution has been proved to exist for any such set of b's, it is impossible for the determinant to be zero. The solution of (10) is therefore unique, and the theorem is proved.

THEOREM 5. *If none of the elements of a determinant are positive except those on the main diagonal and if the sum of the elements on each row is positive, the determinant is positive and the cofactors of all of its elements are positive or zero.*

The truth of the theorem is apparent for a determinant of the second order. Assuming its truth for one of order $N-1$, consider a determinant of order N whose elements are a_{ii} and $-a_{ij}$, $i \neq j$. By the conditions of the theorem

$$a_{ii} - \sum_{j=1}^{N} a_{ij} > 0, \qquad j \neq i;\ i = 1, 2, \cdots, N.$$

The expansion of the determinant in terms of its first row is

$$\Delta = a_{11} A_{11} - \sum_{j=2}^{N} a_{1j} A_{1j}.$$

A_{11} is a determinant of order $N-1$ satisfying the conditions of the theorem. It is therefore positive and the cofactors (in A_{11}) of its elements are positive or zero. Let Δ_j, $j = 2, 3, \cdots, N$, be the determinant obtained by adding to the elements of the $(j-1)$th column of A_{11} the corresponding elements of the first column of Δ. Then

$$\Delta_j = A_{11} + \Delta_j', \tag{14}$$

where Δ_j' is the result of replacing the elements of the $(j-1)$th column of A_{11} by the corresponding elements of the first column of Δ. The cofactors (in Δ_j') of the $(j-1)$th column of Δ_j', being cofactors in A_{11}, are positive or zero. If Δ_j' is expanded in terms of its $(j-1)$th column, all of whose elements are negative or zero, it is seen that Δ_j' is negative or zero. On the other hand,

$$\Delta_j' = (-1)^{j-2} \cdot (-1)^{j-1} A_{1j} = -A_{1j}.$$

Hence A_{1j} for $j = 2, 3, \cdots, N$ is positive or zero. Since Δ_j is a determinant of order $N-1$ for which the hypotheses of the theorem are

fulfilled, Δ_j is positive. From (14), $A_{11} > A_{1j}$ for $j = 2, 3, \cdots, N$. The determinant Δ is therefore greater than

$$A_{11}\left(a_{11} - \sum_{j=2}^{N} a_{1j}\right)$$

and is positive.

Interchanging the first and second rows of Δ changes the sign of Δ and of the cofactors of all its elements. A subsequent interchange of the first two columns restores the signs. It has been proved above that the cofactors of all the elements of the first row in the new determinant are positive or zero. The same is therefore true of the second row of Δ. The other rows can be treated similarly. The theorem is therefore established.

4. The principal existence theorem. Consider a system of partial differential equations which are all of the form

$$Du = F, \tag{15}$$

where D denotes a differential operator. Concerning the system the following assumptions are made:

I. *The unknown functions u, the independent variables x and the equations are finite in number.*

II. *Cotes have been assigned to the unknowns and to the independent variables, the first cote of each of the latter being unity.*

III. *The left members are all distinct.*

IV. *The function in each right member may have as its arguments the independent variables and those derivatives* whose first cote does not exceed the first cote of the corresponding left member.*

V. *The functions in the right members can be expanded in Taylor's series in the region under consideration.*

The derivatives which are left members of the given system and those which can be obtained from them by differentiation are called *principal* for the system. The other derivatives are *parametric*.

Without loss of generality it is assumed that the initial values of the independent variables are zero. Equations (15) are to be satisfied by series of the type (3). The totality of equations (15) whose left members are derivatives of a particular unknown, say u_1, furnishes a set of monomials by means of which the expansion of u_1 is separated in the form (5). The sum of the terms corresponding to the complementary monomials will be called the *initial determination of u_1*. A complete set of monomials and an initial determination are obtained in this way for each unknown. All the initial determinations being regarded as arbitrarily

* The term "derivatives" is to be interpreted as including the unknowns themselves.

given, it is desired to determine the remaining terms of the expansions so that they converge and satisfy the given system.

The series for u must satisfy all equations obtained by differentiating and combining (15). Let y', of first cote c', denote the left member of an equation (15), and let y, of first cote c, be any derivative in the corresponding right member. The equation obtained by differentiating with respect to x will have for left member $\partial y'/\partial x$ whose first cote is $c'+1$. The derivative y will cause $\partial y/\partial x$ and in general y, whose first cotes are respectively $c+1$ and c, to be present in the new right member. Since $c' \geqq c$, property IV is preserved under differentiation.

Add to the original system equations arising from it by differentiation so that the sets of monomials corresponding to left members are the complete sets given by (5). This step can be taken so as to preserve property III. The resulting system then has properties I–V, and in addition

VI. *The set of monomials corresponding to the left members involving any unknown is complete.*

Denote the minimum first cote of the left members of the system satisfying I–VI by a and the maximum by A. There is a unique equation whose left member is any given principal derivative of first cote a. Call the totality of the equations whose left members are principal derivatives of first cote a the system S_a. Any principal derivative of first cote $a+1$ either will be found among the left members of the given system or can be obtained from one of them by differentiation with respect to a multiplier: the expansions (5) will indicate just which one to differentiate. The set of equations whose left members are the set of all principal derivatives of first cote $a+1$ will be called the system S_{a+1}. In the same way are obtained the systems $S_{a+2}, \cdots, S_A, S_{A+1}$, whose left members are the principal derivatives of first cote equal to the subscript. All the systems $S_a, \cdots, S_{A+1}$ have properties I–V. In addition, S_{A+1} is linear in all derivatives of first cote $A+1$, because such derivatives can only enter as the result of differentiating a function of a derivative of first cote A.

It is assumed that

VII. *The systems $S_a, S_{a+1}, \cdots, S_A$, in which the independent variables have been replaced by zero and the parametric derivatives by their values for $x = 0$ given by the initial determinations, uniquely determine the initial values of the principal derivatives whose first cote does not exceed A. The system S_{A+1} has property* V *for these values.*

In equations (15) subject each unknown to a transformation

$$u = \bar{u} + \varphi + \sum \frac{a_{i_1 i_2 \cdots i_n}}{i_1!\, i_2! \cdots i_n!} x^{i_1} x^{i_2} \cdots x^{i_n}, \tag{16}$$

in which φ is the initial determination and the a's are the values for $x = 0$ of the principal derivatives whose first cote does not exceed A. Transposition of the known terms to the right gives a system in $\overline{u}$ with exactly the same types of initial determination as before, because the sets of monomials are the same. In order that u have the initial determination φ, it is necessary and sufficient that $\overline{u}$ have its initial determination zero. In the same way, the principal derivatives of $\overline{u}$ of first cote not exceeding A must vanish for $x = 0$ in order that the corresponding derivatives of the u's reduce to the a's. Without loss of generality, it can therefore be assumed that

VIII. *The initial determinations of the unknowns are zero and the principal derivatives of first cote not exceeding A reduce to zero for* $x = 0$.

From the first equation of (7) it is seen that the equality of the first cotes of two derivatives of the same unknown means that the derivatives are of the same order. The system S_{A+1}, which is of the form

$$D_i u_\alpha = \sum p_{i\alpha j\beta} D_j u_\beta + q_{i\alpha}, \tag{17}$$

therefore has the following properties:

(i) The derivatives of any unknown u_α which are left members are all of the same order g_α, and they arise by differentiating a complete set.

(ii) The right members are linear in the derivatives of order g_α of each u_α. The coefficients p of these derivatives and the terms independent of them q may involve the derivatives of each u_α up to those of order $g_\alpha - 1$.

A necessary condition upon any solution of (15) is that it satisfy S_{A+1}. If S_{A+1} is regarded as a system proposed for solution, its parametric derivatives include in addition to the parametric derivatives of (15) the principal derivatives of the original system whose first cote does not exceed A. In accordance with VIII it suffices to consider the case in which all the parametric derivatives of S_{A+1} vanish for $x = 0$. It will next be shown that in certain cases the above necessary condition on the developments determines them uniquely.

The existence and convergence of the developments will be proved by an extension of Cauchy's method of dominant functions.*

It is first convenient to define certain constants. The number of unknowns being r, choose r^2 positive constants $\lambda_{\alpha\beta}$ to satisfy

$$\sum_{\beta=1}^{r} \lambda_{\alpha\beta} < 1, \qquad \alpha = 1, 2, \cdots, r. \tag{18}$$

* For an introductory account of dominant functions, consult Goursat, Cours d'analyse mathématique, 3rd edition, vol. I, p. 473; vol. II, p. 354 et seq.

Let $H_{i\alpha\beta}$ be the number of derivatives of u_β which have their first cote equal to $A+1$ and enter into the right member of the equation (17) whose left member is $D_i u_\alpha$. Let the absolute values of the constant terms of the p's be denoted by $P_{i\alpha j\beta}$. Interpreting the totality of derivatives of first cote $A+1$ as the set to which reference is made in the remark following the proof of Theorem 3 and taking as ϵ the largest of the numbers

$$P_{i\alpha j\beta} H_{i\alpha\beta} / \lambda_{\alpha\beta},$$

determine the constants ξ, ζ. Then we are sure that

$$\frac{\lambda_{\alpha\beta} \xi^{i_1} \dots \xi^{i_n} \zeta_\alpha}{H_{i\alpha\beta} \xi^{j_1} \dots \xi^{j_n} \zeta_\beta} > P_{i\alpha j\beta}, \tag{19}$$

provided the derivative $D_j u_\beta$ precedes $D_i u_\alpha$. It is found expedient to assume

IX. *Inequalities* (19) *hold also for any $D_j u_\beta$ of first cote $A+1$ which is in a right member of S_{A+1} and does not precede the corresponding left member $D_i u_\alpha$.*

The satisfaction of this assumption can be kept in mind during the choice of λ, ξ, ζ. If the P's involved in it are all zero, the assumption is always satisfied by the ξ, ζ given by Theorem 3.

Let M be an upper bound to the absolute values of the terms of the expansions p, q in the region under consideration. Choose next r^2 numbers $\mu_{\alpha\beta}$, positive or zero, to satisfy the conditions

$$\frac{\lambda_{\alpha\beta} + \mu_{\alpha\beta}}{H_{i\alpha\beta}} \cdot \frac{\xi^{i_1} \dots \xi^{i_n} \zeta_\alpha}{\xi^{i_1} \dots \xi^{i_n} \zeta_\beta} \geqq M \tag{20}$$

for all combinations of the indices such that $D_j u_\beta$ is a derivative of first cote $A+1$ occurring in the right member of an equation (17) whose left member is $D_i u_\alpha$.

Finally, choose r positive numbers ν_α to satisfy all the relations

$$\nu_\alpha \xi^{i_1} \dots \xi^{i_n} \zeta_\alpha \geqq M, \tag{21}$$

where $D_i u_\alpha$ is any principal derivative of first cote $A+1$.

Consider the system $\mathfrak{S}_{A+1}$

$$D_i U_\alpha = \sum \frac{1}{H_{i\alpha\beta}} \left(\frac{\lambda_{\alpha\beta} + \mu_{\alpha\beta}}{1-\sigma} - \mu_{\alpha\beta} \right) \frac{\xi^{i_1} \dots \xi^{i_n} \zeta_\alpha}{\xi^{j_1} \dots \xi^{j_n} \zeta_\beta} D_j U_\beta + \frac{\nu_\alpha \xi^{i_1} \dots \xi^{i_n} \zeta_\alpha}{1-\sigma}, \tag{22}$$

which is identical with (17) except that the unknowns are here denoted by U's, and the p's and q's are different. In (22)

$$\sigma = \frac{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n + \sum DU}{\varrho},$$

the summation being extended over all derivatives of first cote not exceeding A, and ϱ being the smallest number of those defining the region of convergence of the coefficient p, q.

The function

$$\frac{M}{1 - \dfrac{\sum x + \sum DU}{\varrho}}$$

dominates the p's and q's. Since $\xi_k > 1$, the function $M/(1-\sigma)$ does likewise. It is easily seen that any given p or q is also dominated by

$$\frac{M_1}{1-\sigma}, \tag{23}$$

where $M_1 \geqq M$; and by

$$\frac{M_2 + M_3}{1-\sigma} - M_3 \tag{24}$$

where $M_2 + M_3 \geqq M$ and M_2 exceeds the absolute value of the constant term of the p or q in question.

By taking M_1 in (23) equal to the left member of each condition (21) in turn, each q is seen to be dominated by the corresponding term of (22). Corresponding to each p make in (24)

$$M_2 = \frac{\lambda_{\alpha\beta}}{H_{i\alpha\beta}} \cdot \frac{\xi^{i_1} \ldots \xi^{i_n} \zeta_\alpha}{\xi^{j_1} \ldots \xi^{j_n} \zeta_\beta}, \qquad M_3 = \frac{\mu_{\alpha\beta}}{H_{i\alpha\beta}} \cdot \frac{\xi^{i_1} \ldots \xi^{i_n} \zeta_\alpha}{\xi^{j_1} \ldots \xi^{j_n} \zeta_\beta}.$$

From (19) and (20), it follows that the coefficients in (22) dominate the p's in (17).

Equations (22) have a solution depending on the x's only in the combination

$$\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n. \tag{25}$$

To show this, put $U_\alpha = \zeta_\alpha U'_\alpha$ and let $U'_{\alpha i}$ be the result of differentiating U'_α $i = i_1 + i_2 + \cdots + i_n$ times with respect to the variable (25). Then

$$D_i U_\alpha = U'_{\alpha i} \cdot \xi^{i_1} \ldots \xi^{i_n} \zeta_\alpha.$$

When the U's are regarded as functions of the single variable (25), σ becomes a function σ' of that variable and of the derivatives of U'_α of orders $0, 1, \cdots, g_\alpha - 1$. Equations (22) reduce to

$$U'_{\alpha g_\alpha} = \sum_{\beta=1}^{r} \left(\frac{\lambda_{\alpha\beta} + \mu_{\alpha\beta}}{1 - \sigma'} - \mu_{\alpha\beta} \right) U'_{\beta g_\beta} + \frac{\nu_\alpha}{1 - \sigma'}. \tag{26}$$

For the values

$$x = 0, \quad U'_{\alpha\tau} = 0, \qquad \tau = 0, 1, \cdots, g_\alpha - 1, \tag{27}$$

the functional determinant of (26) has the elements $\delta_{\alpha\beta} - \lambda_{\alpha\beta}$, where $\alpha, \beta = 1, 2, \cdots, r$ and δ is the Kronecker delta. Because of (18), Theorem 5 shows that this determinant does not vanish. The implicit function theorem* states that (26) can be solved for $U'_{\alpha g_\alpha}$ in the neighborhood of (27). Cauchy's existence theorem† for systems of ordinary equations shows that (26) have a unique solution satisfying (27).

It will now be shown that in the expansions of the U''s, whose existence has just been proved, all the coefficients are positive except those which vanish in accordance with (27). Rewrite (26) in the form

$$U'_{\alpha g_\alpha} = \sigma' U'_{\alpha g_\alpha} + \sum_{\beta=1}^{r} (\lambda_{\alpha\beta} + \mu_{\alpha\beta}\, \sigma') U'_{\beta g_\beta} + \nu_\alpha. \tag{28}$$

The initial values of $U'_{\alpha g_\alpha}$ satisfy

$$(U'_{\alpha g_\alpha})_0 - \sum_{\beta=1}^{r} \lambda_{\alpha\beta} (U'_{\beta g_\beta})_0 = \nu_\alpha. \tag{29}$$

By Theorem 5, the determinant of the system is positive and the cofactors of all its elements are positive or zero. Since the ν's are also positive, the coefficients determined by (29) are positive. Assume that this is true of all the coefficients of order less than h_α. Differentiation of (28) the proper number of times gives for the derivatives of order h_α equations in which the coefficients of the derivatives of highest order of the various unknowns are the same as in (28). The remaining terms are polynomials in the derivatives of lower order and the independent variable with positive coefficients. The initial values of the derivatives of order h_α will therefore satisfy equations exactly like (29), whose right members, although not ν_α, are positive. Theorem 5 shows the values in question are positive and the induction is complete.

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* Cf. Goursat, l. c., vol. I, p. 95.

† Cf. Goursat, l. c., vol. II, pp. 358, 359.

System (22) is therefore satisfied by convergent expansions U_α whose coefficients of first cote $A+1$ or higher are all positive.

Return now to the consideration of the system S_{A+1}. By differentiation of its equations with respect to multipliers there arise systems $S_{A+\varrho}$, $\varrho = 2, 3, \cdots$, having the following properties:

(i) The left members of any $S_{A+\varrho}$ are all distinct and comprise all the principal derivatives of first cote $A+\varrho$.

(ii) The right members of any $S_{A+\varrho}$ are linear in the principal derivatives of first cote $A+\varrho$.

(iii) If the p and q of (17) be regarded as expanded, the right members of any $S_{A+\varrho}$ are, except for the terms involving the principal derivatives of first cote $A+\varrho$, combinations by addition and multiplication alone of the coefficients of p and q, and of derivatives of first cote less than $A+\varrho$.

From (22) systems $\mathfrak{S}_{A+\varrho}$ are obtained in exactly the same way.

The systems $S'_{A+\varrho}$, $\mathfrak{S}'_{A+\varrho}$ arising from $S_{A+\varrho}$, $\mathfrak{S}_{A+\varrho}$ by making x and the parametric derivatives zero are conditions on the coefficients of the expansions of the u's and U's. It has already been proved that $\mathfrak{S}'_{A+\varrho}$ are satisfied by positive values of the unknowns.

The equations $\mathfrak{S}'_{A+1}$ and S'_{A+1} are of the forms (9) and (10) respectively, X and x being the coefficients of U, u whose first cote is $A+1$. The A's of $\mathfrak{S}'_{A+1}$ are positive or zero and the B's are surely positive because they are the constant terms in the developments

$$\frac{\nu_\alpha \xi^{i_1} \cdots \xi^{i_n} \zeta_\alpha}{1-\sigma}. \tag{30}$$

These quantities, moreover, are not exceeded by the absolute values of the corresponding quantities in S'_{A+1}. Since $\mathfrak{S}'_{A+1}$ is satisfied by a set of positive values, theorem 4 is applicable and states that S'_{A+1} has a unique solution, and that the coefficients of the developments u so determined do not exceed in absolute value the corresponding coefficients of U.

Assume that the last statement is true applied to all the systems $S'_{A+\varrho}$, where $\varrho = 1, 2, \cdots, k-1$. Then from properties (i) and (ii) given above, S'_{A+k} is of the form (10), the coefficients of u whose first cote is $A+k$ being the unknowns x. From (iii) and the fact that $\mathfrak{S}_{A+1}$ dominates S_{A+1} it follows that the a's and b's of S'_{A+k} do not exceed in absolute value the corresponding A's and B's of $\mathfrak{S}'_{A+k}$. The A's in $\mathfrak{S}'_{A+k}$ are positive or zero, and the B's are positive because each one of them involves as an additive term one of the positive coefficients of the expansion (30). Theorem 4 shows that the equations S'_{A+k} uniquely determine the coefficients of the u's whose first cote is $A+k$, and that the absolute values of the coefficients so determined do not exceed the corresponding

coefficients of the U's. Thus the induction is complete. The developments u converge because the U's do.

Select a particular equation of the set $S_a, S_{a+1}, \cdots, S_{A+1}$, and think of it in the form

$$R \equiv Du - F = 0. \tag{31}$$

If the developments u are substituted in R, the result is a holomorphic function of the x's. Since substituting the expansions for the various derivatives in R and then evaluating for $x = 0$ is equivalent to substituting the corresponding coefficients of the expansions and then making $x = 0$, it follows from the way in which the coefficients were determined that $R(x)$ and all its derivatives with respect to multipliers of Du vanish for $x = 0$. Hence $R(x)$ depends only on the non-multipliers and vanishes when they are put equal to zero. The equation arising from (31) by putting the non-multipliers equal to zero is therefore satisfied identically in the multipliers by the results of evaluating the developments u and their derivatives for the non-multipliers equal to zero. A concise statement of this fact is: the developments u satisfy each equation of $S_a, S_{a+1}, \cdots, S_{A+1}$ on the variety defined by the vanishing of the corresponding non-multipliers.

The results established above can be recapitulated as follows:

THEOREM 6. *If assumptions* I–V, VII *and* IX *are realized for a system of partial differential equations, it is possible to construct unique convergent developments which have given initial determinations of the form prescribed by the left members of the system and which satisfy each equation of* $S_a, S_{a+1}, \cdots, S_{A+1}$ *on the variety defined by the vanishing of the corresponding non-multipliers. If a holomorphic solution satisfying the system and having the given initial determinations exists, it is given by these developments.*

As an example, take the system composed of the single equation

$$\frac{\partial u^2}{\partial x\, \partial y} = F(x, y, r, t), \tag{32}$$

in which

$$r = \frac{\partial^2 u}{\partial x^2}, \qquad t = \frac{\partial^2 u}{\partial y^2}.$$

Let the cotes be

x	y	u,
1	1	0,
1	2	0.

Assumptions I–IV are realized and F is supposed to satisfy V.

The separation (5) gives

$$u = f_{00}(y) + x f_{10}(x) + xy f_{11}(x, y).$$

The complete set of monomials consists of the single monomial xy whose multipliers are x, y. Assumption VI is therefore realized for the original system.

The complementary set consists of 1, which has the multiplier y, and of x, which has the multiplier x. The functions f_{00} and f_{10} in the initial determination can be specified by giving $\varphi(y)$ and $\psi(x)$, where

$$\begin{aligned} u &= \varphi(y) \quad \text{for } x = 0, \\ \frac{\partial u}{\partial x} &= \psi(x) \quad \text{for } y = 0. \end{aligned} \tag{33}$$

From the choice of the cotes, $a = A = 2$. Since x and y are multipliers for the left member (32), the system S_{A+1} consists of the two equations

$$\frac{\partial^3 u}{\partial x^2 \partial y} = F_r \frac{\partial^3 u}{\partial x^3} + F_t \frac{\partial^3 u}{\partial x \partial y^2} + F_x, \tag{34}$$

$$\frac{\partial^3 u}{\partial x \partial y^2} = F_r \frac{\partial^3 u}{\partial x^2 \partial y} + F_t \frac{\partial^3 u}{\partial y^3} + F_y, \tag{35}$$

the subscripts denoting partial differentiation. Because (32) is solved for the only principal derivative of cote 2 and because the right members of (34), (35) do not involve that derivative, assumption VII is surely satisfied.

Choose $\zeta = 1$. The set of monomials in ξ, η $(= \xi_2)$ corresponding to the derivatives of first cote $A+1$ is, arranged in order,

$$\xi^3, \quad \xi^2 \eta, \quad \xi \eta^2, \quad \eta^3.$$

As only the ratio of these is of interest, make $\theta_1 = 1$. The formulas (6) give as the monomials in θ_2

$$\theta_2^3, \quad \theta_2^4, \quad \theta_2^5, \quad \theta_2^6.$$

Here $H = 2$. Let P_1, P_2 be the values $|F_r(0, 0, \psi'[0], \varphi''[0])|$ and $|F_t(0, 0, \psi'[0], \varphi''[0])|$, respectively. There are four inequalities (19) which reduce to the two

$$\frac{2P_2}{\lambda} < \theta_2 < \frac{\lambda}{2P_1}. \tag{36}$$

If IX is to be satisfied, since $\lambda<1$,

$$P_1 P_2<\frac{1}{4}. \tag{37}$$

Conversely, if (37) is satisfied, let $P_1 \leqq P_2$, a condition readily realized. Then $2P_1<1$. It is possible to choose λ satisfying

$$4P_1 P_2<\lambda^2<1, \qquad 2P_1<\lambda,$$

and subsequently a θ_2 which is greater than 1 and satisfies (36). Hence IX is satisfied.

There being no non-multipliers, Theorem 6 in this case states that the series u satisfies (32). Therefore, *if* (37) *is satisfied by F, a unique solution of* (32) *and* (33) *exists for* φ *and* ψ *arbitrary.*

5. **Orthonomic systems.** In general, the application of Theorem 6 to a given system must be supplemented by an investigation to determine whether the developments actually satisfy the system when values different from zero are given to the non-multipliers. In the case of an orthonomic system, this question can be easily disposed of and the theorem of existence stated in more satisfactory form.

An *orthonomic system* is defined as one having the following properties:

(i) The unknown functions u, the independent variables x, and the equations are finite in number.

(ii) Cotes have been assigned to the unknowns and to the independent variables, the first cote of each of the latter being unity.

(iii) The equations are solved for certain of the derivatives of the unknowns, the left members being distinct and the right members not involving any derivative which is a left member.

(iv) All the derivatives in any right member precede the corresponding left member.

(v) The right members considered as functions of all their arguments are developable in Taylor's series in the region under consideration.

An orthonomic system clearly satisfies assumptions I–V.

Let y' be the left member of an equation in an orthonomic system and let y be a derivative in the corresponding right member. Then y' follows y. If the equation is differentiated, the presence of y in the original will cause, in general, y and $\partial y/\partial x$ to appear in the new right member. Since the new left member $\partial y'/\partial x$ follows both of these, (iv) is preserved under differentiation. Consider the complete sets given by the separations (5). If any derivative corresponding to a monomial of the complete sets is wanting among the left members of the system, form by differentiation a single

equation having it as left member. The original system augmented by the equations so obtained will have its left members distinct. Any derivative which is in both a left member and a right member can be removed from the latter. For consider the totality of such derivatives and choose the one of them which precedes all the rest. The equation of which this derivative is the left member has in its right member no derivative which is also a left member: if such were not the case, property (iv) would be violated. By means of that equation, the left member in question can be eliminated from all the right members in which it occurs. Property (iv) is not disturbed in the process. A repetition of the above shows that the original orthonomic system can be replaced by an equivalent orthonomic system for which assumption VI is satisfied.

The reasoning just employed also shows that the systems $S_a, S_{a+1}, \cdots, S_{A+1}$ formed for an orthonomic system are equivalent to orthonomic systems and can be thought of as orthonomic themselves. The systems $S_a, S_{a+1}, \cdots, S_A$ being in solved form, it follows that assumption VII is satisfied.

Since any derivative in a right member of an orthonomic system precedes the corresponding left member, assumption IX is vacuously satisfied.

Theorem 6 shows the existence of developments u which satisfy any orthonomic system on the varieties obtained by equating the non-multipliers to zero. It remains to obtain the conditions that the developments satisfy the system for any values of the non-multipliers.

Suppose that each equation of the orthonomic system under consideration, that is, one satisfying VI, is written in the form (31). The Du will still be referred to as the left member. Let each R be considered to have the same cotes and the same multipliers as the corresponding left member, and let two R's with derivatives of the same u for left members have the same relative rank as those derivatives. Select an equation $R_1 = 0$ which has a non-multiplier x_i. By Theorem 2 the derivative of the left member of R_1 which is contained in $\partial R_1/\partial x_i$ is equal to the first term of DR_2, where R_2 is of higher rank than R_1, and D denotes a derivative with respect to multipliers of R_2. Hence the difference

$$\frac{\partial R_1}{\partial x_i} - DR_2 \tag{38}$$

involves, in addition to the independent variables, only those derivatives of the unknowns which precede the first term of $\partial R_1/\partial x_i$.

Corresponding to each equation of the orthonomic system written in the form (31) there is an identity,

$$Du \equiv R + F. \tag{39}$$

The derivatives of the u's which enter in F are parametric. If the totality of relations (39) is differentiated with respect to multipliers, an infinite sequence J of identities is obtained whose left members are the principal derivatives. Each right member is a derivative of an R plus a certain polynomial in the principal derivatives which precede the principal derivative in the left member, the coefficients of the polynomial involving the independent variables and the parametric derivatives. These identities will give for any principal derivative an expression of the form

$$Du \equiv K(DR)+L(Du, x), \tag{40}$$

where (i) the right member is free from principal derivatives of the u's; (ii) the derivatives of the R's in K all precede the left member with the exception of one which has exactly the same series of cotes as the left member; (iii) K vanishes when the R's are made zero; (iv) L involves the independent variables and the parametric derivatives of the u's alone. To prove this for a given principal derivative, consider the finite set composed of all the identities J whose left members do not follow the derivative. Of the principal derivatives which occur in right members, select the one which precedes all the others. The equation having it as left member is of the form (40) and can be used to eliminate it from the other equations. A repetition of the process gives the desired result.

In the difference (38) substitute for the principal derivatives their expressions from identities (40). Since the principal derivatives enter in (38) rationally and integrally, there results an identity of the form

$$\frac{\partial R_1}{\partial x_i} - DR_2 \equiv K(DR)+L(Du, x), \tag{41}$$

where K and L have the same properties as in (40), and in addition the derivatives of the R's which enter in K all precede $\partial R_1/\partial x_i$.

By placing the R's equal to zero the relation

$$L(Du, x) = 0 \tag{42}$$

is obtained as a necessary condition that the given system be satisfied by the developments u. Since (42) involves only the independent variables and the parametric derivatives, which are subjected only to conditions of inequality, (42) must be identically satisfied.

The totality of conditions (42) *formed by making all possible choices of* R_1 *and* x_i *are called the integrability conditions of the system from which they arose.* If the integrability conditions of the system are satisfied

identically in the independent variables and the parametric derivatives, the system is called *passive*.

Suppose the given system passive. From identities (41) we have

$$\frac{\partial R_1}{\partial x_i} = DR_2 + K(DR). \tag{43}$$

Regard (43) as equations to determine the functions of x which the R's become when the developments u are substituted in them. Theorem 6 applied to the given orthonomic system states additional conditions on the R's, namely, that they are zero when the corresponding non-multipliers are made zero.

By assigning to the R of highest rank for each u an additional cote equal to unity, to the next highest an additional cote 2, and so on, and to the variables an additional cote zero, each DR_2 in (43) is made to precede the corresponding $\frac{\partial R_1}{\partial x_i}$. System (43) is therefore orthonomic and Theorem 6 is applicable. The initial determinations defined by the left members of (43) are the values assumed by the R's for the corresponding non-multipliers equal to zero. Theorem 6 states that there exists at most a unique set of R's satisfying (43) and reducing to zero initially. Since $R \equiv 0$ satisfies both equations and initial conditions, it follows that the R's are identically zero.

THEOREM 7. *Every passive orthonomic system is completely integrable, that is, it is satisfied by a unique set of holomorphic functions which have arbitrarily given initial determinations of the type prescribed by the left members of the system.*

6. **Examples of orthonomic systems.** A special orthonomic system is the canonical form of Kowalevsky

$$\frac{\partial^{k_\alpha} u_\alpha}{\partial x_1^{k_\alpha}} = F_\alpha, \qquad \alpha = 1, 2, \cdots, r, \tag{44}$$

in which any right member may involve any of the derivatives of u_α of order not exceeding k_α except the derivative which is a left member. It can be assumed that the unknowns are numbered so that $k_1, k_2, \cdots, k_r$ form a monotone decreasing sequence. Assign first and second cotes in the following way:

x_1	x_2	$\cdots$	x_n	u_1	u_2	$\cdots$	u_r
1	1	$\cdots$	1	0	k_1-k_2	$\cdots$	k_1-k_r
1	0	$\cdots$	0	0	k_1-k_2	$\cdots$	k_1-k_r.

Let $D_l u_\beta$ be a derivative in a right member of (44). Then

$$l_1+l_2+\cdots+l_n \leqq k_\beta, \tag{45}$$

and when the equality holds, $l_1 < k_\beta$. The cotes of the left members and of $D_l u_\beta$ are respectively

$$k_1, \quad l_1+l_2+\cdots+l_n+k_1-k_\beta,$$
$$k_1, \quad l_1+k_1-k_\beta.$$

The differences (1) are

$$k_\beta-(l_1+l_2+\cdots+l_n), \qquad k_\beta-l_1.$$

If the inequality holds in (45), the first of these is positive. If the equality holds in (45), the first is zero and the second positive. Property (iv) of orthonomic systems is possessed by the present system, and if the functions F_α are subjected to the restriction (v), (44) is orthonomic.

Consider a second example. If in the system

$$\frac{\partial u}{\partial x} = f\left(\frac{\partial w}{\partial x}\right), \quad \frac{\partial v}{\partial x} = g\left(\frac{\partial u}{\partial y}, \frac{\partial w}{\partial x}\right), \quad \frac{\partial v}{\partial y} = h\left(\frac{\partial u}{\partial y}, \frac{\partial w}{\partial x}\right), \quad \frac{\partial w}{\partial y} = k\left(\frac{\partial w}{\partial x}\right), \tag{46}$$

the cotes are assigned as follows

x	y	u	v	w
1	1	1	1	0
0	1	0	2	2,

the derivatives are ordered

$$\frac{\partial w}{\partial x}, \ \frac{\partial w}{\partial y}, \ \frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial y},$$

so that the system is orthonomic. If further the single integrability condition

$$\frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}$$

is satisfied identically in $\partial u/\partial y$, $\partial w/\partial x$, $\partial^2 u/\partial y^2$, $\partial^2 w/\partial x^2$, the system is passive and possesses by Theorem 7 a unique solution with the initial determinations

$$u = \varphi(y), \qquad x = 0,$$
$$v = \text{const.}, \qquad x = y = 0,$$
$$w = \psi(x), \qquad y = 0.$$

The existence of this solution is not stated by the Cauchy-Kowalevsky theorem.

7. **Reduction to orthonomic form.** Consider now a system composed of equations involving the derivatives of certain unknowns; the functions, derivatives, independent variables and equations being finite in number and the functions which equated to zero give the equations being analytic.

The derivatives can be ordered by assigning the following cotes

x_1	x_2	$\cdots$	x_{n-1}	x_n	u_1	u_2	$\cdots$	u_r
1	1	$\cdots$	1	1	0	0	$\cdots$	0
0	0	$\cdots$	0	0	1	2	$\cdots$	r
1	0	$\cdots$	0	0	0	0	$\cdots$	0
0	1	$\cdots$	0	0	0	0	$\cdots$	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0	0	$\cdots$	1	0	0	0	$\cdots$	0
0	0	$\cdots$	0	1	0	0	$\cdots$	0

The above ordering amounts to arranging the derivatives first according to increasing order, then according to increasing index of the unknown, and finally according to increasing rank.

Select an equation which involves the last derivative, according to the above ordering, which occurs in the given system, and solve it for that derivative. Substitute in the other equations. Repeat for the next to the last derivative, and so on. After a finite number of operations, either a relation among the independent variables is deduced, in which case the system is incompatible, or an orthonomic system Σ_1 equivalent to the original is obtained. Form the integrability conditions of Σ_1, and treat them as the original system was treated. Since the integrability conditions involve only the parametric derivatives, none of the left members of the additional equations obtained can be obtained from the left members of Σ_1 by differentiation. Add these equations to the set Σ_1 to form Σ_2. Repetition of this process gives a sequence

$$\Sigma_1, \ \Sigma_2, \ \cdots, \tag{47}$$

such that each set of monomials in Σ_i contains the corresponding set in Σ_{i-1} and in addition monomials which are not multiples of any of those in the corresponding set in Σ_{i-1}. By Theorem 1, the number of monomials in Σ_i, which increases with i, cannot exceed a finite limit. Hence the sequence (47) is finite, so that after a finite number of operations a state is reached when the unsolved equations involve no derivatives. If they involve the x's, the system is incompatible; if they are identically satisfied, the final system Σ is completely integrable.

THEOREM 8. *Any compatible system of partial differential equations can be reduced by a finite number of operations to passive, orthonomic form.*

The reduction to passive, orthonomic form, if applied to a system solved for certain derivatives, will in general change the type of initial determinations. For example, the equation

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2},$$

whose initial determination is $u = \varphi(y)$ for $x = 0$, when put in the orthonomic form is

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}$$

with initial determination $u = \varphi(x)$, $\partial u/\partial y = \psi(x)$ for $y = 0$.

The orthonomic form to which reduction is made above is the canonical form of Tresse* who proved Theorem 8. Riquier's contribution surpasses that of Tresse because it includes Theorem 7 which states the actual existence of solutions when no incompatibility is encountered in the reduction to canonical form.

8. **Discussion of the results.** The principal results of Riquier's work can be summarized as follows:

(i) His orthonomic form is more general than the canonical form of Cauchy and Kowalevsky. For any system in orthonomic form Theorem 7 asserts the existence of solutions with arbitrary initial determinations of the type prescribed by the left members.

(ii) The process employed in the analysis leading to Theorem 8 will give a definite answer to the question; does a given analytic system have a holomorphic solution, no initial determination being prescribed?

(iii) By use of Theorem 6 the existence of solutions with arbitrary initial determinations can be proved for systems more general than the orthonomic, but as a rule the system must be restricted in order that the theorem apply. The restrictions, necessary for the successful application of Theorem 6, may not be necessary for the existence of solutions. Hence the ultimate has not been reached in this direction.

Assumptions II and IV, in addition, are not necessary for the existence of solutions with arbitrary initial determinations. In the case of a single unknown, for example, they prevent the right member from being of higher order than the left. But the equation

$$\frac{\partial u}{\partial x} = y\frac{\partial^2 u}{\partial x\,\partial y} - y^2\frac{\partial^2 u}{\partial y^2} \tag{48}$$

* Acta Mathematica, vol. 18 (1894), p. 9.

has the solution

$$u = \varphi(ye^x)$$

which reduces to an arbitrary function $\varphi(y)$ for $x = 0$. Theorem 6 does not apply since the first cote of the derivatives in the right member is necessarily greater by unity than the first cote of the left member.

Equation (48) can, of course, be replaced by the system

$$\frac{\partial u}{\partial y} = v, \qquad \frac{\partial v}{\partial x} = y\frac{\partial v}{\partial y} + \frac{1}{y}\frac{\partial u}{\partial x}. \tag{49}$$

With the following assignment of cotes

x	y	u	v
1	1	0	0
1	0	0	1,

Theorem 7 predicts a solution with initial determination

$$u = f(x), \qquad y = 1,$$
$$\frac{\partial u}{\partial y} = g(y), \qquad x = 0.$$

By choosing $f(0) = \varphi(1)$ and $g(y) = \varphi'(y)$, a solution is obtained which reduces to $\varphi(y)$ for $x = 0$, but the existence of this solution is not predicted by Riquier's theorems if the initial value of y is zero.

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THE SEQUENCE OF FUNCTIONS WHICH DEFINE A DEFINITE INTEGRAL CONTAINING A PARAMETER.

BY R. L. JEFFERY.

In a paper* on Definite Integrals Containing a Parameter the convergence of

$$\text{(A)} \qquad \sum_{i=1}^{n} f(\xi_i, y)(x_i - x_{i-1})$$

to its limit $F(y)$ was considered with respect to its uniformity in y. In that paper certain conditions are given which are sufficient to insure the uniform convergence of the sum (A) with respect to y, and that independent of the choice of ξ_i on each interval. An example at the end of the paper shows that the convergence need not be uniform in y independent of the choice of ξ_i even though $f(x, y)$ be bounded and continuous in each variable separately. This example points the way to a further sufficient condition, and in the case of a certain class of functions, to a condition which is both necessary and sufficient. The first part of the paper consists in the development of these conditions. In the second part of the paper we proceed to a discussion of the continuity of a function defined by a Riemann-Stieltjes integral, and the uniform convergence of a Riemann-Stieltjes sum similar to (A).

Part I.

The sufficient condition referred to above is stated in the following:

THEOREM 1. *Let the function $f(x, y)$ defined on the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$ be bounded, continuous in y for each x, and a Riemann integrable function of x for each y. Designate by K the two dimensional set of points at which the saltus of $f(x, y) \geqq k$. A sufficient condition for the uniform convergence of the Riemann sum then is, that for each $k > 0$ the part of the corresponding set K which lies on any line $y = y_0$ has zero linear content† on this line, $c \leqq y_0 \leqq d$.*

By virtue of the Heine-Borel theorem, it is clearly sufficient to establish about each value $y = y_0$ an interval throughout which the convergence is uniform.

* Annals of Math., vol. 26 (1925), pp. 173–180. We find that the results of that paper are easily extended to include functions of any number of variables.

† By content we mean the lower limit of the sum of the lengths of a finite number of intervals enclosing the set. See Hobson, *Real Variable*, vol. I, second ed., § 117.

Let $\varepsilon > 0$ be given. Consider the set of points S_{y_0} on the line $y = y_0$ at which the saltus of $f(x, y) \geqq \varepsilon$. Under the conditions of our theorem this set has linear content zero and can, therefore, be enclosed in a finite set of open intervals A where mA, the total length of A, satisfies

$$mA < \varepsilon. \tag{1}$$

Let B be the finite set of closed intervals* complementary to A on the line $y = y_0$. At each point of B the saltus of $f(x, y)$ is less than ε. It then follows that about each point (x, y_0) of B as center a square whose sides are parallel to the coördinate axes can be constructed such that when (x', y') is any point of the square,

$$|f(x, y_0) - f(x', y')| < \varepsilon.$$

Given (x, y_0) any point of the B-intervals, let $2\eta_x$ be the side of the square of which this point is center. We then have

$$|f(x, y_0) - f(x', y')| < \varepsilon \tag{2}$$

for any (x', y') which satisfies $|x - x'| < \eta_x$, $|y_0 - y'| < \eta_x$. Putting $y' = y_0$ in (2) we get

$$|f(x, y_0) - f(x', y_0)| < \varepsilon \tag{3}$$

for $|x - x'| < \eta_x$. Combining (2) and (3) we have

$$|f(x', y_0) - f(x', y')| < 2\varepsilon$$

for $|y_0 - y'| < \eta_x$, and for $|x - x'| < \eta_x$. We now have every point (x, y_0) of the B-intervals interior to an interval of length $2\eta_x$ such that for x on this interval of length $2\eta_x$

$$|f(x, y_0) - f(x, y)| < 2\varepsilon$$

provided $|y - y_0| < \eta_x$. From this infinite set of intervals the Heine-Borel theorem permits us to choose a finite set having the same interior points. Corresponding to this finite set there will be a finite set of values of η_x. If now we fix η smaller than the smallest of this finite set, we have

$$|f(x, y_0) - f(x, y)| < 2\varepsilon \tag{4}$$

for all values of x on the closed B-intervals, provided $|y_0 - y| < \eta$.

* A may, of course, be so chosen that no two intervals abut.

Let (x_{i-1}, x_i) be any sub-division of (a, b) and ξ_i any point on (x_{i-1}, x_i). Let (x_{j-1}, x_j) be the q intervals of this sub-division which are wholly on the B-intervals. It will then follow from (4) that

$$(5) \qquad \left| \sum_{j=1}^{q} f(\xi_j, y_0)(x_j - x_{j-1}) - \sum_{j=1}^{q} f(\xi_j, y)(x_j - x_{j-1}) \right| < 2\varepsilon(b-a)$$

provided $|y_0 - y| < \eta$. If (x_{k-1}, x_k) denote the remainder of the sub-division (x_{i-1}, x_i), and if L and l are the least upper bound and greatest lower bound respectively of $f(x, y)$, we see that

$$(6) \qquad \left| \sum_{k=1}^{n-q} f(\xi_k, y_0)(x_k - x_{k-1}) - \sum_{k=1}^{n-q} f(\xi_k, y)(x_k - x_{k-1}) \right| \leqq (mA + 2p\delta)(L-l),$$

where p is the number of intervals in the set A, and δ is the greatest length of any $x_k - x_{k-1}$. Thus if $x_i - x_{i-1} < \delta$ it follows from (1), (5), and (6) that

$$(7) \qquad \begin{aligned} &\left| \sum_{i=1}^{n} f(\xi_i, y_0)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(\xi_i, y)(x_i - x_{i-1}) \right| \\ &\qquad < (\varepsilon + 2p\delta)(L-l) + 2\varepsilon(b-a) \end{aligned}$$

provided $|y - y_0| < \eta$. By taking δ sufficiently small

$$\sum_{i=1}^{n} f(\xi_i, y_0)(x_i - x_{i-1})$$

can be made arbitrarily near to its limit $F(y_0) = \int_a^b f(x, y_0)\,dx$. Since $F(y)$ is continuous,* $F(y)$ is arbitrarily near to $F(y_0)$ for η sufficiently small and $|y - y_0| < \eta$. From these considerations, the fact that the choice of δ is independent of p, and inequality (7), it follows that for δ sufficiently small and for η sufficiently small

$$\left| F(y) - \sum_{i=1}^{n} f(\xi_i, y)(x_i - x_{i-1}) \right|$$

is arbitrarily small provided $x_i - x_{i-1} < \delta$, and $|y_0 - y| < \eta$. This establishes the uniformity of the convergence in question for a neighbourhood about y_0.

We show by an example that the conditions of Theorem 1 are not necessary. Let α be the set of open intervals on (0, 1) complementary to the non-dense closed set G lying on (0, 1), and having content λ, $0 < \lambda < 1$. On each interval α_i of α as base construct an isosceles triangle T_i with vertex on $y = 1$. This countable set of triangles T we

* W. H. Young, Monatsh. für Math. und Phys., vol. 2 (1910).

suppose arranged in a sequence as follows: T_1, T_1, T_2, T_1, T_2, T_3, T_1, T_2, T_3, T_4, $\cdots$. In this sequence each triangle of the set T occurs an infinite number of times. With points on the sides of the nth ($n = 1, 2, \cdots$) triangle of the sequence and at a distance $1/2 \cdot 3^{n-1}$ above the x-axis as centers, construct rectangles with sides parallel to the y-axis of length $1/3^n$, and each having a side of this triangle as a diagonal line. We have thus defined on each side of each triangle an infinite set of rectangles. Of the whole set of rectangles R defined we notice: no point is common to two rectangles; a line parallel to the x-axis cuts at most two rectangles; a line parallel to the y-axis cuts at most one rectangle.

At each point of $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ exterior to R let $f(x, y) = 0$. On each rectangle R_i of R construct a regular pyramid of unit height, and let the surface of this pyramid define $f(x, y)$ on R_i. The function $f(x, y)$ is bounded and continuous in each variable separately. Furthermore, it is discontinuous with saltus unity at the end-points of α, and hence at each point of G, since a point of G is either an end-point of α or a limit point of such points. Nevertheless, the Riemann sum converges uniformly to its limit $F(y)$. For $y = y_0 > 0$ $f(x, y)$ is a continuous function of the two variables, and hence the convergence is uniform in the neighbourhood of $y_0 > 0$. It remains to show that the convergence is uniform in an interval including $y = 0$. If w is the combined width of the two rectangles cut by the line $y = \bar{y}$, and if (x_{i-1}, x_i) is any sub-division of $(0, 1)$ with norm less than δ, then the sum $\sum f(\xi_i, \bar{y})(x_i - x_{i-1}) \leqq w + 4\delta$. But w approaches zero as $\bar{y}$ approaches zero. Hence this sum can be made arbitrarily small for all y sufficiently near $y = 0$ by taking δ sufficiently small. But $F(y)$ is continuous and $F(0) = 0$. Hence the convergence is uniform on $0 \leqq y \leqq 1$.

This example shows that even when $f(x, y)$ is continuous in each variable separately, the conditions of Theorem 1 are not necessary.

For an extensive class of functions the conditions of Theorem 1 are necessary as well as sufficient. Consider the function $f(x, y) = 2xy/(x^2+y^2)$ for x and y not both zero, and $f(0, 0) = 0$, for which the origin is a point of discontinuity. We notice that the upper limit of $f(x, y)$ at this point is approached along the line $x = y$, and the lower limit is approached along the line $y = 0$. We shall be concerned with functions for which a corresponding situation exists at all points of discontinuity.* To be exact let

* Functions of this type are known to exist such that in every region for which the function is defined the discontinuities form an everywhere dense non-enumerable set. W. H. Young and G. C. Young, Quart. Journ. Math., vol. 41, p. 87. In this case the path of approach for the upper limit in each case is an arc of a parabola, while that for the lower limit is a straight line.

L_0 and l_0 be the upper and lower limit respectively of $f(x, y)$ at a point of discontinuity x_0, y_0. Let each of these limits be approached in at least one of the following three ways:

A. Over a set (x_1, y_0), (x_2, y_0), $\cdots$, on the line $y = y_0$ with (x_0, y_0) as a limit point.

B. Along a curve which can be represented by $x = \varphi(y)$ for all values of y on some interval $0 < y - y_0 < \delta$, and where $\lim_{y \to y_0} \varphi(y) = x_0$.

C. Along a curve which can be represented by $x = \varphi(y)$ for all values of y on some interval $0 < y_0 - y < \delta_0$, where $\lim_{y \to y_0} \varphi(y) = x_0$.

For the class of functions $f(x, y)$ which have all their discontinuities of the kind just described we have the following

THEOREM 2. *If $f(x, y)$ belongs to the foregoing class, is bounded, continuous in y for each x, and Riemann integrable in x for each y, then it is necessary and sufficient for the uniform convergence of the Riemann sum that the set K at which the saltus of $f(x, y) \geqq k$ have zero content on each line $y = y_0$.*

That these conditions are sufficient is proved in Theorem 1. There is left then the problem of showing that they are necessary. This we proceed to do in the following manner.

Suppose first that for some $k > 0$ the part of the corresponding set K on a line $y = y_0$ has linear content $I > 0$. It is then possible to show that if ε is any positive number $< Ik/8$, and (x_{i-1}, x_i) is any sub-division whatever of (a, b), there exists in every neighborhood of y_0 a value y_ε of y and on each (x_{i-1}, x_i) values ξ_i and ξ_i^ε of x for which

$$(1) \qquad \left| \sum_{i=1}^{n} f(\xi_i, y_0)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(\xi_i^\varepsilon, y_\varepsilon)(x_i - x_{i-1}) \right| \geqq \frac{Ik}{8} - \varepsilon.$$

Consider for the moment that (1) holds. The sum $\sum_{i=1}^{n} f(\xi_i, y_0)(x_i - x_{i-1})$ can be made as near to $F(y_0)$ as we please by making $x_i - x_{i-1}$ sufficiently small. The function $F(y)$ being continuous at y_0, can, by keeping y near enough to y_0, be made to differ from $F(y_0)$ by as little as we please. The number ε of inequality (1) being arbitrary, it would then follow that if δ is a positive number, there exist sub-divisions (x_{i-1}, x_i) where $x_i - x_{i-1} < \delta$, values ξ_i^ε of x on (x_{i-1}, x_i), and in every neighborhood of y_0 a value y_ε of y for which

$$\left| F(y_\varepsilon) - \sum_{i=1}^{n} f(\xi_i^\varepsilon, y_\varepsilon)(x_i - x_{i-1}) \right|$$

is less than $Ik/8$ by as little as we please.

It remains to establish inequality (1). Since $f(x, y)$ is continuous in y for each x it follows that at each point (ξ, y_0) of K on the line $y = y_0$

$$l_\xi \leqq f(\xi, y_0) \leqq L_\xi$$

where L_ξ and l_ξ are the upper and lower limits of $f(x, y)$ at the point (ξ, y_0).

Then $L_\xi - l_\xi \geqq k$, and either

$$L_\xi - f(\xi, y_0) \geqq \frac{k}{2}$$

or

$$f(\xi, y_0) - l_\xi \geqq \frac{k}{2}.$$

It follows then that every point of K on $y = y_0$ belongs to at least one of the following six sets.

(a) $f(x, y)$ approaches L_ξ by way of (A)
(b) $f(x, y)$ approaches L_ξ by way of (B)
(c) $f(x, y)$ approaches L_ξ by way of (C)

} and $L_\xi - f(\xi, y_0) \geqq k/2$,

(d) $f(x, y)$ approaches l_ξ by way of (A)
(e) $f(x, y)$ approaches l_ξ by way of (B)
(f) $f(x, y)$ approaches l_ξ by way of (C)

} and $f(\xi, y_0) - l_\xi \geqq k/2$.

The content of the set (a) is zero, and the content of the set (d) is zero, since the saltus of the function $f(x, y_0)$ is greater than or equal to $k/2$ at these points, and $f(x, y_0)$ is Riemann integrable. But the sum of the contents of the six sets is greater than or equal to I; hence the content of at least one of the four sets (b), (c), (e), (f) is greater than or equal to $I/4$. To be definite suppose that the content of (b) is greater or equal to $I/4$. The treatment for each of the other possible cases would follow similar lines.

Let (x_{i-1}, x_i) be any finite sub-division of (a, b). Designate by (x_{j-1}, x_j) the p intervals of this sub-division which contain points of the set (b) on their interiors. It then follows that the total length of (x_{j-1}, x_j) is greater than or equal to $I/4$. Interior to each interval of this set fix a point (ξ_j, y_0) of the set (b). For each of these points the way of approach of $f(x, y)$ to L_ξ is (B). If then we fix a positive number $\eta < k/2$ and such that $\eta I/4 + \eta < \varepsilon$, we can find a number $\delta_1 > 0$ such that for each y on $0 < y - y_0 < \delta_1$, there exists a value $\bar{\xi}_j$ of x interior to (x_{j-1}, x_j) and given by $x = \varphi(y)$ for which

$$f(\bar{\xi}_j, y) - f(\xi_j, y_0) > \frac{k}{2} - \eta.$$

The intervals of the finite sub-division of (a, b) which are not contained in (x_{j-1}, x_j) we designate by (x_{k-1}, x_k), and in each of these intervals fix an arbitrary point ξ_k. Since there are only a finite number of such points, and since $f(x, y)$ is continuous in y for each x, it follows that there exists $\delta_2 > 0$ such that

$$|f(\xi_k, y) - f(\xi_k, y_0)| < \frac{\eta}{b-a}$$

when $|y - y_0| < \delta_2$. Then for δ less than the smaller of the two numbers δ_1 and δ_2, and for any fixed y on $0 < |y - y_0| < \delta$, consider the sum

$$S = \sum_{j=1}^{p} \{f(\bar{\xi}_j, y) - f(\xi_j, y_0)\}(x_j - x_{j-1}) + \sum_{k=1}^{n-p} \{f(\xi_k, y) - f(\xi_k, y_0)\}(x_k - x_{k-1}).$$

The first partial sum on the right is greater than or equal to $(k/2 - \eta)I/4$, while the second lies between $-\eta$ and η. Hence

$$S \geq \left(\frac{k}{2} - \eta\right)\frac{I}{4} - \eta > \frac{kI}{8} - \varepsilon.$$

An obvious change of notation now turns $|S|$ into the left hand side of inequality (1).

Part II.

The continuity of a function defined by a Riemann-Stieltjes integral, and the uniform convergence of a Riemann-Stieltjes sum.

Let $\varphi(x)$ be non-decreasing on (a, b), and $f(x, y)$ be bounded on $a \leqq x \leqq b$, $c \leqq y \leqq d$. We assume that for each y the Riemann-Stieltjes integral of $f(x, y)$ with respect to $\varphi(x)$ exists. A function $F(y)$ is thus defined by the equation

$$F(y) = \int_b^a f(x, y)\, d\varphi(x).$$

The function $F(y)$ is then by definition

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_i, y)\{\varphi(x_i) - \varphi(x_{i-1})\}.$$

We designate this sum by $\sigma(n, \xi_i, y)$ and state without proof the following easily obtained result:

THEOREM 3. *If $f(x, y)$ is continuous in y uniformly with respect to x, then $F(y)$ is continuous and the convergence of $\sigma(n, \xi_i, y)$ is uniform in y.*

If we assume merely that $f(x, y)$ is continuous in y for each x, then the facts as to the continuity of $F(y)$ and the uniformity of the convergence of $\sigma(n, \xi_i, y)$ are not so evident. As a preliminary step in the study of these questions we associate* with each point u on $\alpha = \varphi(a) \leqq u \leqq \beta = \varphi(b)$ a unique point x_u on (a, b) as follows:

Let $x_1, x_2, \cdots$ be the countable set of discontinuities of φ. If u is a point of a closed interval $\{\varphi(x_i - 0), \varphi(x_i + 0)\}$ then $x_u = x_i$. If u does not belong to one of these closed intervals, and the equation $u = \varphi(x)$ is satisfied by a single value of x, then this value of x shall be x_u. If u does not belong to one of these intervals and the equation $u = \varphi(x)$ is satisfied for more than one value of x, then $\varphi(x)$ is constant and equal to u throughout an interval. In this case x_u shall be the lower end-point of this interval.

If $g(x)$ is any function defined on (a, b), then on (α, β) we set $\psi(u) = g(x_u)$, and state

LEMMA 1. *If the* $RS\int_a^b g(x)\, d\varphi(x)$ *exists, then the* $R\int_\alpha^\beta \psi(u)\, du$ *exists, and the two are equal.*

The argument for the existence of $\int_\alpha^\beta \psi(u)\, du$ is essentially that of Hobson.† By making use of the sums $\overline{S}_n$ and $\underline{S}_n$ defined by Hobson‡ taken over a sub-division of (a, b) at points $x_1, x_2, \cdots, x_{n-1}$ which are points of continuity of $\varphi(x)$, it is not difficult to show that

$$\underline{S}_n \leqq \int_\alpha^\beta \psi(u)\, du \leqq \overline{S}_n.$$

But $\int_a^b g(x)\, d\varphi(x)$ is the common limit approached by $\underline{S}_n$ and $\overline{S}_n$ as n becomes infinite. Therefore

$$\int_\alpha^\beta \psi(u)\, du = \int_a^b g(x)\, d\varphi(x).$$

We can now prove

THEOREM 4. *If* $f(x, y)$ *is continuous in* y *for each* x, *then* $F(y)$ *is continuous.*

The function $\psi(u, y) = f(x_u, y)$ is evidently continuous in y for each u, and by Lemma 1,

* The correspondence here given is similar to that used by Lebesgue, Comptes Rendus, vol. 150, p. 86. It is also similar to that employed by Hobson, *Real Variables*, 2nd ed., vol. I, § 378, the difference being that Hobson does not define x_u if for any u the equation $u = \varphi(x)$ is satisfied by more than one value of x.

† Theory of Functions of a Real Variable, 2nd ed., vol. I, § 376.

‡ Loc. cit., vol. II, p. 774, corrections to p. 508 of vol. I.

$$F(y) = \int_\alpha^\beta \psi(u, y)\, du.$$

But the right hand side is continuous* in y. Therefore $F(y)$ is continuous.

With the function $f(x, y)$ satisfying the conditions of Theorem 4, and $\varphi(x)$ non-decreasing, it does not follow that $\sigma(n, \xi_i, y)$ converges uniformly to $F(y)$. To see this we need only take $\varphi(x) = x$, and $f(x, y)$ any function for which the Riemann sum does not converge uniformly.† Nevertheless, it is possible to obtain uniform convergence for a special class of monotone functions provided ξ_i is properly chosen on (x_{i-1}, x_i). This we proceed to discuss.

The function $\varphi(x)$, being non-decreasing on the closed interval (a, b), can be expressed as the sum of two non-decreasing functions,‡ $\varphi(x) = \nu(x) + \varkappa(x)$, where on any closed interval the discontinuities of ν are the discontinuities of φ, and the variation of ν is the variation due to its discontinuities on this interval, and where $\varkappa$ is continuous. Making use of this we prove:

THEOREM 5. *If the component function $\varkappa(x)$ is absolutely continuous, then for a given $\epsilon > 0$ there exists $\delta > 0$ such that when $x_i - x_{i-1} < \delta$ and ξ_i is properly chosen on (x_{i-1}, x_i),*

$$\left| F(y) - \sum_{i=1}^{n} f(\xi_i, y)\, \{\varphi(x_i) - \varphi(x_{i-1})\} \right| < \epsilon,$$

independent of y.

We first show that about any value y_0 of y on (c, d) there is an interval for which the theorem holds. If y_0 is fixed, then

$$|F(y_0) - \sigma(n, \xi_i, y_0)|$$

can be made as small as we please by taking $x_i - x_{i-1}$ sufficiently small. Again, since $F(y)$ is continuous,

$$|F(y_0) - F(y)|$$

can be made arbitrarily small by restricting y to a sufficiently small interval about y_0. The theorem will then hold for an interval about y_0 if we can show that for any sub-division (x_{i-1}, x_i) whatever, there corresponds to each positive number $\epsilon > 0$ a positive number $\eta > 0$ such that when $|y_0 - y| < \eta$

$$|\sigma(n, \xi_i, y_0) - \sigma(n, \xi_i, y)| < \epsilon \tag{1}$$

provided ξ_i is properly chosen on (x_{i-1}, x_i).

* Young, loc. cit.

† For such a function see Annals of Math., loc. cit., p. 179.

‡ Caratheodory, Vorlesungen über reelle Funktionen, p. 153.

The existence of such a number η will now be established. Designate the countable set of discontinuities of φ on (a, b) by $d_1, d_2, \cdots$, and the variation of φ due to a discontinuity d_i by s_i.* Since φ is bounded, we can fix $i = k$ so that

$$\sum_{i=k+1}^{\infty} s_i < \varepsilon .$$

Since $\varkappa(x)$ is absolutely continuous, there exists $\delta > 0$ such that over any set of intervals on (a, b) with total length $< \delta$ the variation† of φ is less than $\varepsilon + s$ where s is the total variation of φ due to its discontinuities on these intervals. With k and δ so determined, we introduce a set of points $G_{\varepsilon\eta}$ as follows:

If ε and η are two positive numbers, a point x belongs to $G_{\varepsilon\eta}$ provided

$$|f(x, y') - f(x, y'')| < \varepsilon$$

for every pair of values y', y'' on (c, d) which satisfy $|y' - y''| < \eta$. The fact that $G_{\varepsilon\eta}$ is measurable, and that for a given ε $mG_{\varepsilon\eta}$ can be made as near to $(b - a)$ as we please by taking η sufficiently small has been established in the paper to which we have referred.‡ We note further, since $f(x, y)$ is continuous in y for each x, that it is possible to take η sufficiently small to include in $G_{\varepsilon\eta}$ any finite number of preassigned x-points on (a, b). We now fix η so that $G_{\varepsilon\eta}$ contains the points $d_1, d_2, \cdots, d_k$, and so that $mG_{\varepsilon\eta}$ differs from $(b - a)$ by not more than the above determined positive number δ.

Now let (x_{i-1}, x_i) be any finite sub-division of (a, b), (x_{j-1}, x_j) the p intervals of this sub-division containing points of $G_{\varepsilon\eta}$, either as interior points or boundry points, and (x_{k-1}, x_k) the remaining intervals of the set (x_{i-1}, x_i). On each of the intervals (x_{j-1}, x_j) fix a point ξ_j of $G_{\varepsilon\beta}$, and on each interval of (x_{k-1}, x_k) fix ξ_k any point whatever. Now consider the sum

$$S = |\sigma(p, \xi_j, y_0) - \sigma(p, \xi_j, y) + \sigma(n - p, \xi_k, y_0) - \sigma(n - p, \xi_k, y)|$$

for $|y_0 - y| < \eta$. Since ξ_j belongs to $G_{\varepsilon\eta}$

$$|\sigma(p, \xi_j, y_0) - \sigma(p, \xi_j, y)| < \varepsilon \{\varphi(b) - \varphi(a)\}.$$

Furthermore, since $mG_{\varepsilon\eta} > b - a - \delta$, and since (x_{k-1}, x_k) contains no points of this set, it follows that the total lenght of (x_{k-1}, x_k) is less

* $s_i = \varphi(d_i + 0) - \varphi(d_i - 0)$.

† The variation of φ over an interval (a', b') is $\varphi(b') - \varphi(a')$.

‡ Annals of Math., loc. cit., p. 174.

than δ, and at the same time these intervals contain none of the points $d_1, \cdots, d_k$, either on their interior or as boundry points. It then follows that

$$|\sigma(n-p, \xi_k, y_0) - \sigma(n-p, \xi_k, y)| < \left(\varepsilon + \sum_{i=k+1}^{\infty} s_i\right)(L-l) < 2\varepsilon(L-l),$$

where L and l are the least upper bound and greatest lower bound of $f(x, y)$. Consequently

$$S < \varepsilon\{\varphi(b) - \varphi(a)\} + 2\varepsilon(L-l)$$

where ε is arbitrary.

We have now shown that corresponding to each y_0 on (c, d) there is a $\delta_0 > 0$ such that (1) holds for some interval about y_0 when $x_i - x_{i-1} < \delta_0$, provided ξ_i is a point of $G_{\varepsilon\eta}$ if there is a point of this set on (x_{i-1}, x_i). Otherwise ξ_i is any point of the interval. The fact that $G_{\varepsilon\eta}$ is independent of y permits us to use the Heine-Borel theorem in obtaining (1) for the whole interval $c \leqq y \leqq d$.

The absolute continuity of the function $\varkappa$ is sufficient to give Theorem 5 for all functions $f(x, y)$ satisfying the conditions of that theorem. One naturally wonders if the absolute continuity of $\varkappa$ is necessary. This question we have so far been unable to settle.

ON THE STRUCTURE OF NORMAL DIVISION ALGEBRAS.†

By A. Adrian Albert.

1. **Introduction.** The chief outstanding problem in the theory of linear associative algebras is the determination of all normal division algebras. In the present paper we shall add to the present meager knowledge of the structure of such algebras by proving certain basic theorems on the structure of algebras satisfying mild postulates. We shall also, in the exposition of this theory, prove certain theorems tacitly assumed by writers in the theory of division algebras, and define new terms to be used here and in subsequent papers.

We shall consider normal linear associative division algebras over a non-modular field F and shall speak of them, simply, as normal division algebras. The general non-modular reference field F shall be understood unless the contrary is explicitly stated.

2. **General Theory.**‡ Theorem 1*. Let A be an associative algebra over a non-modular field F; let $\xi_1, \cdots, \xi_n$ be independent variables in F. The quantity x of A with coördinates $\xi_1, \cdots, \xi_n$ is the root of a uniquely defined rank equation $R(\omega) = 0$; the coefficient of the highest power ω^r in this equation is 1, its other coefficients are polynomials in $\xi_1, \cdots, \xi_n$ with coefficients in F. Also x is not the root of an equation of degree less than r, all of whose coefficients are such polynomials. We call r the *rank* of A and $R(\omega)$ its *rank function*.

As a consequence of this we have

Theorem 2. Every element of an algebra A of rank r satisfies an equation of degree r and with leading coefficient one.

The proof of the above is obvious.

We also have

Theorem 3*. Every algebra A of rank r contains an element satisfying an equation of degree r with leading coefficient one and other coefficients in F and which is irreducible in F.

Definition. An element x shall be said to be of *grade* t if the degree of its minimum equation is t.

† Received November 24, 1928.

‡ The definitions and theorems marked with an asterisk are presupposed results found in the German and English volumes of L. E. Dickson's "Algebras and their Arithmetics".

t is evidently the rank of the algebra $F(x)$ generated by x.

THEOREM 4*. The order of every normal division algebra is the square of an integer.

THEOREM 5*. Every normal division algebra of order n^2 has rank n.

THEOREM 6*. Every normal division algebra A in n^2 units has the property that, if q is in A and has grade n, the only elements of A commutative with q are polynomials in q with coefficients in F.

THEOREM 6'. Let A be a normal division algebra in n^2 units. Then the grade of any element of A is a divisor of n.

Proof. By a known theorem on matrices the rank function of any matrix has the same roots as its minimum function, so that when the latter is irreducible the former function is an exact power of the latter function. The rank function of any element x of A is a power of its minimum function since both of the functions are identically equal to the corresponding functions for the matrix corresponding to x in the representation of A as an equivalent matric algebra. But the rank equation of x is of degree n by Theorem 5. Hence n is a multiple of the degree of the minimum equation of x, i. e., the grade of x is a divisor of n.

THEOREM 7*. The order of a division algebra A is a multiple of the order of any division sub-algebra of A.

THEOREM 8*. If $B = CD$ where C and D are polynomials in ω over a division algebra A, and if $\omega - x$ is a right divisor of B, but not of D, where x is in A, so that $D = Q(\omega - x) + R$, where $R \neq 0$ is free of ω, then $\omega - y$ is a right divisor of C if $y = RxR^{-1}$ is the transform of x by R.

THEOREM 9*. If A is a normal division algebra over F and if $\varphi(\omega) = 0$ of degree p is the minimum equation of an element x_1 of A, then there exist further elements $x_2, \cdots, x_p$ of A which are transforms of x_1 by elements of A such that

$$\varphi(\omega) \equiv (\omega - x_p)(\omega - x_{p-1}) \cdots (\omega - x_2)(\omega - x_1).$$

Also $\varphi(\omega)$ is the product of the same linear factors permuted cyclically.

We also have in the proof of the above theorem the following theorems.

THEOREM 10*. Every transform of an element x of A satisfies its minimum equation and hence has the same minimum equation as x.

THEOREM 11*. Let the minimum equation of an element x of A be $\varphi(\omega) = 0$ and let $\varphi(\omega)$ be of degree p and

$$\varphi(\omega) \equiv LM, \qquad M \equiv (\omega - x_m)(\omega - x_{m-1}) \cdots (\omega - x_1)$$

where $m < p$. Then there exists a transform t of x such that $\omega - t$ does not divide M on the right.

THEOREM 12. *Let $\varphi(\omega) = 0$ be the minimum equation of an element x_1 of A. Then every element of A satisfying $\varphi(\omega) = 0$ is a transform of x_1 by an element of A.*

Proof. Let x_1 have grade p. Let y be an element of A satisfying $\varphi(\omega) = 0$ and not a transform of x_1 by an element of A.

Lemma 1. No transform sys^{-1} of y by s in A is equal to a transform tx_1t^{-1} of x_1 by t in A and hence the remainder on division of $\omega - tx_1t^{-1}$ by $\omega - sys^{-1}$ is $R \neq 0$ and free of ω for every s and t.

Proof. Let $sys^{-1} = tx_1t^{-1}$. Then $y = s^{-1}tx_1t^{-1}s = (s^{-1}t)x_1(s^{-1}t)^{-1}$, a transform of x_1, contrary to hypothesis.

Since y satisfies $\varphi(\omega) = 0$ then by ordinary algebra $\omega - y$ is a right divisor of $\varphi(\omega)$. By Theorem 9, where $x_1, x_2, \cdots, x_p$ are transforms of x_1,

$$\varphi(\omega) \equiv (\omega - x_p) \cdots (\omega - x_2)(\omega - x_1).$$

Let $S_r \equiv (\omega - x_p)(\omega - x_{p-1}) \cdots (\omega - x_r)$. Then $S_1 \equiv \varphi(\omega)$, $S_p \equiv (\omega - x_p)$.

Lemma 2. There exists an element t_r of A for every r, such that $\omega - t_r y t_r^{-1}$ divides S_r.

Proof. Let $t_1 \equiv 1$. The lemma is evidently true for $r \equiv 1$. Let it be true for $r = k$. Then $\omega - t_k y t_k^{-1}$ divides $s_k \equiv s_{k+1}(\omega - x_k)$. By Lemma 1 the remainder on division of $\omega - x_k$ by $\omega - t_k y t_k^{-1}$ is $s_k \neq 0$. By Theorem 8 $\omega - s_k(t_k y t_k^{-1}) s_k^{-1}$ divides s_{k+1}. Let $t_{k+1} \equiv s_k t_k$. The induction is complete.

Applying Lemma 2 to the case $r = p$, we obtain a contradiction of Lemma 1, and the proof is complete.

3. Algebras satisfying special postulates. *Definition. Type of an equation.* An equation $\varphi(\omega) = 0$ with leading coefficient unity and further coefficients in F, irreducible in F, is said to be of *type* R_k if $k-1$ of its distinct complex roots are rational functions with coefficients in F of a kth distinct complex root.

If $\varphi(\omega) = 0$ is of degree n and type R_n then it is said to be of *type* R. If $\varphi(\omega) = 0$ is of type R_k and is not of type R_{k+1} then $\varphi(\omega) = 0$ is said to be of *type* S_k.

Definition. Type of an element of an algebra. An element x of a division algebra A is said to be of *type* R_k if its minimum equation is of type R_k; x is said to be of *type* R if its grade is n and type is R_n; x is said to be of *type* S_k if it is of type R_k and not of type R_{k+1}.

Definition. Type of A, a normal division algebra in n^2 units. A is said to be of type R_k if it contains an element x of grade n and type R_k. A is said to be of *type* R if it is of type R_n. A is said to be of *type* S_k if of type R_k but not of type R_{k+1}.

A. Every equation, element, and normal division algebra is of type R_1, at least, by Theorems 3 and 5. Hence the assumption that an algebra A is of type R_2 is the mildest possible assumption of this kind.

B. Every equation, element, and normal division algebra of type R_k is of type R_i $(i = 1, 2, \cdots, k-1)$.

C. All normal division algebras in 1^2, 2^2, 3^2, and 4^2 units have been shown to be of type R.

D. The structure of all normal division algebras of type R has been shown to be of the following kind:*

Let A be a normal division algebra in n^2 units of type R_n. Then A contains i satisfying the irreducible $\varphi(\omega) = 0$ of degree n and type R_n with distincts roots $\theta_0(i) = i$, $\theta_1(i)$, $\theta_2(i)$, $\cdots$, $\theta_{n-1}(i)$ and elements $j_0 = 1, j_1, \cdots, j_{n-1}$, such that

$$j_r i = \theta_r(i) j_r; \qquad j_r j_s = \alpha_{rs}(i) j_t, \qquad \alpha_{rs}(i) \text{ in } F(i),$$

$(r, s = 0, 1, \cdots, n-1)$ and the elements $i^r j_s$ are linearly independent and form a basis of A.

Also when $\varphi(\omega) = 0$ has a solvable group the algebra A has been determined.

In previous papers on normal division algebras the distinction between ordinary complex roots of the minimum equation of an element of a division algebra A and the roots in the algebra was not clearly given. We have defined type in terms of complex roots. We shall now establish the meaning in terms of roots in A.

THEOREM 13. *If x is of type R_k then there exist k distinct polynomials $\theta_0(x) = x$, $\theta_1(x)$, $\cdots$, $\theta_{k-1}(x)$ such that if $\varphi(\omega) = 0$ is the minimum equation of x then $\varphi[\theta_i(x)] = 0$ $(i = 0, 1, \cdots, k-1)$ and*

$$\varphi(\omega) \equiv LM, \qquad M \equiv (\omega - \theta_{k-1}(x)) \cdots (\omega - \theta_0(x)).$$

Proof. Since x is of type R_k then there exists a complex root ξ of $\varphi(\omega) = 0$ and polynomials $\theta_1(\xi), \cdots, \theta_{k-1}(\xi)$ satisfying $\varphi(\xi) = 0$, since all rational functions of ξ, a root of an algebraic equation with coefficients in F, are known to be expressible as polynomials in the element ξ. Since x and ξ satisfy the same equation the fields $F(x)$ and $F(\xi)$ are equivalent under the correspondence $x \sim \xi$. Hence $\theta_i(x)$ corresponds to $\theta_i(\xi)$ $(i = 1, \cdots, k-1)$ and hence $\varphi[\theta_i(x)]$ corresponds to $\varphi[\theta_i(\xi)] = 0$. Hence $\varphi[\theta_i(x)] = 0$ since the zero element of two simply isomorphic fields corresponds.

* L. E. Dickson "New Division Algebras", Trans. Amer. Math. Soc., vol. 28, pp. 207–231.

By applying Theorem 8 and using the fact that $\theta_i(x) \neq \theta_j(x)$ for $i \neq j$ and the remainders R are all commutative with the $\theta_i(x)$, we obtain the final conclusion.

THEOREM 14. *An element x is of type R_k if and only if there exist k distinct polynomials $x, \theta_1(x), \cdots, \theta_{k-1}(x)$, with coefficients in F, satisfying the minimum equation of x, $\varphi(\omega) = 0$.*

Proof. If the polynomials exist then, by the equivalence of the fields $F(x)$ and $F(\xi)$, where ξ is any complex root of $\varphi(\omega) = 0$, $\xi, \theta_1(\xi), \cdots, \theta_{k-1}(\xi)$ are distinct and satisfy $\varphi(\omega) = 0$ and hence, by the definition, x is of type R_k. The converse is Theorem 13.

THEOREM 15. *An element x of a division algebra A over F is of type S_k if and only if there exist k distinct polynomials $x, \theta_1(x), \cdots, \theta_{k-1}(x)$ satisfying the minimum equation of x and no further element of $F(x)$ satisfies this equation. Moreover if $\varphi(\omega) = 0$ is the minimum equation of x then*

$$\varphi(\omega) \equiv LM, \qquad M = [\omega - \theta_{k-1}(x)] \cdots [\omega - x].$$

Proof. This theorem is an obvious consequence of 13 and 14 and the simple isomorphism of $F(\xi)$ and $F(x)$ where ξ is any complex root of $\varphi(\omega) = 0$.

THEOREM 16. *Let $\varphi(\omega) = 0$ be an equation of degree n with coefficients in F and leading coefficient unity, irreducible in F. Let $\varphi(\omega) = 0$ be of type S_k. Then k is a divisor of n.*

Lemma 1. Let η be a complex root of $\varphi(\omega) = 0$ and let $\theta(\eta)$, a polynomial in η with coefficients in F, be another root. There exists an integer r such that the iteratives $\theta[\theta(\eta)] \equiv \theta^2(\eta), \theta^3(\eta), \cdots, \theta^{r-1}(\eta), \theta^0(\eta) \equiv \eta$ are all distinct and $\theta^r(\eta) = \eta$.

Proof. Consider the iteratives $\theta^i(\eta)$ $(i = 0, 1, 2, \cdots)$. These are not all distinct for if they were then $\varphi(\omega)$ would have an infinity of distinct roots while it has, by the fundamental theorem of algebra, exactly n. Let r be the least integer such that $\theta^r(\eta) = \theta^s(\eta)$ for some $s < r$. Let $r - s = t \leq r$. Hence $\theta^r(\eta) = \theta^{s+t}(\eta) = \theta^t[\theta^s(\eta)] = \theta^s(\eta)$. Since $\varphi(\omega) = 0$ is irreducible its Galois group is transitive and contains at least one substitution carrying $\theta^s(\eta)$ to η. This will leave unaltered the quantity $0 = \theta^s(\eta) - \theta^t[\theta^s(\eta)]$. Hence $0 = \eta - \theta^t(\eta)$. But r was the least integer such that $\theta^r(\eta) = \theta^s(\eta)$ for some $s < r$ and t is one such integer $\leq r$. Hence $t = r$ and $s = 0$. Hence the roots $\theta^0(\eta) = \eta, \theta(\eta), \theta^2(\eta), \cdots, \theta^{r-1}(\eta)$ are all distinct and $\theta^r(\eta) = \eta$.

Lemma 2. If η and ζ are any two complex roots of $\varphi(\omega) = 0$ and if ζ is a polynomial in η then η is a polynomial in ζ.

Proof. Let $\zeta = \zeta(\eta)$. Then, by Lemma 1, there exists an integer r such that $\zeta^r(\eta) = \eta$. Consequently $\eta = \zeta^{r-1}(\zeta)$.

By hypothesis there exists a root ξ of $\varphi(\omega) = 0$ such that the polynomials $\theta_i(\xi)$, $(i = 1, 2, \cdots, k-1)$ are $k-1$ distinct roots of $\varphi(\omega) = 0$, distinct from ξ and such that the remaining $n-k$ roots of $\varphi(\omega) = 0$ are not expressible as polynomials in ξ. Select from the n roots of $\varphi(\omega) = 0$ starting with $\xi_1 = \xi$ a set G of the q roots $\xi_1, \cdots, \xi_q$ with the property that no one of these is a polynomial in any other root in G and such that this set is the largest one with the previous property. This is possible since the number of roots is finite. Hence every root not in the set is a polynomial in one of the roots ξ_i $(i = 1, \cdots, q)$. The equations $\varphi[\theta^i(\omega)] = 0$ are satisfied by one of the roots ξ of the irreducible $\varphi(\omega) = 0$ and hence by all of its roots. Hence $\theta_i(\xi_j)$ $(i = 0, 1, \cdots, k-1;\ j = 1, 2, \cdots, q)$ are all roots of $\varphi(\omega) = 0$. These are all the roots of $\varphi(\omega) = 0$ since any root not in the set is not in G and hence is a polynomial $\psi(\xi_j)$ for some j. Hence $\psi(\xi)$ is a root contrary to the hypothesis that the only roots which were polynomials in ξ were the $\theta_i(\xi)$ while since if $\psi(\xi) = \theta_i(\xi)$ then by the transitivity of the Galois group $\psi(\xi_j) = \theta_i(\xi_j)$ a contradiction. Hence $\varphi(\omega)$ has $kq = n$ roots.

Corollary 1. *Let $\varphi(\omega)$ be an irreducible equation of degree $2p$, p a prime, and of type R_3. Then $\varphi(\omega)$ is of type R_p or R_{2p}.*

Proof. It is of type S_k where $k \geqq 3$ and divides $2p$.

Corollary 2. *Let $\varphi(\omega)$ be an equation of degree p, p a prime, irreducible in F and of type R_2. Then $\varphi(\omega)$ is of type R.*

THEOREM 17. *Every normal division algebra A in p^2 units, p a prime, and of type R_2 is a cyclic algebra.*

Proof. By hypothesis A contains an element q satisfying an irreducible equation of degree p and of type R_2. By Corollary 2 of Theorem 16 this equation is of type R and hence has a regular transitive group on a prime numbers of letters. The only such group is the cyclic group. Hence the equation is cyclic and the algebra is a cyclic algebra.

THEOREM 18. *Let x be an element of rank n and type S_k. Let $\varphi(\omega) = 0$ be its minimum equation and $\theta_1(x), \theta_2(x), \cdots, \theta_{k-1}(x)$ be $k-1$ distinct polynomials in x no one of which is equal to x and every one of which is a root of $\varphi(\omega) = 0$. Let r_i be the least integer such that $\theta_i^{r_i}(x) = x$. Then r_i divides k $(i = 1, 2, \cdots, k-1)$.*

Proof. Let ξ be any complex root of $\varphi(\omega) = 0$. Then $\theta_i(\xi)$ $(i = 1, \cdots, k-1)$ are roots also. Let $\eta_1 = \xi$ and $\eta_2, \cdots, \eta_q$ be a maximal set G of the polynomials $\theta_0(\xi) = \xi, \theta_1(\xi), \cdots, \theta_{k-1}(\xi)$ with the property that $\eta_i \neq \theta_1^m(\eta_j)$ $(i \neq j;\ m = 0, 1, \cdots, r_1-1;\ i, j = 1, 2, \cdots, q)$. Hence every polynomial $\theta_i(\xi)$ not in G is a polynomial $\theta_1^{m_i}(\eta_{j_i})$ for a fixed j_i and m_i. Moreover the roots $\theta_1^m(\eta_j)$ are all instinct for if $\theta_1^{m_1}(\eta_{j_1}) = \theta_1^{m_2}(\eta_{j_2})$ then

$m_1 = r_1 - t$, $\eta_{j_1} = \theta_1^{m_2+t}(\eta_{j_2})$ which is impossible unless $j_1 = j_2$, $m_2 + t = pr_1$ since $t = r_1 - m_1$, $m_2 - m_1 = (p-1)r$, and m_2 and m_1 are both less than r_1, $m_2 = m_1$. Hence every root $\theta_i(\xi)$ is expressible as one of the above $r_1 q$ roots. $\therefore k = r_1 q$. Hence r_1 divides k. Similarly r_i divides k $(i = 2, \cdots, k-1)$.

Definition. Let $\theta(x)$ be a polynomial in x satisfying the minimum equation of x. Let r be the least integer so that $\theta^r(x) = x$. Then r shall be called the *index* of $\theta(x)$.

We shall next consider the theory of the linear independence of elements of a divison algebra A where the coefficients are also in A. These theorems are assumed without proof by most writers in the subject.

Definition. Consider the set $u_1, u_2, \cdots, u_n$ of any whatever elements of A. Let S be a set of any whatever elements of A such that S contains the zero element of A. The elements $u_1, \cdots, u_n$ are said to be *left linearly independent* with respect to the set S if whenever

$$\sum_{q=1}^{n} s_q u_q = 0, \; s_q \text{ in } S, \text{ then } s_q = 0 \; (q = 1, 2, \cdots, n).$$

Definition. The above with right replacing left and $\sum_{q=1}^{n} u_q s_q$ replacing $\sum_{q=1}^{n} s_q u_q$ define right linear indepence.

Definition. If the quantities of S are commutative with u_q then right linear independence and left linear independence are equivalent and the quantities are said to be *linearly independent.*

THEOREM 19. *If u_γ $(\gamma = 1, 2, \cdots, n)$ of A are left linearly independent with respect to S and $n = pr$, $u_\gamma = v_\alpha w_\beta$ $(\gamma = \alpha \cdot \beta; \alpha = 1, 2, \cdots, p; \beta = 1, 2, \cdots, r)$ then the elements w_β are left linearly independent with respect to T, the set composed of all left linear combinations with coefficients in S of the quantities v_α.*

Proof. If left linear independence is defined for the u_γ with respect to S then the zero element of A is in S and hence in T. Hence left linear independence is defined for T.

Let $\sum_{\beta=1}^{r} t_\beta w_\beta = 0$ where the t_β are in T and are not all zero. Let $t_\beta = \sum_{\alpha=1}^{p} s_{\alpha\beta} v_\alpha$. Hence

$$\sum_{\beta=1}^{r} \left(\sum_{\alpha=1}^{p} s_{\alpha\beta} v_\alpha \right) w_\beta = \sum_{\alpha,\beta} s_{\alpha\beta} v_\alpha w_\beta = \sum_{\gamma=1}^{n} s_\gamma u_\gamma$$

where $s_{\alpha\beta} \equiv s_\gamma$, $u_\gamma \equiv v_\alpha w_\beta$ $(\gamma = \alpha \cdot \beta; \alpha = 1, \cdots, p; \beta = 1, \cdots, r)$. Since the elements u_γ are left linearly independent with respect to S,

$$s_\gamma = 0 \ (\gamma = 1, 2, \cdots, n), \ s_{\alpha\beta} = 0 \ (\alpha = 1, 2, \cdots, p; \ \beta = 1, 2, \cdots, r),$$
$$t_\beta = \sum_\alpha s_{\alpha\beta} v_\alpha = 0$$

contrary to hypothesis. Hence the elements w_β are left linearly independent with respect to T.

Conversely we have the theorem

THEOREM 20. *Let $v_\alpha \ (\alpha = 1, 2, \cdots, p)$ be left linearly independent with respect to S and $w_\beta \ (\beta = 1, 2, \cdots, r)$ be left linearly independent with respect to T, the set of all left linearly combinations with coefficients in S of v_α. Then the set of elements*

$$u_\gamma \equiv v_\alpha w_\beta \ (\gamma = \alpha \cdot \beta; \ \alpha = 1, 2, \cdots, p; \ \beta = 1, 2, \cdots, r)$$

are left linearly independent with respect to S.

Proof. Consider $\sum_{\gamma=1}^{n} s_\gamma u_\gamma = 0, \ n = pr$. Let $s_\gamma = s_{\alpha\beta}$ for $u_\gamma = v_\alpha w_\beta$. Then

$$\sum_{\alpha,\beta} s_{\alpha\beta} v_\alpha w_\beta = \sum_\beta \Big(\sum_\alpha s_{\alpha\beta} v_\alpha \Big) w_\beta = 0. \quad \therefore \sum_\alpha s_{\alpha\beta} v_\alpha = 0$$

for every β. Hence $s_{\alpha\beta} \equiv s_\gamma = 0 \ (\gamma = 1, 2, \cdots, pr)$. This proves the theorem.

We shall apply this theory first to the case of the direct product of two division algebras.

Corollary 3. *Let $A = B \otimes C$, where B and C are division algebras over F. Then the basal elements of C are linearly independent with respect to the algebra B and conversely.*

Proof. Let the basal elements of B be $v_\alpha \ (\alpha = 1, 2, \cdots, p)$ and those of C be $w_\beta \ (\beta = 1, 2, \cdots, r)$.

Since the product is a direct product the basal elements of A are

$$u_\gamma \equiv v_\alpha w_\beta \ (\gamma = \alpha\beta; \ \alpha = 1, 2, \cdots, p; \ \beta = 1, 2, \cdots, r).$$

Since these elements form a basal set they are linearly independent with respect to F. Hence, using the Theorem 19 with $S = F$ and $T = B$, the elements w_β are left linearly independent with respect to B. Since the product is a direct product the elements of B and C are commutative. Hence the basal elements of C are linearly independent with respect to the algebra B. By symmetry the basal elements v_α of B are linearly independent with respect to C.

THEOREM 21. *Let $A = B \otimes C$, where B is an associative normal division algebra of order n^2 over F, of type R_n, and C is an associative normal division algebra of order m^2 over F, of type R_m. If A is an associative division algebra then A is a normal division algebra of type R_{mn}.*

Proof. Let i be an element of B satisfying the equation

$$\varphi(\omega) \equiv \omega^n + \alpha_1 \omega^{n-1} + \cdots + \alpha_n = 0, \qquad \alpha_q \text{ in } F \ (q = 1, 2, \cdots, n)$$

irreducible in F and such that $\theta_p(i)$ $(p = 1, 2, \cdots, n-1)$, $\theta_0(i) = i$, are polynomials in i with coefficients in F which are distinct and satisfy $\varphi(\omega) = 0$. Then there exist elements $j_q (q = 1, 2, \cdots, n-1)$, $j_0 = 1$ in B such that $j_q i j_q^{-1} = \theta_q(i)$ and the elements $i^p j_q$ are linearly independent with respect to F and form a basal set of algebra B.

We have a similar condition in C with i replaced by k, j_q by l_q, θ_q by ψ_q and n by m.

As a consequence of Theorem 20 and Corollary 3 we have the following lemmas.

LEMMA 1. The elements $i^p j_q$ are left linearly independent with respect to algebra C.

LEMMA 2. The elements j_q are left linearly independent with respect to the set T composed of all left linear combinations with coefficients in C of the elements $i^p (p = 0, 1, \cdots, n-1)$.

Let x be commutative with every element of A; then $x = \sum_{p,q} a_{pq} i^p j_q$ where a_{pq} are in C. Since $xi = ix$, we have

$$i \sum_{p,q} a_{pq} i^p j_q = \sum_{p,q} a_{pq} i^p j_q i = \sum_{p,q} i^p \theta_q(i) j_q,$$

so that $\sum_q \left[\sum_p a_{pq} i^p (i - \theta_q(i))\right] j_q = 0$ and therefore by Lemma 2

$$\sum_p a_{pq} i^p [i - \theta_q(i)] = 0 \qquad (q = 1, 2, \cdots, n-1).$$

But if $q \neq 0$ then $i - \theta_q(i) \neq 0$ and has an inverse in A. Hence $\sum_p a_{pq} i^p = 0$ $(q = 1, 2, \cdots, n-1)$. By Lemma 1 applied with respect merely to elements $i^p j_0 = i^p$ we have $a_{pq} = 0$ $(p = 0, 1, \cdots, n-1;\ q = 1, 2, \cdots, n-1)$. Hence $x = \sum_p a_{p0} i^p$. But

$$x j_q = j_q x = \sum_p a_{p0} j_q i^p = \left\{\sum_p a_{p0} [\theta_q(i)]^p\right\} j_q,$$

$$\therefore \sum_p a_{p0} \{[\theta_q(i)]^p - i^p\} = 0 \qquad (q = 0, 1, \cdots, n-1),$$

$$\therefore \sum_{q=0}^{n-1} \sum_p a_{p0} \{[\theta_q(i)]^p - i^p\} = \sum_p a_{p0} \left\{\sum_{q=0}^{n-1} [\theta_q(i)]^p - n i^p\right\}.$$

But $\sum_{q=0}^{n-1} [\theta_q(i)]^p$ is symmetric in the roots of (1) and hence is in F for all values of p. Hence by Lemma 1 applied to the elements i^p, considering

the coefficients of $i^p (p = 1, 2, \cdots, n-1)$ which are a_{p0} respectively, we have $a_{p0} = 0$ $(p = 1, 2, \cdots, n-1)$. Hence $x = a_{00}$ in C. But x is commutative with every element of C and C is normal. Hence a_{00} is in F. Hence A is normal.

Consider the field $F(i, k)$. This field is of order nm by Theorem 20. Hence it contains an element t, by ordinary algebraic number theory, satisfying an irreducible equation of degree mn in F, such that all elements of the field are polynomials in t with coefficients in F. Since t is in $F(i, k)$ it is a polynomial in k with coefficients in $F(i)$ and hence

$$t = \sum_{\beta=0}^{m-1} a_\beta k^\beta, \qquad a_\beta \text{ in } F(i).$$

Since $j_q i j_q^{-1} = \theta_q(i)$ in $F(i)$ and hence $j_q a_\beta j_q^{-1} = a_\beta[\theta_q(i)]$, an element of $F(i)$ for $q = 0, 1 \cdots, n-1$, $\beta = 0, 1, \cdots, m-1$. Also $j_q k = k j_q$ since j_q is in B and k is in C and $A = B \otimes C$. Hence $j_q t j_q^{-1} = A_{q0}$ $(q = 0, 1, \cdots, n-1)$ and element of $F(i, k)$ and hence a polynomial in t with coefficient in F.

Similarly

$$l_r t_{q0} l_r^{-1} \equiv t_{qr} \qquad \begin{matrix} (q = 0, 1, \cdots, n-1) \\ r = 0, 1, \cdots, m-1 \end{matrix}$$

where t_{qr} are in $F(i, k)$ and hence are nm polynomials in t. They are obviously roots of the minimum equation of t since $t_{qr} = (l_r j_q)\, t\, (l_r j_q)^{-1}$ a transform of t. They are distinct for if $(l_{r_1} j_{q_1})\, t (l_{r_1} j_{q_1})^{-1} = (l_{r_2} j_{q_2})\, t (l_{r_2} j_{q_2})^{-1}$ then, since i is a polynomial in t, we have

$$(l_{r_1} j_{q_1})\, i\, (l_{r_1} j_{q_1})^{-1} = (l_{r_2} j_{q_2})\, i\, (l_{r_2} j_{q_2})^{-1} = j_{q_1} i j_{q_1}^{-1} = \theta_{q_1}(i) = j_{q_1} i j_{q_2}^{-1} = \theta_{q_2}(i)$$

which is not true unless $q_1 = q_2$. Similarly we prove $r_1 = r_2$ making use of k. Hence the nm polynomials t_{qr} are $n \cdot m$ distinct polynomial roots of the minimum equation of t, of degree mn. Hence t is of type R_{nm}. But the order of A is $(nm)^2$. Hence A is of type R_{nm}.

THEOREM 23.* *If an associative division algebra A over F contains a normal division algebra B, A can be expressed as the direct product of B and another algebra C over F.*

Corallary 4. *If a normal associative division algebra A over F contains a normal division algebra B, A can be expressed as the direct product of B and a normal algebra C.*

THEOREM 24. *Let A be an associative normal division algebra in n^2 units over F, containing an element i of type S_k and grade h. Then A contains elements*

* See Wedderburn, Trans. Amer. Math. Soc., vol. 22 (1921), p. 132.

$$\theta_0(i) = i, \quad \theta_1(i), \cdots, \theta_{k-1}(i), \quad j_0 = 1, j_1, \cdots, j_{k-1}$$

with the properties that $\theta_q(i)$ $(q = 0, 1, \cdots, k-1)$ *are distinct polynomials in* i*, with coefficients in* F*, satisfying the minimum equation of* i*, and* A *contains no more such polynomials;* $j_q(i) = \theta_q(i) j_q$ $(q = 0, 1, \cdots, k-1)$*; the elements* $i^p j_q$ $(p = 0, 1, \cdots, n-1;\ q = 0, 1, \cdots, k-1)$ *are linearly independent and form a basal set of an algebra* Σ *of order* kn *which is a division sub-algebra of* A*;* Σ *is a normal division of order* k^2 *over the field composed of all polynomials in* i *symmetric in the functions* $\theta_q(i)$*,* $(q = 0, 1, \cdots, k-1)$*; this field contains an element* u *generating it and of grade* n/k*; every element of* Σ *is commutative with* u *and these are the only elements of* A *commutative with* u*.*

Proof. The existence of the elements $\theta_q(i)$ is shown in Theorem 15. This also shows the nonexistence of further polynomials with these properties. There exist elements j_q $(q = 0, 1, \cdots, n-1)$ such that $j_q i j_q^{-1} = \theta_q(i)$, by Theorem 12, and we may evidently take $j_0 = 1$.

If the elements $i^p j_q$ $(p = 0, 1, \cdots, n-1;\ q = 0, 1, \cdots, k-1)$ are not linearly independent, then the quantities j_q are not left linearly independent with respect to $F(i)$, since the elements i^p are linearly independent with respect to F due to the fact that i has grade n and we may apply Theorem 21. Hence r of the elements j_q are left linearly independent with respect to $F(i)$ while the $r+1$st one is expressible left linearly in terms of them so that:

$$j_{r+1} = \sum_{q=0}^{r} \psi_q(i) j_q, \qquad 0 \leqq r < k-1.$$

But

$$j_{r+1} i = \theta_{r+1}(i) j_{r+1},$$

$$\theta_{r+1} \sum_{q=0}^{r} \psi_q(i) j_q = \left(\sum_{q=0}^{r} \psi_q(i) j_q \right) i = \sum_{q=0}^{r} \psi_q(i)\, \theta_q(i) j_q.$$

This shows that

$$\sum_{q=0}^{r} \psi_q(i) [\theta_q(i) - \theta_{r+1}(i)] j_q = 0.$$

But the elements j_q $(q = 0, \cdots, r)$ are left linearly independent with respect to $F(i)$. Thus $\psi_q(i) [\theta_q(i) - \theta_{r+1}(i)] = 0$ $(q = 0, 1, \cdots, r)$. But the polynomials $\theta_q(i)$ $(q = 0, 1, \cdots, k-1)$ are all distinct so that $\theta_q(i) - \theta_{r+1}(i) \neq 0$ $(q = 0, 1, \cdots, r)$. Hence $\psi_q(i) = 0$ $(q = 0, 1, \cdots, r)$ and $j_{r+1} = 0$ contrary to its origin.

Hence the elements $i^p j_q$ are linearly independent with respect to F. To prove the linear set composed of all linear combinations of them with

coefficients in F is an algebra we need only show that $j_r j_s$ is in the set for every r and s since then the multiplication table of units will be complete.

$$(j_r j_s) i (j_r j_s)^{-1} = j_r \theta_s(i) j_r^{-1} = \theta_s[\theta_r(i)],$$

a polynomial in i satisfying its minimum equation and hence one of the polynomials $\theta_q(i)$. Hence

$$(j_r j_s) i (j_r j_s)^{-1} = \theta_t(i) = j_t i j_t^{-1}, \quad i = (j_t^{-1} j_r j_s) i (j_t^{-1} j_r j_s)^{-1}.$$

Consequently $j_t^{-1} j_r j_s$ is commutative with i and by Theorem 6

$$j_t^{-1} j_r j_s = \beta_{rs}(i). \quad \therefore \; j_r j_s = j_t \beta_{rs}(i) = \alpha_{rs}(i) j_t$$

an element of the linear set. Hence the linear set is an algebra of order nk with basal elements $i^p j_q$ and is thus a division sub-algebra of A.

Every division algebra is expressible as a normal division algebra over the field of all of its elements which are commutative with every element of it. Let this field be generated by u for the algebra Σ, and Σ is a normal division algebra of order m^2 over $F(u)$. Since u is commutative with i it is a polynomial in i. Since u is commutative with j_q for each q we have $u = u(i) = j_q u j_q^{-1} = u[\theta_q(i)]$ $(q = 0, 1, \cdots, k-1)$. Hence u is symmetric in the roots $\theta_q(i)$ and it follows that all elements of $F(u)$ have this property.

The element i of Σ satisfies the equation

$$\varphi(\omega) \equiv \omega^k + s_1 \omega^{k-1} + \cdots + s_k = 0$$

where $s_1, \cdots, s_k$ are the properly signed elementary symmetric functions of the k roots $\theta_q(i)$. Each of these quantities and consequently every symmetric function of the quantities $\theta_q(i)$ is in $F(u)$, since each is left unaltered under multiplication by any quantity $\sum_q \psi_q(i) j_q$ on the left and division by the same quantity on the right and is hence commutative with every element of Σ. But the equation of least degree with leading coefficient unity and further coefficients in $F(u)$, satisfied by i has degree at least k and hence is $\varphi(\omega) = 0$ since it has k distinct roots $\theta_q(i)$. The elements $i^p j_q$ $(p, q = 0, 1, \cdots, k-1)$ are linearly independent with respect to the field $F(u)$ since the elements j_q are left linearly independent with respect to $F(u)$ by Theorem 20 and the elements i^p are left linearly independent with respect to $F(u)$ by the fact that $\varphi(\omega) = 0$ is the equation of least degree with coefficients in $F(u)$ satisfied by i. Also every polynomial in i is expressible, by means of $\varphi(\omega) = 0$ as a linear combination

with coefficients in $F(u)$ of the powers i_p $(p = 0, 1, \cdots, k-1)$. Hence every element of Σ is expressible linearly with coefficients in $F(u)$ in terms of the elements $i^p j_q (p, q = 0, 1, \cdots, k-1)$ which are linearly independent with respect to $F(u)$. Hence Σ is a normal division algebra of order k^2 over $F(u)$. But the order of Σ is kn. Hence the order of $F(u)$ is n/k (n divides k by Theorem 17, since the order of an algebra, over an algebraic field with respect to the original field F, is the product of the order of the algebra with respect to the algebraic field and the order of that field).

Finally let x be commutative with u. If x is in Σ then it is obviously commutative with u. If not then consider the algebra S composed of all polynomials in elements of Σ and the element x. It is a sub-algebra of A. Its order is obviously greater than kn, the order of Σ. It is a normal division algebra over the field K of all elements of S commutative with every element of S. Since every element of K is therefore cummutative with i and with j_q it is a polynomial in i symmetric in the $\theta_q(i)$ and hence is in $F(u)$, the field consisting of all symmetric functions with coefficients in F of the polynomials $\theta_q(i)$. Hence S is a normal division algebra over $F(u)$. But S contains Σ a normal division algebra over $F(u)$. Hence, by Theorem 23, $S = B \otimes \Sigma$ where B is a division algebra. But every element of B, in the direct product, is commutative with every element of Σ and hence with i. Hence every element of B is a polynomial in i with coefficients in F. An obvious consequence of this is that every element of S is a sum of products of polynomials in i by elements of Σ. All such elements are elements of Σ and hence $S = \Sigma$ contrary to hypothesis that x is not in Σ but is in S.

We have obtained the partial structure of any normal division algebra containing an element x whose minimum equation has a special property. The case $k = n$ of this theorem gives the complete structure of the algebra and has been previously studied.* While these algebras are the only algebras which have been constructed we shall define a broader class of algebras as known algebras and shall, in future determination problems, attempt to show that the algebras determined are known in this new sense.

Definition. Known algebras. Let Σ be any associative division algebra over F. Let to every element A of Σ correspond a unique element A' of Σ, and let $s \neq 0$ be a self corresponding element of Σ. Let $A'' \equiv (A')'$, $A''' \equiv (A'')', \cdots$. Let $A^{(p)} = sAs^{-1}$ for every A of Σ. Let $\alpha' = \alpha$, $(AB)' = A'B'$, $(A+B)' = A'+B'$ for every α of F, and A and B of Σ. Call any algebra Σ with these properties a base algebra.

* L. E. Dickson, Trans. Amer. Math. Soc., vol. 28, pp. 207–231.

L. E. Dickson has shown that, given any base algebra Σ, there exists an element z such that the algebra Γ, of elements

$$A_0 + A_1 z + \cdots + A_{p-1} z^{p-1}$$

and multiplication table

$$zA = A'z, \qquad z^p = s,$$

where $A, A_0, \cdots, A_{p-1}$ range independently over all elements of Σ, is an associative algebra over F.

Hence to every base algebra there corresponds a Γ algebra. Let G_1 be the set of all algebraic fields over F. All such algebras are known. Let H_1 be the set of all base algebras obtained by considering all possible self-correspondences of algebras of G_1. Let G_2 be the set of all algebras which are division algebras obtained by considering the direct products of all algebras obtained from H_1 and all algebras of G_1, and taking all division algebras of this set. Considering all self-correspondences of the algebras of G_2 we define the set H_2. Continuing similarly we define the sets G_k for all k's. The algebras of all G_k will be said to be known algebras and any algebra will be said to be determined if we can show that it belongs to one of the sets G_k.

With this as the definition of known division algebras, and with the obvious conclusion that if A has $n^2 m^2$ units and is a Γ algebra over Σ as base algebra, where Σ has $n^2 m$ units, then p of the definition is equal to m by the hypothesis about orders, we shall prove the following theorems:

THEOREM 27. *Let A be a normal division algebra in $n^2 m^2$ units containing a sub-algebra S which is a known normal division algebra in n^2 units over an m-ic field $F(q)$. Then A is a known algebra over S as Σ algebra if the minimum equation of q is cyclic.*

Proof. Let $\Phi(\omega) = 0$ be the minimum equation of q, cyclic by hypothesis. Let $\theta(q) \neq q$ satisfy $\Phi(\omega) = 0$, such that $q, \theta(q), \cdots, \theta^{m-1}(q)$ are distinct roots of $\Phi(\omega) = 0$, and $\theta^m(q) = q$. By Theorem 12 there exists an element z of A such that $zq = \theta(q)z$. z is not in S since all elements of S are commutative with q. Then $z^i q = \theta^i(q) z^i (i = 0, 1, \cdots, m-1)$, $z^m q = q z^m$. Let the basal elements of S with respect to $F(q)$ be $u_1 = 1, u_2, \cdots, u_{n^2}$. Then the elements $1, z, z^2, \cdots, z^{m-1}$ are left linearly independent with respect to S since if not then for $0 < k < m-1$, the elements $1, z, z^2, \cdots, z^k$ are left linearly independent with respect to S while

$$z^{k+1} = \sum_{\alpha=0}^{k} s_\alpha z^\alpha,$$

s_α in S not all zero,

$$z^{k+1} q = \theta^{k+1}(q) z^{k+1} = \left(\sum_{\alpha=0}^{k} s_\alpha z^\alpha\right) q = \sum_{\alpha=0}^{k} s_\alpha \theta^\alpha(q) z^\alpha$$

$$= \theta^{k+1}(q) \sum_{\alpha=0}^{k} s_\alpha z^\alpha = \sum_{\alpha=0}^{k} [\theta^{k+1}(q) s_\alpha] z^\alpha,$$

$$\therefore \sum_{\alpha=0}^{k} [\theta^{k+1}(q) - \theta^\alpha(q)] s_\alpha z^\alpha = 0 \quad \therefore [\theta^\alpha(q) - \theta^{k+1}(q)] s_\alpha = 0.$$

$\alpha = 0, 1, \cdots, k$ which is impossible since not all of the s_α are zero while the $\theta^\alpha(q) - \theta^{k+1}(q)$ are all different from zero.

Since the $n^2 m$ units $q^\alpha u_\beta$ ($\alpha = 0, 1, \cdots, m-1$; $\beta = 1, 2, \cdots, n^2$) are basal elements of S and hence linearly independent with respect to F, the $n^2 m^2$ elements $q^\alpha u_\beta z^\gamma$ ($\alpha, \gamma = 0, 1, \cdots, m-1$; $\beta = 1, 2, \cdots, n^2$) are linearly independent, by Theorem 21. Hence they form a basis of A since they are $(nm)^2$ in number and every element of A is expressible in the form $s_0 + s_1 z + \cdots + s_{m-1} z^{m-1}$ with s_α in S.

Lemma 1. The only elements of A commutative with q are elements of S.

Proof. Let $x = s_0 + s_1 z + \cdots + s_{m-1} z^{m-1}$ be commutative with q. Then

$$xq = s_0 q + s_1 \theta(q) z + \cdots + s_{m-1} \theta^{m-1}(q) z^{m-1} = qx$$

Hence

$$s_0(q-q) + s_1[\theta(q)-q] z + \cdots + s_{m-1}[\theta^{m-1}(q)-q] z^{m-1} = 0.$$

By the left linear independence of the elements z^α ($\alpha = 0, 1, \cdots, m-1$) with respect to S, $s_\alpha[\theta^\alpha(q)-q] = 0$ ($\alpha = 0, 1, \cdots, m-1$). But for $\alpha \neq 0$ $\theta^\alpha(q) - q \neq 0$. Since S is a division algebra, $s_\alpha = 0$ ($\alpha = 1, 2, \cdots, m-1$) and $x = s_0$ in S.

Hence $z^m = s$ in S. It remains to show that $zX = X'z$ for X in S and X' in S. Let $zXz^{-1} = X'$ where X is any element of S. Since $z^m = s$, $z^{m-1} = sz^{-1}$, $z^{-1} = s^{-1} z^{m-1}$. Since q is commutative with any element of S it is commutative with s^{-1}. Hence $X'q = zXz^{-1}q = zXs^{-1}z^{m-1}q$ $= zXs^{-1}\theta^{m-1}(q) z^{m-1} = z\theta^{m-1}(q) Xs^{-1} z^{m-1} = \theta^m(q) zXz^{-1} = qX'$. This shows that X' is commutative with q and X' is in S by the lemma.

Obviously since A is assumed to be a normal associative algebra S has the properties that $(A+B)' = A'+B'$, $(AB)' = A'B'$, $\alpha' = \alpha$ for every A, B of S and α of F. Also by the definition $zXz^{-1} = X'$, $X^{(m)} = z^m X z^{-m}$ for every X of S. Hence A is a known algebra.

To prove the converse of this theorem we shall prove a result on the rank of a normal algebra over a commutative division algebra.

THEOREM 28. *Let A be a normal division algebra in n^2 units over a field $F(q)$ of order m. Then A is of rank nm with respect to F.*

Proof. A is of rank n with respect to $F(q)$. The element x, the general element of A, satisfies the rank equation of A with respect to F. This equation has degree t and the algebra A has the property that every element of A satisfies an equation of degree t in F and some element of A is of grade t with respect to F.

Let i be of grade t with respect to F. The field $K = F(i, q)$ has the property that every element of K is expressible in the form $\sum_{\substack{\lambda=0,\cdots,m-1 \\ \mu=0,\cdots,n-1}} \alpha_{\lambda\mu}\, i^\lambda\, q^\mu$ since q is the root of an equation of degree m in F and i is the root of an equation of degree n in $F(q)$. Hence the order of K is $\leq nm$. But $F(i) \leqq F(i, q)$. Hence $t \leqq nm$.

Conversely let i be of grade n with respect to $F(q)$. Consider the field $F(i, q)$. The elements $q^\mu i^\lambda$ $(\mu = 0, 1, \cdots, m-1;\ \lambda = 0, 1, \cdots, n-1)$ are linearly independent with respect to F by the property that i is of grade n with respect to $F(q)$ and by Theorem 21. Hence $F(i, q)$ is of order nm with respect to F, and contains an element r of grade nm. Hence r is of grade $nm \leqq t$. Hence t is equal to nm.

As a consequence of this theorem we obtain the converse of Theorem 27.

THEOREM 29. *Let A be a normal division algebra in $n^2 m^2$ units over F. Let A contain S which is a known normal division algebra over an m-ic field $F(q)$. Let A be a known algebra over S as Σ algebra. Then the minimum equation of q is cyclic.*

Proof. If $m = 1$ the theorem is trivial. Let $m > 1$. By hypothesis there exists an element z of A such that $zX = X'z$ for every X of S, where X' is in S. Also every element of A may be written in the form

$$a = s_0 + s_1 z + \cdots + s_{m-1} z^{m-1}, \qquad z^m = s$$

where $s_0, \cdots, s_{m-1}, s$ are in S, and the elements $1, z, \cdots, z^{m-1}$ are left linearly independent with respect to S. Let a set of basal units of S with respect to F be $v_1, \cdots, v_{n^2 m}$. Let $zv_i = v_i' z$ $(i = 1, 2, \cdots, n^2 m)$. The elements v_i' form a basis of S since any linear dependence between them may be transformed into a linear dependence between the v_i by transforming by z^{-1} and they are $n^2 m$ in number. Since q is commutative with each of $v_1, \cdots, v_{n^2 m}$, q' is commutative with each $v_1', \cdots, v_{n^2 m}'$ and hence with every element of S, where $q' = zqz^{-1}$. But S is a normal algebra over $F(q)$. Hence q' is in $F(q)$. Let $q' = \theta(q)$. Then the elements $q, \theta(q), \cdots, \theta^{m-1}(q)$ are all roots $z^\alpha q z^{-\alpha}$ of the minimum equation of q. It remains only to show that they are all distinct. Since $z^\alpha q = \theta^\alpha(q)\, z^\alpha$, let $\theta^\alpha(q) = \theta^\beta(q)$ where $0 \leqq \alpha < \beta \leqq m-1$. It follows that $z^\alpha q z^{-\alpha} = z^\beta q z^{-\beta}$, $z^{\beta-\alpha} q = q z^{\beta-\alpha}$ where $m > \beta - \alpha > 0$. Let r be the least integer

such that $z^r q = qz^r$. Then r is a divisor of m since if $m = ra + b$ where $0 < b < r$ then, since $z^m q = qz^m$ and $z^{ra} q = qz^{ra}$, $z^{m-ra} q = qz^{m-ra} = z^b q = qz^b$ contrary to the hypothesis that r is the least integer such that $z^r q = qz^r$ and $b < r$. Hence $b = 0$, $m = ra$.

Consider the linear set T of all left linear combinations with coefficients in S of $1, z^r, z^{2r}, \cdots, z^{(a-1)r}$. It is an algebra since $z^r x = x^{(r)} z^r$, $z^{ar} = z^m = s$, for x in S. Hence it is a normal division algebra over the field of all of its elements commutative with every element of it. Let x be any such element. Then, since S is of rank nm by Theorem 29 and contains an element i of grade nm whence by Theorem 6, the only elements of A commutative with i are polynomials in i with coefficient in F, $xi = ix$ and x is a polynomial in i and is in S. But x is commutative with every element of T and hence with every element of S. Hence, since S is normal over $F(q)$, x is in $F(q)$. Hence T is a normal division algebra in $n^2 a$ units over $F(q)$. Hence $a = c^2$, $c > 1$, and, by Theorem 29, T is of rank nmc with respect to F and contains an element of grade nmc. But A is of rank nm with respect to F. Hence $c = 1$, a contradiction. Hence the polynomials $\theta^\alpha(q)$ are distinct and q satisfies a cyclic m-ic in F.

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TENSORS WITH INVARIANT COMPONENTS.*

By M. S. Knebelman.†

Introduction. One of the fundamental theorems of algebraic invariant theory requires the knowledge of the form of the most general tensor — absolute or relative — whose components have the same values in every coördinate system. The form of the most general absolute tensor with invariant components has been correctly surmized and to the writer's knowledge three papers have appeared within the past few years attempting to prove the generality of this form. Neither of these papers however contains a valid proof and it is the object of this note to obtain the most general tensor with invariant components, first when the tensor is absolute and second, when the tensor is relative of integral — positive or negative — weight, the only relative tensors of fractional weight with invariant components being zero tensors, as is proved in § 3.

A slight change in the wording of the theorems gives the most general tensor-fields with invariant components.

1. **Absolute tensors.** Let $i_1, \cdots, i_r$ be an ordered set of r numbers each on the range 1, 2, $\cdots$, n and let $(i_r)_\alpha$ represent a permutation of the i's it being agreed that $(i_r)_1$ or simply (i_r) is the arrangement of the i's in the natural order of their subscripts, the other $(r!-1)$ permutations being arranged in some fixed chronological order. Let $T^{(i_r)}_{(j_p)}$ be an absolute tensor in a number space of n dimensions, having the property that each one of its components has the same value in every coördinate system. If x and $\bar{x}$ are any two coördinate systems and we write $\partial \bar{x}^i / \partial x^j \equiv u^i_j$, a tensor with invariant components must satisfy the equations

$$T^{(i_r)}_{(h_p)} u^{h_1}_{j_1} \cdots u^{h_p}_{j_p} = T^{(k_r)}_{(j_p)} u^{i_1}_{k_1} \cdots u^{i_r}_{k_r} \tag{1.1}$$

for all values of the u's, (1.1) being nothing more than the law of transformation of an absolute tensor having the desired property. Let a be a particular number of the range 1, 2, $\cdots$, n and let the transformation of coördinates be such that

$$u^i_j = \delta^i_j, \quad u^a_j = u^i_a = 0, \quad u^a_a = c \qquad (i, j \neq a). \tag{1.2}$$

* Received October 19, 1928.

† National Research Fellow.

Now suppose that in (1.1) the index a occurs s_1 times among the i's and s_2 times among the j's. Then under (1.2) equations (1.1) become

$$T^{(i_r)}_{(j_p)} c^{s_2} = T^{(i_r)}_{(j_p)} c^{s_1}$$

and since this is to hold for all values of c it follows that either $T^{(i_r)}_{(j_p)} = 0$ or $s_1 = s_2$. Hence every index that occurs contravariantly must occur an equal number of times covariantly and therefore the total number of covariant and contravariant indices must be the same, otherwise the tensor is a zero tensor, a case which is of no interest and which we exclude from consideration. We may combine these results into

THEOREM 1. *If a tensor $T^{(i_r)}_{(j_p)}$ is to have invariant components r must equal p and the components, whose covariant indices are not a permutation of the contravariant ones, vanish.*

We now assume $p = r$ and pick out the coefficients of $u^{h_1}_{k_1} u^{h_2}_{k_2} \cdots u^{h_r}_{k_r}$ in (1.1) obtaining — we have interchanged the indices j, h, k cyclically —

$$T^{(i_r)}_{(j_r)_\alpha} \delta^{(h_1}_{k_1} \cdots \delta^{h_r)_\alpha}_{k_r} = T^{(h_r)_\alpha}_{(k_r)} \delta^{i_1}_{(j_1} \cdots \delta^{i_r}_{j_r)_\alpha}, \tag{1.3}$$

where α is summed from 1 to $r!$ and $(h_1 \cdots h_r)_\alpha$ is the same permutation of $h_1 \cdots h_r$ as $(j_r)_\alpha$ is of $j_1 \cdots j_r$. Equations (1.3) are the necessary and sufficient conditions that $T^{(i_r)}_{(j_r)}$ must satisfy in order that the tensor have invariant components.* We may therefore consider (1.3) as a set of equations to be solved for the unknowns $T^{(i_r)}_{(j_r)}$. We first let $(j_1 \cdots j_r)$ be any set of indices on the range $1, \cdots, n$ and choose $i_1 \cdots i_r$ so that $(i_1 \cdots i_r) \neq (j_1 \cdots j_r)_\alpha$, this choice being obviously possible except in the trivial case $n = 1$, which we assume not to be the case. Then since $\delta^i_j = 0$ if $i \neq j$ equations (1.3) become

$$T^{i_1 \cdots i_r}_{(j_1 \cdots j_r)_\alpha} \delta^{(h_1}_{k_1} \cdots \delta^{h_r)_\alpha}_{k_r} = 0. \tag{1.4}$$

* These conditions are obviously necessary; their sufficiency may be proved as follows: let u^j_h be an arbitrary set of n^2 numbers; multiply (1.3) by $u^{j_1}_{h_1} \cdots u^{j_r}_{h_r}$. Since $u^{j_1}_{(k_1} \cdots u^{j_r}_{k_r)_\alpha} = u^{(j_1}_{k_1} \cdots u^{j_r)_\alpha}_{k_r}$ we get $T^{(i_r)}_{(j_r)_\alpha} u^{(j_1}_{k_1} \cdots u^{j_r)_\alpha}_{k_r} = T^{(h_r)_\alpha}_{(k_r)} u^{i_1}_{(h_1} \cdots u^{i_r}_{h_r)_\alpha}$ and applying to each set of dummy indices the permutation α^{-1} which is permissible since they are dummy indices—we get

$$r!\, T^{(i_r)}_{(j_r)} u^{j_1}_{k_1} \cdots u^{j_r}_{k_r} = r!\, T^{(h_r)}_{(k_r)} u^{i_1}_{h_1} \cdots u^{i_r}_{h_r}$$

which is (1.1).

But by Theorem 1, the only solution admitted by (1.4) is $T^{i_1 \cdots i_r}_{j_1 \cdots j_r} = 0$. Hence, since $i_1 \cdots i_r$ are fixed, the equations

$$A_{(j_1 \cdots j_r)\alpha}\, \delta^{h_1}_{k_1} \cdots \delta^{h_r}_{k_r} = 0$$

have a matrix whose rank R is equal to the number of unknowns.* Returning to (1.3), suppose we replace the components of the tensor on the right hand side by an arbitrary set of n^{2r} constants $c^{h_1 \cdots h_r}_{k_1 \cdots k_r}$. The resulting equations for the determination of $T^{(i_r)}_{(j_r)\alpha}$ will be either contradictory or will have a *unique* solution since the rank of the unaugmented matrix is equal to the number of unknowns. An arbitrary choice of the c's will make the equations contradictory; to find the c's which do not lead to contradiction we solve these equations by Kramer's rule, obtaining

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_r} = C^\alpha(c)\, \delta^{i_1}_{(j_1} \cdots \delta^{i_r}_{j_r)\alpha}, \tag{1.5}$$

where C^α must be a scalar function of the c's since $T^{(i_r)}_{(j_r)}$ is a tensor. Therefore the n^{2r} constants $c^{(h_r)}_{(k_r)}$ must be linearly dependent on $r!$ constants C^α. Hence in order that (1.3) shall be consistent we must have $c^{(i_r)}_{(j_r)}$, or its equal $T^{(i_r)}_{(j_r)}$, satisfy the equations

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_r} = C^\alpha\, \delta^{i_1}_{(j_1} \cdots \delta^{i_r}_{j_r)\alpha} \tag{1.6}$$

which gives the most general form of a tensor with invariant components, C^α being a set of $r!$ arbitrary constants.

2. **An alternative proof.** If $r \leqq n$ we can choose in equations (1.3) $h_1 = k_1 = 1$, $h_2 = k_2 = 2$, $\cdots$, $h_r = k_r = r$. Then among the quantities $\delta^{(h_1}_{k_1} \cdots \delta^{h_r)\alpha}_{k_r}$ there exists only one which does not vanish, namely $\delta^{(h_1}_{k_1} \cdots \delta^{h_r)_1}_{k_r} \equiv \delta^{h_1}_{k_1} \cdots \delta^{h_r}_{k_r} = 1$. Hence (1.3) become

$$T^{(i_r)}_{(j_r)_1} \equiv T^{i_1 \cdots i_r}_{j_1 \cdots j_r} = T^{(1\,2 \cdots r)\alpha}_{1\,2 \cdots r}\, \delta^{i_1}_{(j_1} \cdots \delta^{i_r}_{j_r)\alpha}.$$

Therefore denoting the $r!$ components $T^{(1\,2 \cdots r)\alpha}_{1\,2 \cdots r}$ by C^α we get equations (1.6). To prove the generality of (1.6) for the case $r > n$ we shall use

* If the set $(j_1 \cdots j_r)$ contains s_1 ones, s_2 twos, $\cdots$, s_n n's, $\sum_{i=1}^{n} s_i = r$, the number of unknowns and the rank of the above equations is

$$R = r!/s_1!\, s_2! \cdots s_n!.$$

induction: assume that (1.6) is the most general form of a tensor with invariant components for *all values of* n and for $1, 2, \cdots, r$ pairs of indices. Let $T^{(i_{r+1})}_{(j_{r+1})}$ be a tensor with invariant components; it must satisfy equations (1.3) (with r replaced by $r+1$) which when contracted for $h_1 j_1, h_2 j_2, \cdots, h_r j_r$ and for k_{r+1} and h_{r+1} give, upon replacing $k_1 \cdots k_r$ by $j_1 \cdots j_r$ and j_{r+1} by h,

$$\begin{aligned} &T^{i_1 \cdots i_r k}_{j_1 \cdots j_r k}\, \delta^{i_{r+1}}_{h} + T^{i_1 \cdots k i_{r+1}}_{j_1 \cdots j_r k}\, \delta^{i_r}_{h} + \cdots + T^{k i_2 \cdots i_{r+1}}_{j_1 \cdots j_r k}\, \delta^{i_1}_{h} \\ &\quad = n\, T^{i_1 \cdots i_{r+1}}_{j_1 \cdots j_r h} + T^{i_1 \cdots i_{r+1}}_{j_1 \cdots h j_r} + \cdots + T^{i_1 \cdots i_{r+1}}_{h j_2 \cdots j_r j_1}. \end{aligned} \tag{2.1}$$

Considering (2.1) as equations for the determination of the components of the contracted tensor we see that since the contracted tensor is uniquely determined by the components $T^{(i_{r+1})}_{(j_{r+1})}$, we must have

$$T^{i_1 \cdots k \cdots i_r}_{j_1 \cdots k \cdots j_r} = n A^{(i_r)}_{(j_r)} + B^{(i_r)}_{(j_r)},$$

where A and B are tensor functions of $T^{(i_{r+1})}_{(j_{r+1})}$. But according to our assumption A and B, having only r pairs of indices must be of the form (1.6). Hence

$$T^{i_1 \cdots i_{\mu-1} k i_{\mu+1} \cdots i_r}_{(j_r) k} = (n \cdot c^{\alpha}_{\mu} + e^{\alpha}_{\mu})\, \delta^{i_1}_{(j_1} \cdots \delta^{i_r}_{j_r)\alpha}, \tag{2.2}$$

where $\mu = 1, \cdots, r+1$ and $\alpha = 1, \cdots, r!$

We now contract (1.3) (with r replaced by $r+1$) for h_{r+1} and k_{r+1}, getting

$$n\, T^{(i_{r+1})}_{(j_r)\alpha\, j_{r+1}}\, \delta^{(h_1}_{k_1} \cdots \delta^{h_r)\alpha}_{k_r} + T^{(i_{r+1})}_{(j_{r+1})\bar{\alpha}}\, \delta^{(h_1}_{k_1} \cdots \delta^{h_r)\bar{\alpha}}_{k_r} = T^{(h_1 \cdots h_r h)\beta}_{k_1 \cdots k_r h}\, \delta^{i_1}_{(j_1} \cdots \delta^{i_{r+1}}_{j_{r+1})\beta},$$

where $(j_{r+1})\bar{\alpha}$ is a permutation of (j_{r+1}) with j_{r+1} *not in the last place*, $(h_1 \cdots h_r)\bar{\alpha}$ being the permutation of the h's corresponding to that of $j_1 \cdots j_r$ in $(j_{r+1})\bar{\alpha}$. The $(r+1)!$ permutations of $h_1 \cdots h_r h$ may be divided into $r+1$ groups in each of which h occupies a different fixed position and each group then is of the form (2.2); hence

$$\begin{aligned} &n\, T^{(i_{r+1})}_{(j_r)\alpha\, j_{r+1}}\, \delta^{(h_1}_{k_1} \cdots \delta^{h_r)\alpha}_{k_r} + T^{(i_{r+1})}_{(j_{r+1})\bar{\alpha}}\, \delta^{(h_1}_{k_1} \cdots \delta^{h_r)\bar{\alpha}}_{k_r} \\ &\quad = \sum_{\mu=1}^{r+1} (n\, c^{\alpha}_{\mu} + e^{\alpha}_{\mu})\, \delta^{(h_1}_{(k_1} \cdots \delta^{h_r)\beta}_{k_r)\alpha}\, \delta^{i_1}_{(j_1} \cdots \delta^{i_{\mu-1}}_{j_{\mu-1}}\, \delta^{i_\mu}_{j_{\overline{r+1}}}\, \delta^{i_{\mu+1}}_{j_{\mu+1}} \cdots \delta^{i_{r+1}}_{j_\mu)\beta} \end{aligned} \tag{2.3}$$

when $j_{\overline{r+1}}$ is intended to indicate that in each permutation of the $(r+1)$ j's, j_{r+1} is held fixed. When (2.3) are contracted for $h_1 k_1, h_2 k_2, \cdots, h_r k_r$, we shall get n^r out of those terms *only* in which $h_1 \cdots h_r$ occur in the same

order as $k_1 \cdots k_r$, i. e., when $\alpha = \beta$; and since (2.3) are assumed to be true for all values of n we get by equating coefficients of n^{r+1}

$$T^{(i_{r+1})}_{(j_{r+1})} = \sum_{\mu=1}^{r+1} c^{\alpha}_{\mu}\, \delta^{i_1}_{(j_1} \cdots \delta^{i_\mu}_{j_\mu} \cdots \delta^{i_{r+1}}_{j_{r+1})_\alpha}$$

or designating the $(r+1)!$ constants c^{α}_{μ} by C^{β}, $\beta = 1, \cdots, (r+1)!$ we get

$$T^{(i_{r+1})}_{(j_{r+1})} = C^{\beta}\, \delta^{i_1}_{(j_1} \cdots \delta^{i_{r+1}}_{j_{r+1})_\beta}.$$

Hence our assumption being true for $r = 1, \cdots, n$ as was proved in the beginning of this paragraph, it is true for any value of r.

3. **Relative tensors.** The most general form of a relative tensor with invariant components can be now easily obtained from the absolute one. Let $T^{(i_r)}_{(j_s)}$ be a relative tensor of weight w having the same components in every coördinate system. We must then have—if $\Delta = \left| \frac{\partial \bar{x}}{\partial x} \right|$ and $\frac{\partial \bar{x}^i}{\partial x^j} = u^i_j$

$$T^{(i_r)}_{(j_s)}\, u^{h_1}_{j_1} \cdots u^{h_s}_{j_s} \cdot \Delta^w = T^{(k_r)}_{(j_s)}\, u^{i_1}_{k_1} \cdots u^{i_r}_{k_r}. \tag{3.1}$$

The determinant Δ is an irreducible polynomial in the variables u^i_j and therefore if w is fractional equations (3.1) can not be satisfied except by $T^{(i_r)}_{(j_s)} = 0$. We therefore assume that w is an integer and use the transformation determined by $u^i_j = c\,\delta^i_j$. Equations (3.1) then become

$$T^{(i_r)}_{(j_s)}\, c^{s+nw} = T^{(i_r)}_{(j_s)}\, c^r \tag{3.2}$$

and since (3.2) are to hold for all values of c, we must have

$$r = s + nw \tag{3.3}$$

or the tensor must be a zero tensor. We now assume that (3.3) hold and that w is positive. Then the tensor $T^{i_1 \cdots i_{s+nw}}_{j_1 \cdots j_s} \cdot \varepsilon_{h_{11} \cdots h_{1n}} \cdot \varepsilon_{h_{21} \cdots h_{2n}} \cdots \varepsilon_{h_{w1} \cdots h_{wn}}$ is an absolute tensor with invariant components and according to the results of § 1 must have the form $\bar{C}^{\alpha}\, \delta^{i_1}_{(j_1} \cdots \delta^{i_s}_{j_s}\, \delta^{i_{s+1}}_{h_{11}} \cdots \delta^{i_r}_{h_{wn})_\alpha}$, where α is summed from 1 to $r!$. When the two expressions above are multiplied by $\varepsilon^{h_{11} \cdots h_{1n}} \cdots \varepsilon^{h_{w1} \cdots h_{wn}}$ and the corresponding indices are summed out we get

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_s} = C^{\alpha}\, \delta^{i_1}_{(j_1} \cdots \delta^{i_s}_{j_s}\, \delta^{i_{s+1}}_{h_{11}} \cdots \delta^{i_r}_{h_{wn})_\alpha}\, \varepsilon^{h_{11} \cdots h_{1n}} \cdot \varepsilon^{h_{21} \cdots h_{2n}} \cdots \varepsilon^{h_{w1} \cdots h_{wn}} \tag{3.4}$$

since* $\varepsilon_{h_1 \cdots h_n} \varepsilon^{h_1 \cdots h_n} = n!$, C^{α} being $\frac{\bar{C}^{\alpha}}{(n!)^w}$.

* For the definition of $\varepsilon^{h_1 \cdots h_n}$ and its properties that are used here see Oswald Veblen's Cambridge Tract No. 24, Chap. I.

In a similar manner we can show that if w is negative, the tensor must take the form

$$(3.5)\quad T^{i_1\cdots i_r}_{j_1\cdots j_s} = C^\alpha \delta^{i_1}_{j_1} \cdots \delta^{i_r}_{j_r} \delta^{h_{11}}_{j_{r+1}} \cdots \delta^{h_{wn}}_{j_s)_\alpha} \varepsilon_{h_{11}\cdots h_{1n}} \varepsilon_{h_{21}\cdots h_{2n}} \cdots \varepsilon_{h_{w1}\cdots h_{wn}}.$$

Combining these results, we have

THEOREM 2. *The only non-vanishing relative tensors with invariant components are those of integral weight satisfying* (3.3) *and having the form* (3.4) *if the weight is positive and* (3.5) *if the weight is negative.*

The preceding results are still valid if we replace the word tensor by tensor-field provided that the constants C^α are replaced by absolute scalar functions $C^\alpha(x^1, \cdots, x^n)$.

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LINEAR DIFFERENTIAL EQUATIONS OF INFINITE ORDER, WITH POLYNOMIAL COEFFICIENTS OF DEGREE ONE.*

By I. M. Sheffer.

1. **Introduction.** The writer has shown† how linear differential equations of infinite order arise from the consideration of systems of linear equations in infinitely many unknowns, and has discussed the case of systems of differential equations whose coefficients are constants. In the present paper there is treated the case where the coefficients are polynomials of degree not exceeding one. The existence theorems for the general case of polynomial coefficients of *bounded degree* have been obtained by von Koch, Perron and Hilb.‡ The treatment of von Koch involves the consideration of the coefficients of certain power series which he sets up; the method does not appear to define precisely the number of linearly independent solutions (of the homogeneous equation). Perron proves the existence theorems relating to such equations, using some theorems he has proved on the theory of infinitely many linear equations in infinitely many unknowns.§ His method is very neat, insofar as concerns the existence proofs. Hilb, in his three papers on the subject, uses the theory of quadratic forms in infinitely many variables and the theory of Laplace's transformation. He obtains the existence theorems, and is also able to express the solutions in series.

In considering the case of degree one, we shall indicate to what extent the equation can be solved by means of an expansion in a series of characteristic functions, and how by an operational method the problem can be led back to the discussion of a linear differential equation of *finite order,* of Fuchsian type.

In § 2 we are led in a very natural manner to the consideration of a function $D(t)$ whose behavior at certain critical points determines the character of the solutions and, to some extent, the number of solutions.

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† Ann. of Math., vol. 30 (1929), p. 250-264. We shall refer to this paper as Annals I.

‡ H. von Koch, Arkiv för Mat., Astr. och Fys., vol. 15 (1921), No. 26. Perron, Math. Ann., vol. 84 (1921), pp. 31–42. Hilb, Math. Ann., vol. 82 (1921), pp. 1–39; vol. 84 (1921), pp. 16–30; pp. 43–52.

§ Perron, Math. Ann., vol. 84 (1921), pp. 1–15.

In this section we treat the case where $D(t)$ is analytic in the region considered. We are then able to give a complete discussion of the equation.

When $D(t)$ is not analytic throughout a certain region, the operational method that we use apparently ceases to apply.* We then introduce, in § 3, the Hurwitz polynomials, in expansions of which we obtain a series-solution of an auxiliary equation. To discuss the convergence we introduce a function $f(t;x)$ whose Taylor coefficients are the Hurwitz polynomials. Some properties of $f(t;x)$ are given, and by means of this function we can establish the existence of at least one solution of the auxiliary equation.

A study of characteristic functions is made in § 4. On introducing a parameter into the equation (where here we take a simple case to start with) we arrive at a set of characteristic values which furnish us corresponding solutions of the homogeneous equation. We consider a range of functions which can be developed in series of these characteristic functions, and in this way obtain a solution of the equation for a subclass of functions. The method of characteristic functions does not seem to yield convergence for all functions with which we are concerned. But even when divergent we are guided formally (§ 5) to a new way in which we may apply the operational method used in Annals I. This method enables us to reduce the problem to a linear differential equation of finite order, of Fuchsian type. We can then solve completely this case.

In § 6 we take up a slightly more general case, which can be treated similarly; and finally in § 7 we treat the general case of polynomial coefficients of degree one.

2. **The function $D(t)$ analytic.** We consider a differential equation of infinite order, whose coefficients are polynomials of degree not exceeding one:

$$\text{(I)} \qquad \sum_{n=0}^{\infty} (a_n + b_n x)\, y^{(n)}(x) = 0,$$

$$\text{(II)} \qquad \sum_{n=0}^{\infty} (a_n + b_n x)\, y^{(n)}(x) = g(x).$$

Set

$$\text{(1)} \qquad A(t) = \sum_{0}^{\infty} a_n t^n, \qquad B(t) = \sum_{0}^{\infty} b_n t^n.$$

We assume that $A(t)$, $B(t)$ are analytic, $|t| < R$, and that $g(x)$ is of exp. val. $= q < R$. Then (I) and (II) can equally well be written†

* Later we return to show that this method, suitably modified, does apply.

† For the meaning of the terms used, and for the proof of the validity of certain operations that we shall make, see Annals I.

By $A[y]$ is understood the differential operator $a_0 y(x) + \cdots + a_n y^{(n)}(x) + \cdots$.

(I) $$A[y]+xB[y] = 0,$$

(II) $$A[y]+xB[y] = g(x).$$

We shall want

LEMMA 1: *If $D(t)$ is analytic, $|t|<R$, and $h(x)$ is of exp. val. $<R$, then*

$$D[xh] = D'[h]+xD[h].$$

Proof: $(xh)^{(n)} = xh^{(n)}+nh^{(n-1)}$, so that

$$D[xh] = \sum_0^\infty d_n(xh)^{(n)} = xD[h]+D'[h].$$

COROLLARY: *If $B(t)$ is also analytic, $|t|<R$, then*

$$D[xB[h]] = BD'[h]+xBD[h].$$

Let $D(t)$ be analytic, $|t|<R$, and operate with it on (I). Using Cor., Lemma 1, we obtain $(AD+BD')[y]+xBD[y] = 0$. Let us choose $D(t)$ as a solution of

(2) $$A(t)\,D(t)+B(t)\,D'(t) = 0,$$

and assume that (2) defines $D(t)$ to be analytic, $|t|<R$. Then (I) reduces to $xBD[y] = 0$, or

(I′) $$BD[y] = 0.$$

Let $\xi_1, \cdots, \xi_r$ be the distinct zeros of $B(t)$ in $|t|<R$. It is clear from equation (2) that any zeros that $D(t)$ may possess (in $|t|<R$) are among the ξ_i's. Let the order of the zeros of $B(t)\,D(t)$ at $t=\xi_i$ be p_i. Then the most general solution of (I′) is*

$$y(x) = \sum_{i=1}^{r}(c_{i,p_i-1}\,x^{p_i-1}+\cdots+c_{i0}\,x^0)\,e^{\xi_i x}.$$

We wish to determine the number of linearly independent solutions of (I). To do this we substitute into (I) the expression†

(i) $$y(x) = (c_{i,p_i-1}\,x^{p_i-1}+\cdots+c_{i0}\,x^0)\,e^{\xi_i x}.$$

Recalling (Annals I, Lemma 3) that

$$E[x^s e^{\xi x}] = e^{\xi x}\left[E(\xi)x^s+\binom{s}{1}E'(\xi)\,x^{s-1}+\cdots+\binom{s}{s}E^{(s)}(\xi)\,x^0\right],$$

we have $E[y] = e^{\xi_i x}P(x)$, $P(x)$ a polynomial of degree $\leqq p_i-1$.

* Annals I, Theorem 2.

† We may (see Annals I) consider the number of linearly independent solutions involving each ξ_i separately, and then sum over the number of distinct zeros.

If we replace E by A, B successively, we find that $A[y]+xB[y]=e^{\xi_i x}Q(x)$, where $Q(x)$ is a polynomial of degree $\leqq p_i$. Since the right hand side is to be zero, every coefficient in $Q(x)$ must vanish. This gives us the equations

$$
\begin{aligned}
&c_{i,p_i-1}\,B(\xi_i) = 0,\\
&c_{i,p_i-1}\left[\binom{p_i-1}{n}B^{(n)}(\xi_i)+\binom{p_i-1}{n-1}A^{(n-1)}(\xi_i)\right]\\
&\qquad + c_{i,p_i-2}\left[\binom{p_i-2}{n-1}B^{(n-1)}(\xi_i)+\binom{p_i-2}{n-2}A^{(n-2)}(\xi_i)\right] \qquad \text{(a)}\\
&\qquad + \cdots + c_{i,p_i-n-1}\,B(\xi_i) = 0\\
&\qquad\qquad\qquad\qquad (n = 1, 2, \cdots, p_i-1),\\
&c_{i,p_i-1}\binom{p_i-1}{p_i-1}A^{(p_i-1)}(\xi_i)+c_{i,p_i-2}\binom{p_i-2}{p_i-2}A^{(p_i-2)}(\xi_i)+\cdots+c_{i0}\,A(\xi_i) = 0
\end{aligned}
$$

as necessary and sufficient conditions that $y(x)$ (involving only ξ_i) satisfy (I).

To take up the question of solving equations (a), we need some relations concerning $A(t)$, $B(t)$ and their derivatives, at $t=\xi_i$. Let ξ_i be a zero of order n of $B(t)$: $B(t)=(t-\xi_i)^n[\beta_0+\beta_1(t-\xi_i)+\cdots]$, $\beta_0\neq 0$. If we solve (2) we find

$$D(t) = e^{-\int\frac{A(t)}{B(t)}dt}, \qquad (3)$$

and since our assumption is that $D(t)$ is analytic, $|t|<R$, it follows that $A(t)$ must have a zero at $t=\xi_i$ of order at least $n-1$:

$$A(t) = (t-\xi_i)^{n-1}[\alpha_0+\alpha_1(t-\xi_i)+\cdots]$$

and $-\alpha_0/\beta_0$ must be a positive integer or zero, say s: $s\beta_0+\alpha_0=0$ and $(s+n)$ is the order of the zero of $B(t)\,D(t)$ at $t=\xi_i$: $s+n=p_i$. But $(n-1)!\,\alpha_0 = A^{(n-1)}(\xi_i)$, and $n!\,\beta_0 = B^{(n)}(\xi_i)$. Therefore

$$sB^{(n)}(\xi_i)+nA^{(n-1)}(\xi_i) = 0, \quad \text{with } B^{(n)}(\xi_i)\neq 0. \qquad \text{(ii)}$$

We can now return to equations (a). Two cases arise:

1. $A^{(n-1)}(\xi_i)\neq 0$. Then s is positive. The first n equations of (a) drop out, as does also (using (ii)) the $(n+1)$st. Therefore in the succeeding equations, c_{i,p_i-1} may be chosen arbitrarily, after which $c_{i,p_i-2}, c_{i,p_i-3}, \cdots, c_{i,p_i-s} = c_{i,n}$ are uniquely determined from the $(n+2)$d equation down to the next to last. This leaves $c_{i,n-1}, c_{i,n-2}, \cdots, c_{i,0}$ to be determined to satisfy the last equation. Now the coefficient of $c_{i,n-1}$ in this equation is $A^{(n-1)}(\xi_i)\neq 0$. We can then solve for $c_{i,n-1}$, so that $c_{i,n-2}, \cdots, c_{i,0}$ are arbitrary. Together

with the arbitrariness of c_{i,p_i-1}, we see that corresponding to $t = \xi_i$ there are precisely n linearly independent solutions of (I), in Case 1.

2. $A^{(n-1)}(\xi_i) = 0$. Then $s = 0$. That is, $p_i = n$. In this case *all* the equations of (a) drop out, so that $c_{i,0}, \cdots, c_{i,n-1}$ are all arbitrary. This again gives n linearly independent solutions of (I) corresponding to $t = \xi_i$.

Hence if we let i run through the values $i = 1, 2, \cdots, r$, we obtain in all m linearly independent solutions of (I), where $m =$ number of zeros of $B(t)$ in $|t| < R$. That is:

THEOREM 1: *Let $A(t)$, $B(t)$ be analytic, $|t| < R$, and let $B(t)$ have m zeros in $|t| < R$. If (2) defines $D(t)$ to be analytic, $|t| < R$, then the homogeneous equation (1) has precisely m linearly independent solutions of exp. val. $< R$. Moreover, if $\xi_1, \cdots, \xi_r$ are the distinct zeros of $B(t)$, of order $n_1, \cdots, n_r$, respectively, then these solutions can be so chosen that n_i of them correspond to the zero ξ_i.*

Turning now to the non-homogeneous equation (II) we have

THEOREM 2: *Let the same hypotheses be made on $A(t)$, $B(t)$, and let $g(x)$ be of exp. val. $q < R$. If (2) defines $D(t)$ to be analytic, $|t| < R$, then a necessary and sufficient condition that the non-homogeneous equation (II) have a solution $y(x)$ of exp. val. $< R$ is that $G(x) = D[g]$ vanish at the origin. If this condition is satisfied, then there is a solution $y(x)$ of exp. val. $\leqq q$, and the general solution is obtained by adding in the general solution of the homogeneous equation (I).**

Proof: *Necessity:* Let y (of exp. val. $< R$) satisfy (II). If we operate on (II) with D we obtain, on using (2):

$$\text{(II')} \qquad xBD[y] = G(x), \quad \text{or} \quad \text{(II'')} \quad BD[y] = G(x)/x,$$

where $G(x) = D[g]$. Since $BD[y]$ is analytic at $x = 0$, we have $G(0) = 0$.

Sufficiency: $G(x)/x = D[g]/x$ is of exp. val. $\leqq q$. (II''), and hence (II'), has a solution $y_0(x)$ of exp. val. $\leqq q$. Moreover, to $y_0(x)$ we can add any solution of $BD(y) = 0$. If we use the relation $xBD[y_0] = D[xB[y_0]] - D'B[y_0]$ (Cor., Lemma 1), (II') becomes, on using (2),

$$\text{(a)} \qquad D[A[y_0] + xB[y_0] - g(x)] = 0.$$

Then, using the notation given in the proof of Theorem 1,

$$\text{(iii)} \qquad A[y_0] + xB[y_0] - g(x) = \sum_{i=1}^{n} (d_{i,p_i-n_i-1}\, x^{p_i-n_i-1} + \cdots + d_{i0}\, x^0)\, e^{\xi_i x}.$$

* We remark that it is not necessary to require that $D(t)$ be analytic in the whole range $|t| < R$, to obtain a solution of exp. val. $\leqq q$; it suffices if $D(t)$ is analytic in the interior of any circle (center at the origin) whose radius is greater than q.

Application to (a) of Cor., Lemma 3 of Annals I tells us that $A[y_0]+xBD[y_0]-g(x)$ cannot involve any zero ξ_i of $B(t)$ which is not also a zero of $D(t)$. We need consider, then, only those ξ_i's which are zeros of both $B(t)$ and $D(t)$; and as usual, we take each ξ_i separately. Furthermore, since $A[y_0]+xB[y_0]-g(x)$ is of exp. val. $\leqq q$, we need consider only those zeros which lie in $|\xi_i| \leqq q$. Let ξ_i be such a zero, and consider (i) $y_i(x) = (c_{i,l_i-1}x^{l_i-1}+\cdots+c_{i0}x^0)e^{\xi_i x}$. We wish to show that the c_i's and l_i can be chosen to satisfy the equation

$$\text{(b)} \qquad A[y_i]+xB[y_i] = -(d_{i,p_i-n_i-1}x^{p_i-n_i-1}+\cdots+d_{i0}x^0)e^{\xi_i x}.$$

On substituting (i) into (b) and equating coefficients of like terms, we arrive at a set of equations (c) similar* to the equations (a) of Theorem 1.

Let us choose $l_i = p_i$. The first (n_i+1) equations of (c) drop out, as in the proof of Theorem 1. In the (n_i+k)th equation $(k \geqq 2)$, the coefficient of c_{i,p_i-k} is found to be $\binom{p_i-k}{n_i-1}[(1-k)B^{(n_i)}(\xi_i)]/n_i$, which is $\neq 0$. We can then find c's to satisfy the equations (c),† and therefore we have a solution $y_i(x)$ of (b). If we add these solutions $y_i(x)$ $(i = 1, \cdots, r)$ to the function $y_0(x)$, we annihilate the terms of the righthand member of (iii), and so obtain a solution $y(x)$ of (II). This solution is of exp. val. $\leqq q$. This completes the proof.

It may happen that $D(t)$ is analytic at some, but not all, of the zeros of $B(t)$. In such a case we have

THEOREM 3: *If in the equation*

$$\text{(I)} \qquad A[y]+xB[y] = 0,$$

$A(t)$, $B(t)$ are analytic, $|t|<R$, and if at some zero $t = \xi$ of $B(t)$ of order p, the function $D(t)$ defined by

$$\text{(2)} \qquad A(t)D(t)+B(t)D'(t) = 0$$

is analytic, then there exist precisely p linearly independent solutions of (I) *of the form $y(x) = e^{\xi x}Q(x)$, $Q(x)$ a polynomial.*

Proof: We know that $A(t)$ has a zero of order at least $p-1$ at $t = \xi$, and that there exists an integer s positive or zero such that

$$\text{(i)} \qquad sB^{(n)}(\xi)+nA^{(n-1)}(\xi) = 0, \qquad B^{(n)}(\xi) \neq 0.$$

* Equations (c) are obtained from (a) as follows: In the left members of (a) replace p_i by l_i, and n by n_i. On the right, replace the values 0, 0, ⋯, 0 by 0, ⋯, 0, $-d_{i,p_i-n_i-1}$, ⋯ ⋯, $-d_{i,1}$, $-d_{i,0}$.

† In the last equation we use the fact that $A^{(n_i-1)}(\xi_i) \neq 0$.

If we assume a solution $y(x) = e^{\xi x}(c_k x^k + \cdots + c_0 x^0)$ and substitute into (I) we arrive at a set of equations like the equations (a) of Theorem 1. The argument now proceeds as before, to the desired conclusion.

3. The function $f(t; x)$ and the Hurwitz polynomials. When we turn to equations (I), (II) of § 2 where $D(t)$ as defined by (2) is not everywhere analytic in $|t| < R$,* the situation is different, as we shall see later. But in this section we wish to point out one property that is common to the two cases. And that is that *if* $B(0) \neq 0$, *we can always solve*

$$A[y] + xB[y] = g(x) \tag{II}$$

to within a constant; i. e., given $g(x)$ of exp. val. $q < R$, then $y(x)$ exists such that $A[y] + xB[y]$ differs from $g(x)$ by at most a constant.

That this is true for the case that $D(t)$ is everywhere analytic in $|t| < R$ follows from Theorem 2, since by hypothesis $B(0) \neq 0$.† Let us then establish it in the contrary case.

On differentiating (II) we obtain the equation

$$H[y] + xK[y] = h(x) \tag{1}$$

where $H(t) = tA(t) + B(t)$, $K(t) = tB(t)$, $h(x) = g'(x)$. With (1) we consider the equation

$$H[y_n] + xK[y_n] = x^n. \tag{2}$$

Since $B(0) \neq 0$, there exists a unique polynomial $y_n(x)$, whose degree is n, satisfying (2). We propose to obtain some fundamental properties of these polynomials to which we may give the name of *Hurwitz polynomials*.‡

On integrating (2) we obtain

$$A[y_n] + xB[y_n] = x^{n+1}/(n+1) + l_{n+1}, \quad n = 0, 1, \cdots. \tag{3}$$

Now $D(t) = \sum_0^\infty d_n t^n$ is analytic for $|t|$ sufficiently small, and since $x^{n+1}/(n+1) + l_{n+1}$ is of exp. val. zero, we may operate on (3) with D:

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* Or in $|t| < q + \varepsilon$, $\varepsilon > 0$ arbitrary, if we are dealing with (II) where $g(x)$ is of exp. val. $q < R$.

† For: $B(0) \neq 0$ implies $D(0) \neq 0$.

‡ In a very interesting paper in Acta Math., vol. 20 (1896/97), pp. 285–312, Hurwitz proves the existence of an entire function solution of the difference equation $u(x+1) - u(x) = g(x)$, $g(x)$ entire, by introducing the polynomial solutions $u_n(x)$, of degree $(n+1)$, of $u(x+1) - u(x) = x^n$.

$$BD[y_n] = (1/x)\,(D[x^{n+1}/(n+1)+l_{n+1})])_0^x$$
$$= n!\left[d_0\frac{x^n}{(n+1)!}+d_1\frac{x^{n-1}}{n!}+\cdots+d_n\frac{x^0}{1!}\right], \tag{4}$$

so that $\sum_0^\infty BD[y_n]\,t^n/n! = D(t)\,(e^{tx}-1)/tx$. We may operate term-wise with $1/BD$, obtaining the following result:

LEMMA 2: *The series*

$$f(t;x) = \sum_0^\infty y_n(x)\,t^n/n! \tag{5}$$

is uniformly convergent, $|x| \leqq X$ *arbitrary,* $|t| \leqq \sigma < \varrho$ *where* ϱ = *absolute value of smallest zero of* $B(t)$ *in* $|t| < R$; *and*

$$f(t;x) = D(t)\cdot\frac{1}{BD}\left[\frac{e^{tx}-1}{tx}\right],\ |t|<\varrho. \tag{6}$$

Furthermore, $f(t;x)$ *is of exp. val.* $\leqq |t|$ *for every* t *in* $|t|<\varrho$.

On expanding (6) we find

COROLLARY 1:

$$f(t;x) = D(t)\int_0^t e^{tx}\,dt/t\,B(t)\,D(t). \tag{7}$$

(7) was established for the region $|t|<\varrho$. But we can continue* the definition of $f(t;x)$ by (7) to all points of $|t|<R$ with the (possible) exception of the zeros of $B(t)$. We denote by T the region $|t|<R$ with the zeros of $B(t)$ deleted. Then from (7): $f(t;x)$ *is of exp. val.* $\leqq |t|$ *in* T.

If we differentiate (7) with regard to t and use the relation $D' = -AD/B$, we arrive at a differential equation for f:

COROLLARY 2:

$$A(t)\,f(t;x) + B(t)\frac{\partial f(t;x)}{\partial t} = e^{tx}/t. \tag{8}$$

Let us now form $A[f]+xB[f]$, using (6). On simplifying this reduces to $A[f]+xB[f] = e^{tx}/t - D(t)/tD(0)$, so that on differentiating with respect to x we return to the operators H, K:

COROLLARY 3:

$$H[f]+xK[f] = e^{tx}. \tag{9}$$

This holds for every t in T.†

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* $f(t;x)$ may then be multiple-valued. In such case we consider any one of its branches.

† That this relation holds *formally* is easily shown: we have but to multiply each member of (2) by $t^n/n!$ and sum (formally) from $n=0$ to $n=\infty$.

COROLLARY 4: *From* (5) *we have by Cauchy's formula*:

$$y_n(x) = \frac{n!}{2\pi i}\int_C \frac{f(t;x)}{t^{n+1}}\,dt, \tag{10}$$

C being a closed curve surrounding $t=0$ *and lying wholly in* $|t|<\varrho$.

Since $y_n(x)$ satisfies (2), it follows that a formal solution of (1) $H[y]+xK[y]=h(x)$ is given by

$$y(x) = \sum_0^\infty h_n\, y_n(x), \tag{11}$$

where $h(x) = \sum_0^\infty h_n x^n$. Let us take up the question of convergence.

Let T' be any *closed* region lying wholly in T. To every $\varepsilon>0$ corresponds an M, independent of t in T', such that

$$f(t;x) \ll M e^{(p+\varepsilon)x}, \tag{12}$$

where p is the maximum distance of points of T' from $t=0$. An inspection of (7) shows this.

Let ϱ again be the absolute value of the smallest zero of $B(t)$. Suppose $h(x)$ is of exp. val. $q<\varrho$. Then $|h_n| \leqq N(q+\varepsilon)^n/n!$. Choose T' as the region $|t|\leqq\varrho'$, where $q+\varepsilon<\varrho'<\varrho-\varepsilon$. Then $f(t;x)\ll M e^{(\varrho'+\varepsilon)x}$. Now choose the contour of (10) as the circle $|t|=\varrho'$. Then $|y_n(x)| \leqq M e^{(\varrho'+\varepsilon)|x|}\, n!/\varrho'^n$, and $\sum_0^\infty |h_n\, y_n(x)| \leqq MN e^{(\varrho'+\varepsilon)|x|} \sum_0^\infty \left(\frac{q+\varepsilon}{\varrho'}\right)^n$. The series on the right converges, so that $y(x)=\sum_0^\infty h_n\, y_n(x)$ converges uniformly, $|x|\leqq X$ arbitrary, and is (clearly) of exp. val. $\leqq \varrho'+\varepsilon$. But $\varepsilon>0$ is arbitrary, and $\varrho'>q$ is arbitrary. Therefore $y(x)$ is of exp. val. $\leqq q$.

$y(x)$ *satisfies* (1). For: (10) and (12) enable us to show that the formal processes are legitimate.

If now $h(x)$ is of exp. val. $q\geqq\varrho$, the formal series (11) may not converge. To insure that a solution of (1) exists even in this case, we proceed to set up a new formal series which we shall prove to be a convergent series.

We assume that $h(x)$ is of exp. val. $q<R$. Then $|h_n| \leqq N(q+\varepsilon)^n/n!$ Choose ϱ' in $q+\varepsilon<\varrho'<R-\varepsilon$, and consider the region T': $|t|\leqq\varrho'$ with small circles whose centers are at the zeros of $B(t)$ deleted. Then (from (12)) $f(t;x)\ll M e^{(\varrho'+\varepsilon)x}$. Finally consider the contour C: $|t|=\varrho'$, and define the functions $z_n(x)$, $n=0, 1, \cdots$ by the relations

$$z_n(x) = \frac{n!}{2\pi i}\int_C \frac{f(t;x)}{t^{n+1}}\,dt. \tag{13}$$

From (12) and (13) we find $|z_n(x)| \leqq Me^{(\varrho'+\varepsilon)|x|}\, n!/\varrho'^n$, and $\sum_0^\infty |h_n z_n(x)| \leqq MNe^{(\varrho'+\varepsilon)|x|} \sum_0^\infty \left(\frac{q+\varepsilon}{\varrho'}\right)^n$, so that

$$y(x) = \sum_0^\infty h_n z_n(x) \tag{14}$$

converges uniformly in $|x| \leqq X$ arbitrary, and represents an entire function of exp. val. $\leqq q$. (12) and (13) enable us to justify operating on (14) term-by-term with the operator $H[y] + xK[y]$, so that (14) is a solution of (1).

We have shown that (1) always has a solution, if $h(x)$ is of exp. val. $< R$. From this follows that (II) can always be solved to within a constant, as was asserted.*

4. **Characteristic functions and certain series developments in terms of them.** We take up the question of the solution of the equations

$$A[y] + xB[y] = 0, \tag{I}$$

$$A[y] + xB[y] = g(x) \tag{II}$$

from another point of view. *We may without loss of generality assume that $B(t)$ is a polynomial.*†

Let us take up the case $B(t) = t - \xi$, where $|\xi| < R$, $A(t)$ being analytic in $|t| < R$. We introduce a parameter which takes on certain *characteristic values* corresponding to which we obtain solutions of the homogeneous equation. Consider the equation

$$\lambda y + {}_1A[y] + x(y' - \xi y) = 0, \tag{1}$$

where ${}_1A(t) = A(t) - A(0)$. There exists a denumerable infinity of values of λ, say λ_n, for which the corresponding operating function $D_n(t)$ is analytic in $|t| < R$.

* If $B(0) = 0$, a transformation of the form $y(x) = e^{sx}u(x)$, where s is a suitably chosen small constant, will yield a new equation in which the "new" $B(t)$ does not vanish at the origin. We need not then consider this case.

† For: Suppose $B(t)$ has a finite number, m, of zeros in $|t| < R$. Let $P(t)$ be the polynomial of degree m with these zeros, so that $B(t) = P(t)C(t)$, $C(t) \neq 0$, $|t| < R$. We may then operate on (I) or (II) with $1/C$, the resulting equation being of the form $G[y] + xP[y] = h(x)$ or zero. On the other hand, if $B(t)$ has infinitely many zeros in $|t| < R$, there can be no cluster point *in* $|t| < R$. We may then consider $|t| < R'$, where R' is very close to R, and less than R. In $|t| < R'$, $B(t)$ has only a finite number of zeros, which reduces to the first case. It is clear that the equation obtained after operating is completely equivalent to the original equation.

Proof: Since $D(t) = e^{-\int \frac{\lambda + {}_1A(t)}{t-\xi} dt}$, the condition for analyticity is that $-(\lambda + {}_1A(\xi)) = n$, $n = 0, 1, \cdots$. We set $\lambda = \lambda_n$ for the value n:

$$\lambda_n = -[n + {}_1A(\xi)], \quad n = 0, 1, \cdots. \tag{2}$$

For $\lambda = \lambda_n$, our previous theory informs us that there is precisely one solution $y(x) = u_n(x)$ of exp. val. $< R$, of (1), and that this is of the form

$$u_n(x) = e^{\xi x} P_n(x), \tag{3}$$

where $P_n(x)$ is a polynomial. *$P_n(x)$ is of degree n.*

Proof: Suppose the degree of $P_n(x)$ is s. Substituting (3) into (1):

$$\lambda_n e^{\xi x} P_n(x) + {}_1A[e^{\xi x} P_n(x)] + x(e^{\xi x} P_n'(x)) = 0. \tag{1'}$$

On expanding and cancelling the common factor $e^{\xi x}$ we find that $s = n$ in virtue of (2), and that the following relation holds:

$$-nP_n(x) + \frac{{}_1A'(\xi)}{1!} P_n'(x) + \cdots + \frac{{}_1A^{(n)}(\xi)}{n!} P_n^{(n)}(x) + xP_n'(x) = 0. \tag{4}$$

We wish to investigate the solutions of (II) in terms of expansions in series of the u_n-functions. It is necessary, for this purpose, to know more about the polynomials $P_n(x)$. (They are determined only up to a multiplicative constant.)

The necessary information is afforded us by returning to the operating function $D(t)$ defined by the lefthand member of (I). As we know,

$$D(t) = e^{-\int \frac{A(t)\,dt}{t-\xi}} = (t-\xi)^{-A(\xi)} e^{-\int_{t=\xi}^{t} \left[\frac{A'(\xi)}{1!} + \frac{A''(\xi)}{2!}(t-\xi) + \cdots\right] dt}.$$

Define

$$C(t-\xi) = e^{\int_{t=\xi}^{t} \frac{A(t)-A(\xi)}{t-\xi} dt}. \tag{5}$$

Then $D(t) = (t-\xi)^{-A(\xi)}[C(t-\xi)]^{-1}$. Since $A(t)$ is analytic, $|t| < R$, the same is true of $C(t-\xi)$. Moreover, if we set

$$C(u) = c_0 + c_1 u + c_2 u^2 + \cdots, \tag{6}$$

this series converges in a circle about $u = 0$ of radius at least as great as $\varrho = R - |\xi|$.

Let us now introduce the set of polynomials $Q_n(x)$ defined by

$$Q_n(x) = c_0 \frac{x^n}{n!} + c_1 \frac{x^{n-1}}{(n-1)!} + \cdots + c_0 \frac{x^0}{0!}. \tag{7}$$

It is an immediate consequence of (7) that

$$C(u)\, e^{ux} = \sum_0^\infty Q_n(x)\, u^n, \qquad |u| < \varrho. \tag{8}$$

Furthermore,

$$Q_n'(x) = Q_{n-1}(x), \qquad Q_0(x) = c_0 = 1, \tag{9}$$

and

$$Q_n(x) = \frac{1}{2\pi i} \int_C \frac{C(u)\, e^{ux}}{u^{n+1}}\, du, \qquad n = 0, 1, \cdots, \tag{10}$$

where C is a contour around $u = 0$ lying wholly within $|u| < \varrho$.

We can now establish the following relation:*

LEMMA 3:

$$Q_n(x) \equiv P_n(x).$$

Proof: If we attempt to determine the coefficients of $P_n(x)$ by substituting a polynomial of degree n into (4), we find that the coefficients are uniquely determined—up to the multiplicative constant. It suffices then to show that $Q_n(x)$ satisfies (4). Let

$$L_n(x) \equiv -n\, Q_n(x) + \frac{{}_1A'(\xi)}{1!}\, Q_n'(x) + \cdots + \frac{{}_1A^{(n)}(\xi)}{n!}\, Q_n^{(n)}(x) + x\, Q_n'(x).$$

We shall show that $L_n(x) \equiv 0$. Since $Q_n^{(m)}(x) = 0$, $m > n$, we may write

$$L_n(x) = -n\, Q_n(x) + \sum_{k=1}^{\infty} \frac{{}_1A_{(\xi)}^{(k)}}{k!}\, Q_n^{(k)}(x) + x\, Q_n'(x).$$

On using (9) and (10) this gives

$$L_n(x) = \frac{1}{2\pi i} \int_C \frac{\Phi(u)\, du}{u^{n+1}},$$

where $\Phi(u) = e^{ux + \int_{u=0}^{u} \frac{A(u+\xi) - A(\xi)}{u} du} \cdot [-(n + {}_1A(\xi)) + {}_1A(u+\xi) + xu]$.

We need then to show that $d^n \Phi(0)/du^n = 0$. Set

$$\Psi(u) = e^{ux + \int_0^u [A(u+\xi) - A(\xi)]\, du/u}.$$

* It is understood that the multiplicative constant in $P_n(x)$ is properly chosen.

Then

$$\Psi'(u) = \psi(u)\left[x + \frac{1}{u}(A(u+\xi) - A(\xi))\right],$$

$$\text{and } \Phi(u) = -n\Psi(u) + u\Psi'(u).$$

Using Leibnitz's Rule for differentiation, we then find that $d^n\Phi(0)/du^n = 0$. That is, $L_n(x) \equiv 0$.

We have then for $Q_n(x)$ the relation

$$(11)\quad -nQ(x) + \frac{{}_1A'(\xi)}{1!}Q_n'(x) + \cdots + \frac{{}_1A^{(n)}(\xi)}{n!}Q_n^{(n)}(x) + xQ_n'(x) = 0;$$

and we may take as the solution $u_n(x)$ of (1) the expression

$$(12)\qquad u_n(x) = e^{\xi x}Q_n(x).$$

We consider now the expansion of a function $f(x)$ in a Q_n-series. The definition of $C(u)$ shows that $[C(u)]^{-1}$ is analytic in $|u| < \varrho$. Set

$$(13)\qquad 1/C(u) = \sigma_0 + \sigma_1 u + \sigma_2 u^2 + \cdots.$$

Then from (8) we have

$$(14)\qquad x^n/n! = \sigma_0 Q_n(x) + \sigma_1 Q_{n-1}(x) + \cdots + \sigma_n Q_0(x).$$

Suppose now that $f(x) = \sum_0^\infty f_n x^n$. Then formally,

$$f(x) = \sum_{n=0}^{\infty} n!\, f_n[\sigma_0 Q_n(x) + \cdots + \sigma_n Q_0(x)]$$

$$\sim \sum_{n=0}^{\infty} Q_n(x)[\sigma_0 n!\, f_n + \sigma_1 (n+1)!\, f_{n+1} + \cdots],$$

so that

$$f(x) \sim \sum_{n=0}^{\infty} \alpha_n Q(x), \qquad \alpha_n \sim \sigma_0 n!\, f_n + \sigma_1 (n+1)!\, f_{n+1} + \cdots.$$

It is easy to justify these operations for the case where $f(x)$ is of exp. val. $< \varrho$. We shall not stop over this, but shall rather obtain a "closed" form for α_n, after which we shall consider conditions under which the expansion of $f(x)$ is valid. Let us define

$$(15)\qquad F(x) = \sum_0^\infty n!\, f_n x^n = \sum_0^\infty F_n x^n, \qquad F_n = n!\, f_n,$$

which is convergent for $|x|<1/\lambda$ if $f(x)$ is of exp. val. $=\lambda$. Then $F_n=\frac{1}{2\pi i}\int_C F(u)\,du/u^{n+1}$, so that formally,

$$\alpha_n=\frac{1}{2\pi i}\int_C F(u)\left[\sigma_0+\frac{\sigma_1}{u}+\frac{\sigma_2}{u^2}+\cdots\right]\frac{du}{u^{n+1}}.$$

That is,

$$\alpha_n=\frac{1}{2\pi i}\int_C\frac{F(u)\,du}{C\left(\frac{1}{u}\right)u^{n+1}}. \tag{16}$$

We can now prove

LEMMA 4: *If* $f(x)=\sum_0^\infty f_n x^n$ *is of exp. val.* $\lambda<\varrho=R-|\xi|$, *and* $F(x)$ *is defined by* (15), *then*

$$f(x)=\sum_0^\infty \alpha_n Q_n(x), \tag{17}$$

where α_n *is given by* (16), C *being a circle, center* $u=0$, *radius* r (*in* $1/\varrho<r<1/\lambda$). *Moreover, the convergence is uniform in any bounded region.*

Proof: $1/C(u)$ is analytic, $|u|<\varrho$, so that $1/C\left(\frac{1}{u}\right)$ is analytic, $|u|>1/\varrho$, and in particular on C. Also, $F(u)$ is analytic in $|u|<1/\lambda$ and therefore on C. Consequently α_n as given by (16) exists. Now $\sum_0^\infty Q_n(x)/u^n$ converges uniformly, $|x|\leqq X$ arbitrary, $|u|=r>1/\varrho$. We may therefore integrate term-wise:

$$\sum_0^\infty \alpha_n Q_n(x)=\frac{1}{2\pi i}\int_C\frac{F(u)}{uC\left(\frac{1}{u}\right)}\sum_0^\infty\frac{Q_n(x)}{u^n}\,du,$$

which from (8) gives

$$\frac{1}{2\pi i}\int_C\frac{F(u)\,e^{x/u}\,du}{u},$$

$$=\frac{1}{2\pi i}\int_C\left[\frac{F_0}{u}+F_1+F_2u+\cdots\right]e^{x/u}\,du=\sum_0^\infty\frac{F_n}{n!}x^n=f(x).$$

This gives (17). Now from (16), $|\alpha_n|\leqq M/r^n$, and from (8), $|Q_n(x)|\leqq N/(\varrho-\varepsilon)^n$, $\varepsilon>0$ arbitrary, $|x|\leqq X$ arbitrary. Therefore $\sum_0^\infty|\alpha_n Q_n(x)|\leqq MN\sum_0^\infty 1/[r(\varrho-\varepsilon)]^n$, which converges. This proves uniformity of convergence.

We have incidentally proved

LEMMA 5: *If* $f(x) = \sum_0^\infty f_n x^n$ *is of exp. val.* λ, *and* $F(x) = \sum_0^\infty n! f_n x^n$, *then*

$$f(x) = \frac{1}{2\pi i}\int_C \frac{F(u)e^{x/u}\,du}{u},$$

where C *is a contour surrounding* $u = 0$ *and lying in* $|u| < 1/\lambda$.

Lemma 4 can be applied to prove

THEOREM 4: *If* $f(x)$ *is of exp. val.* $\lambda < R$, *the equation*

$$A[y] + xy' = f(x) \tag{18}$$

*has a solution**

$$y(x) = \sum_0^\infty \left(\frac{\alpha_n}{A(0)+n}\right) Q_n(x), \tag{19}$$

of exp. val. $= \lambda$, *provided* $D(t)$ *is not analytic*† *at* $t = 0$. *Moreover,* (19) *is the only solution of* (18) *of exp. val.* $< R$, *so that the homogeneous equation*

$$A[y] + xy' = 0 \tag{20}$$

has no solution of exp. val. $< R$ *other than* $y(x) \equiv 0$.

Proof: $A(0) + n \neq 0$ since $D(t)$ is not analytic at $t = 0$. From (10) and (16) we see that series (19) is uniformly convergent, $|x| \leqq X$ arbitrary. Now from (1), since $u_n(x) \equiv Q_n(x)$,

$$A[Q_n] + xQ_n'(x) = [A(0) - \lambda_n]\, Q_n(x) = [A(0) + n]\, Q_n(x).$$

Consequently,‡

$$A[y] + xy' = \sum_0^\infty \alpha_n Q_n(x),$$

which is $f(x)$. Hence (19) is a solution of (18).

Let $R_n(x) = |c_0|\frac{x^n}{n!} + \cdots + |c_n|\frac{x^0}{0!}$, and $V(u) = \sum_0^\infty |c_n| u^n$. Then $y(x) \ll NV(1/r)e^{x/r}$, which is of exp. val. $= 1/r$. Hence $y(x)$ is of exp. val. $\leqq 1/r$, and since r can be chosen as close to $1/\lambda$ as we please, $y(x)$ must be of exp. val. $\leqq \lambda$. But if exp. val. $y(x) = \mu < \lambda$, then $A[y] + xy'$ is of exp. val. $\leqq \mu < \lambda$, which is impossible. Hence exp. val. $y(x) = \lambda$.

* In (18), $B(t) \equiv t$, so that $\xi = 0$, and therefore $\varrho = R$. Consequently every function $f(x)$ of exp. val. $\lambda < R$ has a uniformly convergent expansion: $f(x) = \sum_0^\infty \alpha_n Q_n(x)$, where α_n is given by (16), C being a circle with center at $t = 0$ and radius r: $1/R < r < 1/\lambda$.

† If $D(t)$ is analytic at $t = 0$, it is analytic throughout $|t| < R$, so the method of § 2 applies.

‡ This operation is easily justified.

There remains to prove uniqueness. Suppose the contrary; then $y(x) \not\equiv 0$ exists, of exp. val. $\mu < R$, satisfying (20). $y(x)$ has a Q_n-expansion: $y(x) = \sum_0^\infty \beta_n Q_n(x)$. Substituting into (20):

$$A[y] + xy' = \sum_0^\infty \beta_n [A(0) + n] Q_n(x) \equiv 0,$$

or, on setting $\gamma_n = \beta_n [A(0) + n]$,

$$\text{(a)} \qquad g(x) = \sum_0^\infty \gamma_n Q_n(x) \equiv 0.$$

Now $|\gamma_n| \leqq nN/r^n$, where $1/R < r < 1/\mu$. Consequently (a) is uniformly convergent and of exp. val. $\leqq \mu$. If we express the left hand member of (a) as a power series and equate coefficients (as we may), we obtain the set of equations

$$\text{(b)} \qquad c_0 \gamma_i + c_1 \gamma_{i+1} + c_2 \gamma_{i+2} + \cdots = 0, \qquad i = 0, 1, 2, \cdots.$$

But this tells us that $J(x) = \sum_0^\infty \gamma_n x^n/n!$ satisfies the equation*

$$\text{(c)} \qquad C[J(x)] = 0.$$

Now $C(u) \not\equiv 0$, $|u| < R$, so that (Theorem 1 of Annals I) the only solution of (c), of exp. val. $< R$, is the function zero. That is, $J(x) \equiv 0$, and $\gamma_n \equiv 0$. Now $A(0) + n \not\equiv 0$, $n = 0, 1, \cdots$, so that $\beta_n = 0$, $n = 0, 1, \cdots$, and $y(x) \equiv 0$. This completes the proof.

We now consider the equation

$$\text{(21)} \qquad A[y] + x(y' - \xi y) = g(x).$$

Corresponding to it we have

THEOREM 5: *If in the equation* (21), $A(t)$ *is analytic,* $|t| < R$, ξ *is in* $|\xi| < R$, $D(t)$ *is not analytic*† *at* $t = \xi$, *and* $g(x) = e^{\xi x} f(x)$ *where* $f(x)$ *is of exp. val.* $< \varrho = R - |\xi|$, *then there exists a solution* $y(x)$ *of exp. val.* $< R$, *given by*

$$y(x) = e^{\xi x} \sum_0^\infty \left[\frac{\alpha_n}{A(\xi) + n} \right] Q_n(x),$$

where $f(x) = \sum_0^\infty \alpha_n Q_n(x)$.

Proof: The proof is like that of Theorem 4.

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* $J(x)$ is of exp. val. $\leqq \mu < R$, since $g(x)$ is of exp. val. $\leqq \mu$. Also, $C(u)$ is analytic in $|u| < R$. We may then express the lefthand member of (c) in a power series. On doing this we find that the coefficients of $x^0/0!, x^1/1!, \cdots$ are precisely the lefthand members of equations (b). Therefore (c) holds.

† Then $A(\xi) + n \neq 0$, $n = 0, 1, \cdots$.

If $g(x) = e^{\xi x} f(x)$ where exp. val. of $f(x)$ is $\geqq \varrho$, the Q_n-expansion does not seem to give convergence. We must then proceed differently.

5. **Solution of the problem for $B(t) \equiv t - \xi$.** We are still concerned with the equations

$$A[y] + x(y' - \xi y) = f(x), \tag{21}$$

$$A[y] + x(y' - \xi y) = 0. \tag{22}$$

We had *formally*, $e^{-\xi x} f(x) = \sum_0^\infty a_n Q_n(x)$, so that a formal solution of (21) is

$$y(x) \sim \sum_0^\infty \left[\frac{a_n}{A(\xi)+n}\right] e^{\xi x} Q_n(x) = \sum_0^\infty \gamma_n u_n(x), \text{ where } \gamma_n = a_n/[A(\xi)+n].$$

Now (formally again)

$$\sum_0^\infty \gamma_n Q_n(x) = C[\mathcal{A}(x)], \quad \text{where}^* \ \mathcal{A}(x) = \sum_0^\infty \gamma_n x^n/n!.$$

This suggests that a possible solution of (21) is $y(x) = e^{\xi x} C[\mathcal{A}(x)]$. In fact we have

THEOREM 6: *In equation* (21) $A[y] + x(y' - \xi y) = f(x)$, *let* $f(x)$ *be of exp. val.* $< R$, ξ *in* $|\xi| < R$, *and let* $g(x) = e^{-\xi x} f(x)$ *be of exp. val.* $\lambda < \varrho = R - |\xi|$; *and let* $D(t)$ *fail to be analytic at* $t = \xi$, *so that* $A(\xi) \neq -n$, $n = 0, 1, \cdots$. *Then* (21) *has a solution*

$$y(x) = e^{\xi x} C[\mathcal{A}(x)],$$

of exp. val. $< R$, *where*

$$\mathcal{A}(x) = \sum_0^\infty G_n x^n/(\alpha+n), \qquad \alpha = A(\xi),$$

$$G(x) = \frac{1}{C}[g(x)] = \sum_0^\infty G_n x^n, \qquad C(t) = e^{\int_0^t [A(t+\xi) - A(\xi)]\, dt/t}.$$

Proof: $G(x)$ is of exp. val. $\leqq \lambda$, and therefore $y(x)$ is of exp. val. $\leqq \lambda + |\xi| < R$. We can therefore write $E[y] = e^{\xi x} E_\xi C[\mathcal{A}(x)]$, provided $E(t)$ is analytic, $|t| < R$. Here $E_\xi(t) = E(t+\xi)$. Set $H(x) = A[y] + x(y' - \xi y)$. Then $H = e^{\xi x} C A_\xi [\mathcal{A}] + x e^{\xi x} TC[\mathcal{A}]$, $T(t) = t$. Therefore

$$\frac{1}{C}[e^{-\xi x} H] = (A_\xi - TC'/C)[\mathcal{A}] + xT[\mathcal{A}].$$

But $A_\xi(t) - T(t)C'(t)/C(t) = A(\xi) = \alpha$, so that $\frac{1}{C}[e^{-\xi x} H] = \alpha \mathcal{A}(x) + x\mathcal{A}'(x)$. And from the definition of $\mathcal{A}(x)$, we obtain $\alpha \mathcal{A}(x) + x\mathcal{A}'(x) = \sum_0^\infty G_n x^n = G(x)$.

* We see this by writing $Q_n(x) = c_0 x^n/n! + \cdots + c_n x^0/0!$ in $\sum_0^\infty \gamma_n Q_n(x)$ and equating coefficients of the same powers of x.

Therefore $e^{-\xi x} H(x) = C[G] = g(x)$. Hence $A[y] + x(y' - \xi y) = e^{\xi x} g(x) = f(x)$. $y(x)$ is therefore a solution.

Let us further consider the operational method. Set $B(t) = t - \xi$, and $K(t) = C(t - \xi)$. From the definition of $C(t-\xi)$, $K(t)$ is analytic in $|t| < R$, and $K(t) \not\equiv 0$, $|t| < R$. Operate on (21) with $1/K$:

$$\left(\frac{A}{K} + B\left(\frac{1}{K}\right)'\right)[y] + x \cdot \frac{B}{K}[y] = \frac{1}{K}[f] = g(x), \text{ say.}$$

But $A(t)/K(t) + B(t)(1/K(t))' = A(\xi)/K(t)$, so that on setting $A(\xi) = \alpha$, $w(x) = \frac{1}{K}[y]$, (21) reduces to a differential equation of *finite order*:

$$\text{(a)} \qquad x w'(x) + (\alpha - \xi x) w(x) = g(x).$$

Solving, we find

$$\text{(b)} \qquad w(x) = \frac{e^{\xi x}}{x^{\alpha}} \int g(x) x^{\alpha - 1} e^{-\xi x} \, dx,$$

which is an entire function. In order to know something of the exp. val. of $w(x)$ it is desirable to consider its Taylor coefficients w_n: $w(x) = \sum_0^\infty w_n x^n$. To that end we substitute this power series into (a) and equate coefficients. We find

$$w_n = \frac{g_n}{\alpha + n} + \frac{g_{n-1}\,\xi}{(\alpha+n)(\alpha+n-1)} + \cdots + \frac{g_0\,\xi^n}{(\alpha+n)(\alpha+n-1)\cdots\alpha}, \qquad n = 0, 1, \cdots,$$

where $g(x) = \sum_0^\infty g_n x^n$, so that (as a computation shows) $|w_n| \leqq M[(n+1)\mu^n]/n!$, where $\mu < R$ is the larger of $\lambda + \varepsilon$, $|\xi|$, $\varepsilon > 0$ being arbitrary but fixed. This means that $w(x)$ is of exp. val. $\leqq \mu < R$.

We can now prove

THEOREM 7: *In the equation* (21) $A[y] + x(y' - \xi y) = f(x)$, *let* $A(t)$ *be analytic in* $|t| < R$, ξ *in* $|\xi| < R$, $f(x)$ *of exp. val.* $\lambda < R$. *Furthermore,* $D(t)$ *is not to be analytic at* $t = \xi$, *so that* $A(\xi) + n \not\equiv 0$, $n = 0, 1, \cdots$. *Then* (21) *has one and only one solution* $y(x)$ *of exp. val.* $< R$, *and this solution is given by*

$$\text{(23)} \qquad y(x) = K[w(x)]$$

*where**

$$\text{(24a)} \qquad w(x) = \sum_0^\infty w_n x^n,$$

* The case where $D(t)$ is analytic at $t = \xi$, and therefore in $|t| < R$ has no interest for us at present, since it is completely discussed in § 2.

$$w_n = \frac{g_n}{\alpha+n} + \frac{g_{n-1}\,\xi}{(\alpha+n)(\alpha+n-1)} + \cdots + \frac{g_0\,\xi^n}{(\alpha+n)(\alpha+n-1)\cdots\alpha},$$

(24b)

$$\alpha = A(\xi), \qquad K(t) = C(t-\xi), \qquad g(x) = \sum_0^\infty g_n x^n = \frac{1}{K}[f].$$

The homogeneous equation (22) $A[y] + x(y' - \xi y) = 0$ *has then no solution of exp. val.* $< R$ *other than* $y(x) \equiv 0$.

Furthermore, the exp. val. of $y(x)$ *is determined as follows:*

If $\lambda \geqq |\xi|$, *then* $y(x)$ *is of exp. val.* λ. *If* $\lambda < |\xi|$, *then* $y(x)$ *is of exp. val.* λ *if and only if* $D[f]_{x=0} = 0$; *otherwise* $y(x)$ *is of exp. val.* $= |\xi|$.

The function $w(x)$ *is of the same exp. val. as* $y(x)$, *and satisfies the differential equation*

$$x\,w'(x) + (\alpha - \xi x)\,w(x) = g(x). \tag{25}$$

Proof: We have shown that $w(x)$ as defined by (24) satisfies (25), and that it is of exp. val. $< R$. Hence $y(x)$ given by (23), is uniquely defined (since $K(t) \not\equiv 0$, $|t| < R$), and is of exp. val. $< R$. Also, since $w_0, w_1, \cdots$ are uniquely determined, there cannot be more than one solution $y(x)$ of exp. val. $< R$. Consequently that part of the theorem which does not refer to exp. val. will be proved if we show that (23) *is* a solution of (21). Now $y(x) = K[w(x)]$, $K(t) = C(t-\xi)$. Let $H(x) = A[y] + x(y' - \xi y)$. Then $H(x) = AK[w] + xBK[w]$, $B(t) = t - \xi$. Operate with $1/K$:

$$\frac{1}{K}[H] = (A - BK'/K)[w] + xB[w] = \alpha w + x(w' - \xi w),$$

and this is, by (25), $g(x)$: $\frac{1}{K}[H] = g(x)$, or $H = K[g(x)]$, which is $f(x)$. That is, $A[y] + x(y' - \xi y) = f(x)$, so that $y(x)$ is a solution of (21).

We now consider the exp. val. of $y(x)$. $y(x)$ cannot be of exp. val. $< \lambda$ since then the lefthand member of (21) would likewise be, and could not be equal to $f(x)$. Hence exp. val. $y(x) \geqq \lambda$.

Suppose $\lambda \geqq |\xi|$, and let exp. val. $y(x)$ be $\mu\ (< R)$. If $\mu > \lambda$ we can choose R' in $|\xi| < R' < \mu$ and so obtain a solution of (21) of exp. val. $< R'$, and this solution must by uniqeness be identical with $y(x)$; a contradiction. Hence $\mu = \lambda$, and $y(x)$ is of exp. val. λ.

Suppose $\lambda < |\xi|$. If $D[f(x)]_{x=0} = 0$, then in considering the region $|t| < R'$, $\lambda < R' < |\xi|$, we obtain a solution* $y(x)$ of exp. val. $= \lambda$. If $D[f(x)]_{x=0} \not\equiv 0$, there is no solution of exp. val. $< |\xi|$ so the exp. val. of $y(x)$ is $\geqq |\xi|$. By considering $|t| < R'$, where $|\xi| < R' < R$, we then see

* See Theorem 2.

that $y(x)$ must be of exp. val. $= |\xi|$. The theorem is now completely proved.

6. **The case** $B(t) \equiv (t-\xi_1)(t-\xi_2)$. Let us now consider the equations

$$A[y] + xB[y] = 0, \tag{1}$$

$$A[y] + xB[y] = g(x), \tag{2}$$

where $A(t)$ is analytic, $|t| < R$, $g(x)$ is of exp. val. $< R$, and $B(t) \equiv (t-\xi_1)(t-\xi_2)$, $|\xi_1| < R$, $|\xi_2| < R$. We shall first assume that $\xi_1 \neq \xi_2$. We wish to examine the characteristic functions of the equation

$$\lambda + {}_1A[y] + xB[y] = 0,$$

where ${}_1A(t) = A(t) - A(0)$. We obtain *two sets* of characteristic functions, according as we choose λ to make $D(t)$ analytic at $t = \xi_1$ or at $t = \xi_2$.

The condition that $D(t)$ be analytic at $t = \xi_i$ is that $-[\lambda + {}_1A(\xi_i)]/(\xi_i - \xi_j) = n$ (positive integer or zero). Let us, for, definiteness, consider the zero $t = \xi_1$ and the corresponding characteristic functions $u_n(x)$. We define

$$\lambda_n = -[n(\xi_1 - \xi_2) + {}_1A(\xi_1)], \tag{3}$$

so that

$$\lambda_n u_n + {}_1A[u_n] + xB[u_n] = 0, \qquad n = 0, 1, \cdots. \tag{4}$$

As was shown in § 4, $u_n(x) = e^{\xi x} P_n(x)$ where $P_n(x)$ is a polynomial of degree n, and when we substitute this into (4) we obtain

$$\begin{aligned}[\lambda_n + {}_1A(\xi_1)]\, P_n(x) + \frac{{}_1A'(\xi_1)}{1!} P_n'(x) + \cdots + \frac{{}_1A^{(n)}(\xi_1)}{n!} P_n^{(n)}(x) \\ + x[P_n''(x) + (\xi_1 - \xi_2) P_n'(x)] = 0.\end{aligned} \tag{5}$$

We connect the polynomials $P_n(x)$ with the function $D(t)$, modifying the method used in § 4. Let us suppose* that $|\xi_1| < |\xi_2|$, and set $T(t) = t - \xi_2$. Operate on (4) with $1/T$ (which is analytic, $|t| \leqq |\xi_1|$):

$$(\lambda_n/T + {}_1A/T - B/T^2)[u_n] + x(u_n' - \xi_1 u_n) = 0,$$

or

$$\Gamma_n[u_n] + x(u_n' - \xi_1 u_n) = 0, \tag{6}$$

where

$$\Gamma_n(t) = \lambda_n/(t-\xi_2) + \Theta(t), \qquad \Theta(t) = [{}_1A(t) - (t-\xi_1)/(t-\xi_2)]. \tag{7}$$

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* This assumption is for the purpose of arriving at definite results. Once obtained, these results will be shown to hold without this restriction.

Now keep n fixed, and consider

$$\Gamma_n[y]+x(y'-\xi_1 y) = 0. \tag{8}$$

There will exist (by § 4) characteristic numbers λ_{mn} $(m = 0, 1, \cdots)$ which make the corresponding D-function analytic at $t = \xi_1$:

$$\lambda_{mn}\, u_{mn}(x) + {}_1\Gamma_n[u_{mn}(x)] + x(u'_{mn}(x) - \xi_1 u_{mn}(x)) = 0, \tag{9}$$

with ${}_1\Gamma_n(t) = \Gamma_n(t) - \Gamma_n(0)$. Clearly, $u_{mn}(x) \equiv u_n(x)$ for $m = n$. Corresponding to (8) we have

$$D_n(t) = e^{-\int \frac{\Gamma_n(t)\,dt}{t-\xi_1}} = (t-\xi_1)^{-\Gamma_n(\xi_1)}\,[C_n(t-\xi_1)]^{-1},$$

where

$$C_n(t-\xi_1) = e^{\int_{t=\xi_1}^{t} [\Gamma_n(t)-\Gamma_n(\xi_1)]\frac{dt}{t-\xi_1}}. \tag{10}$$

On rewriting the integrand we obtain

$$C_n(t-\xi_1) = \left(\frac{t-\xi_2}{\xi_1-\xi_2}\right)^{\frac{-\lambda_n}{\xi_1-\xi_2}} \cdot e^{\int_{t=\xi_1}^{t} \frac{\theta(t)-\theta(\xi_1)}{t-\xi_1}\,dt}. \tag{11}$$

As n increases by unity, λ_n increases by $-(\xi_1-\xi_2)$. Hence if we set

$$\Phi(t-\xi_1) = \left(\frac{t-\xi_2}{\xi_1-\xi_2}\right)^{\frac{-\lambda_0}{\xi_1-\xi_2}} \cdot e^{\int_{t=\xi_1}^{t} \frac{\theta(t)-\theta(\xi_1)}{t-\xi_1}dt}, \tag{12}$$

then

$$C_n(t-\xi_1) = \left(\frac{t-\xi_2}{\xi_1-\xi_2}\right)^n \Phi(t-\xi_1), \tag{13}$$

and Φ is independent of n. This relation exhibits the nature of the dependence of $C_n(t-\xi_1)$ on n. Let us set

$$u_{mn}(x) = e^{\xi_1 x}\, P_{mn}(x). \tag{14}$$

By the results of § 4,

$$P_{mn}(x) = c_0^{(n)}\frac{x^m}{m!} + c_1^{(n)}\frac{x^{m-1}}{(m-1)!} + \cdots + c_m^{(n)}\frac{x^0}{0!}, \tag{15}$$

where*

$$C_n(t-\xi_1) = c_0^{(n)} + c_1^{(n)}(t-\xi_1) + \cdots. \tag{16}$$

The series (16) converges at least in $|t-\xi_1| < \varrho$, ϱ being the smaller of $R-|\xi_1|$, $|\xi_1-\xi_2|$. We then obtain, by Cauchy's formula,

* The multiplicative constant in $P_{mn}(x)$ is properly chosen.

$$P_{mn}(x) = \frac{1}{2\pi i}\int_C \frac{C_n(t-\xi_1)\, e^{(t-\xi_1)x}}{(t-\xi_1)^{m+1}}\, dt, \tag{17}$$

C being a contour surrounding $t=\xi_1$ and lying in $|t-\xi_1|<\varrho$. In particular, on choosing $m=n$:

$$P_n(x) = \frac{1}{2\pi i}\int_C e^{(t-\xi_1)x}\cdot\frac{\Phi(t-\xi_1)}{(t-\xi_1)^{n+1}}\left(\frac{t-\xi_2}{\xi_1-\xi_2}\right)^n dt. \tag{18}$$

We obtained (18) on the assumption that $|\xi_1|<|\xi_2|$. We shall show that it is valid in any case. Let us now make no assumption on $\xi_1 \neq \xi_2$. We wish to prove that $P_n(x)$ as defined by (18) makes $u_n(x)=e^{\xi_1 x}P_n(x)$ the characteristic function of (4) for λ_n. It is clear that $P_n(x)$ as so defined is a polynomial of degree n. Since (5) defines a polynomial of degree n uniquely (up to a constant multiplier) our point will be established if we show that $P_n(x)$ given by (18) satisfies (5). To this effect let $L_n(x)$ denote the lefthand member of (5). We wish to prove that $L_n(x)\equiv 0$. Now*

$$\begin{aligned}
L_n(x) &= \frac{1}{2\pi i}\int_C \frac{\Phi(t-\xi_1)\, e^{(t-\xi_1)x}}{(t-\xi_1)^{n+1}}\left(\frac{t-\xi_2}{\xi_1-\xi_2}\right)^n \\
&\quad\times\left[\lambda_n+\sum_{n=0}^{\infty}\frac{{}_1A^{(n)}(\xi_1)}{n!}(t-\xi_1)^n + x\{(t-\xi_1)^2+(\xi_1-\xi_2)(t-\xi_1)\}\right] \\
&= \frac{1}{2\pi i}\int_C e^{(t-\xi_1)x+\int_{t=\xi_1}^{t}\frac{\Theta(t)-\Theta(\xi_1)}{t-\xi_1}dt}\cdot\left(\frac{t-\xi_2}{\xi_1-\xi_2}\right)^{\frac{-\lambda_n}{\xi_1-\xi_2}} \\
&\quad\times[(\lambda_n-A(0))+A(t)+x\{(t-\xi_1)^2+(\xi_1-\xi_2)(t-\xi_1)\}]\frac{dt}{(t-\xi_1)^{n+1}}.
\end{aligned}$$

On reducing and setting

$$\begin{aligned}
K(t) &= [{}_1A(t)-{}_1A(\xi_1)]/(t-\xi_2)+x(t-\xi_1), \\
H(t) &= (t-\xi_2)K(t)+(n-1)(t-\xi_1), \\
S(t) &= e^{\int_{t=\xi_1}^{t}\frac{H(t)\,dt}{(t-\xi_1)(t-\xi_2)}}[H(t)-(n-1)(t-\xi_2)-(\xi_1-\xi_2)],
\end{aligned}$$

we obtain

$$L_n(x) = \frac{1}{2\pi i}\int_C \frac{S(t)\,dt}{(t-\xi_1)^{n+1}}.$$

$L_n(x)$ will be zero if $(d^n S/dt^n)_{t=\xi_1}=0$; and this last follows on applying Leibnitz's Rule. Hence $L_n(x)\equiv 0$, as was to be shown.

* Since $P_n^{(k)}(x)=0$, $k>n$, we may insert in the integrand the terms $\sum_{k=n+1}^{\infty}\frac{{}_1A^{(k)}(\xi_1)}{k!}P_n^{(k)}(x)$.

Since we made no assumption on $\xi_1 \neq \xi_2$, a similar relation holds for the zero $t = \xi_2$.

Let us now determine α_n in the formal development

$$f(x) \sim \sum_0^\infty \alpha_n P_n(x). \tag{19}$$

For convenience we set $u = t - \xi_1$, so that

$$P_n(x) = \frac{1}{2\pi i} \int_C \frac{C(u) e^{ux}}{u^{n+1}} du, \tag{18'}$$

where now C is a contour around $u = 0$ lying wholly in $|u| < \varrho_1$. Then,

$$\begin{aligned} \sum_0^\infty P_n(x) z^n &= \frac{1}{2\pi i} \int_C \frac{\Phi(u) e^{ux}}{u} \sum_{n=0}^\infty \left(\frac{z(u + \xi_1 - \xi_2)}{u(\xi_1 - \xi_2)} \right)^n du \\ &= \frac{1}{2\pi i} (\xi_1 - \xi_2) \int_C \frac{\Phi(u) e^{ux} du}{(\xi_1 - \xi_2) u - z(u + \xi_1 - \xi_2)}. \end{aligned}$$

The only singularity in and on C is at $u = z(\xi_1 - \xi_2)/(\xi_1 - \xi_2 - z)$. Set

$$w = z(\xi_1 - \xi_2)/(\xi_1 - \xi_2 - z). \tag{20}$$

w will be inside C for $|z|$ sufficiently small. Then

$$\sum_0^\infty z^n P_n(x) = \left(\frac{\xi_1 - \xi_2}{\xi_1 - \xi_2 - z} \right) e^{\left(\frac{z(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - z} \right) x} \cdot \Phi \left(\frac{z(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - z} \right). \tag{21}$$

(21) is valid for $|z|$ sufficiently small. From (21) we derive

$$P_n(x) = \frac{1}{2\pi i} \int_C \left(\frac{\xi_1 - \xi_2}{\xi_1 - \xi_2 - u} \right) e^{\left(\frac{u(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - u} \right) x} \cdot \Phi \left(\frac{u(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - u} \right) \frac{du}{u^{n+1}}. \tag{22}$$

Then

$$f(x) \sim \sum_0^\infty \alpha_n P_n(x) \sim \frac{1}{2\pi i} \int_C \left(\frac{\xi_1 - \xi_2}{\xi_1 - \xi_2 - u} \right) e^{\left(\frac{u(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - u} \right) x} \times \Phi \left(\frac{u(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - u} \right) \sum_0^\infty \frac{\alpha_n}{u^{n+1}} du.$$

If we introduce w, defined by (20), we have

$$\sum_0^\infty \left(\frac{w(\xi_1 - \xi_2)}{\xi_1 - \xi_2 + w} \right)^n P_n(x) = \left(\frac{\xi_1 - \xi_2 + w}{\xi_1 - \xi_2} \right) \Phi(w) e^{wx}. \tag{23}$$

If in (23) we expand the lefthand member in powers of w, we obtain

$$\sum_{n=0}^{\infty} w^n \left[P_0(x)\,\delta_{0n} + \binom{n-1}{0}\left(\frac{-1}{\xi_1-\xi_2}\right)^{n-1} P_1(x) + \cdots + \binom{n-1}{n-1}\left(\frac{-1}{\xi_1-\xi_2}\right)^{0} P_n(x)\right],$$

where $\delta_{0n} = 0$ if $n \neq 0$ and 1 if $n = 0$. By (23) this must be equal to $[(\xi_1 - \xi_2 + w)/(\xi_1 - \xi_2)]\,\Phi(w)e^{wx} = \sum_0^{\infty} q_n(x)\,w^n$, say. On equating coefficients:

$$P_0(x) = \Phi(0) = q_0(x),$$

$$\left(\frac{-1}{\xi_1-\xi_2}\right)^{n-1} P_1(x) + \binom{n-1}{1}\left(\frac{-1}{\xi_1-\xi_2}\right)^{n-2} P_2(x) + \cdots \tag{24}$$

$$\cdots + \binom{n-1}{n-1}\left(\frac{-1}{\xi_1-\xi_2}\right)^{0} P_n(x) = q_n(x), \quad n > 0.$$

Let us set

$$Q(w) = \sum_0^{\infty} q_n w^n = \left(1 + \frac{w}{\xi_1-\xi_2}\right)\Phi(w). \tag{25}$$

Then

$$q_n(x) = q_0\frac{x^n}{n!} + q_1\frac{x^{n-1}}{(n-1)!} + \cdots + q_n\frac{x^0}{0!}, \tag{26}$$

$$e^{wx}\,Q(w) = \sum_0^{\infty} q_n(x)\,w^n. \tag{27}$$

Now $Q(0) \neq 0$, so that

$$x^n/n! = \gamma_0 q_n(x) + \gamma_1 q_{n-1}(x) + \cdots + \gamma_n q_0(x), \tag{28}$$

$$1/Q(w) = \sum_0^{\infty} \gamma_n w^n. \tag{29}$$

Substituting into (28) the values of the $q_n(x)$'s given by (24), we obtain the expansion of $x^n/n!$ in a P_n-series:

$$x^0/0! = \gamma_0 P_0(x),$$

$$x^n/n! = [\gamma_0]P_n(x) + \left[\binom{n-1}{1}\left(\frac{-1}{\xi_1-\xi_2}\right)\gamma_0 + \gamma_1\right]P_{n-1}(x) + \cdots$$

$$\cdots + \left[\binom{n-1}{n-1}\left(\frac{-1}{\xi_1-\xi_2}\right)^{n-1}\gamma_0 + \binom{n-2}{n-2}\left(\frac{-1}{\xi_1-\xi_2}\right)^{n-2}\gamma_1 + \cdots + \gamma_{n-1}\right]P_1(x) \tag{30}$$

$$+ [\gamma_n]P_0(x).$$

Now let

$$f(x) = \sum_0^{\infty} f_n x^n, \qquad F(x) = \sum_0^{\infty} n!\,f_n x^n, \qquad F_n = n!\,f_n.$$

Then, using (30), and simplifying, we find that $f(x) = \sum_0^\infty a_n P_n(x)$, where

$$(31) \qquad a_n = \frac{1}{2\pi i} \int_C \frac{Ft}{Q\left(\frac{1}{t}\right)} \cdot \frac{dt}{t^{n+1}\left[1+\frac{1}{t(\xi_1-\xi_2)}\right]^n}.$$

We can now prove

LEMMA 6:* *If* $f(x) = \sum_0^\infty f_n x^n$ *is of exp. val.* $\lambda < \varrho_1$, *where* ϱ_1 *is the smaller of* $R - |\xi_1|$, $|\xi_2 - \xi_2|$, *and if* $F(x) = \sum_0^\infty F_n x^n$ *with* $F_n = n! f_n$, *then the series*

$$(32) \qquad f(x) = \sum_0^\infty a_n P_n(x)$$

converges uniformly in $|x| \leqq X$ *arbitrary. Here* a_n *is given by* (31), C *being the circle* $|t| = r$, $1/\varrho_1 < r < 1/\lambda$.

Proof: The formal operations are easily justified, and lead to the desired result.

We can now obtain solutions of (2) for a sub-class of functions $g(x)$, as was done in § 4; but since the method of characteristic functions does not give convergence for all functions $g(x)$ that we consider, we may pass over this.

We proceed to investigate the equations (1) and (2) by an operational method. Consider the equation

$$(i) \qquad A[y] + x(y'' - (\xi_1+\xi_2)\, y' + \xi_1\, \xi_2\, y) = 0 \quad \text{or} \quad g(x).$$

We suppose that $D(t)$ is not analytic at both $t = \xi_1, \xi_2$. Recalling that $D(t)$ is defined by

$$(33) \qquad A(t)\, D(t) + B(t)\, D'(t) = 0,$$

we see that we can write

$$(34) \qquad D(t) = H(t)\,(t-\xi_1)^{\frac{-A(\xi_1)}{\xi_1-\xi_2}} \cdot (t-\xi_2)^{\frac{-A(\xi_2)}{\xi_2-\xi_1}}$$

where $H(t)$ is analytic and $\neq 0$ in $|t| < R$. In fact,

$$(35) \qquad H(t) = e^{-\int \left[A(t) - \left\{\frac{A(\xi_1)}{\xi_1-\xi_2}(t-\xi_2) + \frac{A(\xi_2)}{\xi_2-\xi_1}(t-\xi_1)\right\}\right] \frac{dt}{(t-\xi_1)(t-\xi_2)}}.$$

Operate on (i) with H: $(AH + H'B)[y] + xBH[y] = 0$ or $H[g]$. Now

$$A(t)\, H(t) + H'(t)\, B(t)$$
$$= H(t)\,[A(\xi_1)\,(t-\xi_2)/(\xi_1-\xi_2) + A(\xi_2)\,(t-\xi_1)/(\xi_2-\xi_1)].$$

* A similar result holds of course for the zero $t = \xi_2$.

Consequently, if we set

$$w(x) = H[y], \tag{36}$$

we have the following equation for $w(x)$:

$$\begin{aligned} &x w''(x) + \left[\frac{A(\xi_1) - A(\xi_2)}{\xi_1 - \xi_2} - x(\xi_1 + \xi_2)\right] w'(x) \\ &+ \left[\xi_1 \xi_2 x - \left(\frac{\xi_2 A(\xi_1) - \xi_1 A(\xi_2)}{\xi_1 - \xi_2}\right)\right] w(x) = 0 \quad \text{or} \quad H[g]. \end{aligned} \tag{37}$$

Now this equation is a linear differential equation of finite order; in fact a second order equation of Fuchsian type. We have thus been able to reduce the problem to an equivalent finite order equation by applying a suitable operator. The theory of equations of second order can now be utilized. We shall not stop over the details.

7. The general case: reduction to a finite order equation. In the preceding section it has been shown that for particular choices of $B(t)$ the equations

$$A[y] + x B[y] = 0, \tag{I}$$

$$A[y] + x B[y] = g(x) \tag{II}$$

can be treated by the method of characteristic functions (at least for a sub-class of functions $g(x)$) and by an operational method. The general case, that in which $B(t)$ is an arbitrary polynomial,* can be similarly handled. We shall not go into the question of characteristic functions. (The previous work indicates how to proceed.) But we shall show that the operational method applies, We have, in fact

THEOREM 8. *Let $A(t)$, $B(t)$ be analytic, $|t| < R$, and let $B(t)$ have a finite number, m, of zeros in $|t| < R$. Suppose $g(x)$ is of exp. val. $q < R$. Then equations* (I), (II) *can be transformed into linear differential equations of finite order (mth order) and the resulting equations are equivalent to the original ones in that to every solution of exp. val. $< R$ of an original equation corresponds a solution of exp. val. $< R$ of the transformed equation and conversely.*

Proof: As was remarked in § 4, we can suppose that $B(t)$ is a polynomial: $B(t) = (t - \xi_1)^{p_1} \cdots (t - \xi_r)^{p_r}$, $\xi_i \neq \xi_j$, $p_1 + \cdots + p_r = m$. Let $D(t)$ be defined by

$$A(t)\, D(t) + B(t)\, D'(t) = 0. \tag{1}$$

CASE 1. $B(t)$ has only simple zeros: $P_i = 1$, $i = 1, 2, \cdots, r (= m)$. Then

* That this is the general case is shown in a footnote at the beginning of § 4.

$$D(t) = e^{-\int \frac{A(t)\,dt}{(t-\xi_1)\cdots(t-\xi_m)}} = H(t)\prod_{i=1}^{m}(t-\xi_i)^{\frac{-A(\xi_i)}{P_i(\xi_i)}}, \tag{2}$$

where

$$P_i(t) = B(t)/(t-\xi_i), \tag{3}$$

$$H(t) = e^{-\int\left[A(t)-\sum_{i=1}^{m}\frac{A(\xi_i)}{P_i(\xi_i)}P_i(t)\right]\frac{dt}{B(t)}}. \tag{4}$$

We observe that $H(t)$ is analytic and $\neq 0$ in $|t|<R$.

Let us operate on (I), (II) with H. Using the relation

$$A(t)\,H(t)+B(t)\,H'(t) = H(t)\sum_{i=1}^{m}[A(\xi_i)/P_i(\xi_i)]\,P_i(t),$$

and setting

$$w(x) = H[y], \tag{5}$$

we obtain the equation

$$\sum_{i=1}^{m}\frac{A(\xi_i)}{P_i(\xi_i)}P_i[w]+xB[w] = 0 \quad \text{or} \quad H[g]. \tag{6}$$

Since $B(t)$ is a polynomial of degree m, (6) is a linear differential equation of order m, of Fuchsian type.

Now $H(t)\neq 0$, $|t|<R$. Consequently by (5), $w(x)$, $y(x)$ are each uniquely determined by the other, provided their exp. val. (which is the same for both) is $<R$. The theorem is thus proved in this case.

Case 2: $B(t)$ has multiple zeros:

$$B(t) = \prod_{i=1}^{r}(t-\xi_i)^{p_i}, \qquad \xi_i \neq \xi_j \qquad p_1+\cdots+p_r = m.$$

Let

$$Q_i(t) = B(t)/(t-\xi_i)^{p_i} \tag{7}$$

$$R_i(t) = A(\xi_i)+\frac{A'(\xi_i)}{1!}(t-\xi_i)+\cdots+\frac{A^{(p_i-1)}(\xi_i)}{(p_i-1)!}(t-\xi_i)^{p_i-1}, \tag{8}$$

$$R_i(t)/Q_i(t) = \sum_{n=0}^{\infty}c_n^{(i)}(t-\xi_i)^n. \tag{9}$$

Then

$$D(t) = H(t)\,K(t)\prod_{i=1}^{r}(t-\xi_i)^{-c_{p_i-1}^{(i)}}, \tag{10}$$

where*

* Such values of i for which $p_i = 1$ contribute nothing to $K(t)$.

$$H(t) = e^{-\int \left[A(t) - \sum_{i=1}^{r} R_i(t)\right] \frac{dt}{B(t)}}, \tag{11}$$

$$K(t) = e^{\sum_{i=1}^{r} \left[\frac{c_0^{(i)}}{p_i - 1} \frac{1}{(t-\xi_i)^{p_i-1}} + \cdots + \frac{c_{p_i-2}^{(i)}}{1} \frac{1}{(t-\xi_i)} \right]}. \tag{12}$$

$H(t)$ is analytic and $\neq 0$ in $|t| < R$. Operate on (I), (II) with H, and use the relation $A(t)\,H(t) + B(t)\,H'(t) = H(t) \sum_{i=1}^{r} R_i(t)$:

$$\sum_{i=1}^{r} R_i[w] + x\,B[w] = 0 \quad \text{or} \quad H[g], \tag{13}$$

where

$$H[y] = w(x). \tag{14}$$

(13) is a linear differential equation of mth order, of Fuchsian type, and since (14) gives $y(x)$, $w(x)$ each uniquely in terms of the other, the theorem is established.

To equations (6) and (13) can now be applied the known theory of Fuchsian equations.

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THE EQUILIBRIUM POINT OF GREEN'S FUNCTION FOR AN ANNULAR REGION.*

By Deborah May Hickey.

Introduction. The behavior of the Green's function in a plane annular region may be studied in the following way. Let us denote by S a plane region of a uniform material that conducts electricity, the boundaries of which are two concentric circles, C_1 and C. Let the positive terminal of a battery be connected at a point M_0 in S and the negative terminal connected to the boundaries which are kept at zero potential. A current of electricity flows from the pole to the boundaries. Equipotential lines can be mapped which are those of the Green's function. For, denote by $g(M_0, M)$ the Green's function for S and by ϱ the radius of a small circle about M_0 such that $\log \varrho^{-1}$ equals a constant c. If M is on this small circle, then $g(M_0, M) = \alpha$, approximately a constant. Also the potential function for this region $v(M)$ is V, approximately a constant on this circle. Hence g and v are constants on the complete boundary of the region obtained from S by leaving out the small circle, therefore g is a linear function of v since $g - \frac{\alpha}{V} v = 0$ on the complete boundary. Then g and v have the same equipotential lines throughout.

In mapping the equipotential lines it is noticed that on the side opposite M_0 in S the current lines, which are orthogonal to the equipotential lines, tend to turn aside, leaving a kind of isolated point. Such a point, where the field intensity is zero, is called an equilibrium point. Theoretically if F denotes the field intensity at any point in S and F_x and F_y its components in the x and y directions, the necessary and sufficient condition for an equilibrium point is that both F_x and F_y vanish at the point.

W. S. Vaughn† found from an experimental investigation that the equilibrium point did not change sensibly when the pole was moved inward or outward along a radius. He develops the condition for an equilibrium point in terms of Fourier series and in terms of the σ-function as given below in Sections 1 and 2. Denoting the radii of the inner and the outer

* Received July 19, 1928.

† W. S. Vaughn, Thesis for the degree of Master of Arts, Rice Institute, 1925, The behavior of the Green's function at the point of equilibrium, manuscript in the Library of the Rice Institute. A continuation of this investigation was suggested to the author by Professor G. C. Evans.

circular boundaries of S by R_1 and R respectively, he shows that for $R_1 = 1$, $R = e$, the equilibrium point, contrarily to what might be expected from the experiment, is not identically fixed. For a particular position of the pole, we have the following

THEOREM. *If the pole is on the circle of geometric mean radius, the equilibrium point is also.*

The theorem is almost immediately evident. In fact, suppose we make an inversion in the points of S with respect to the circle C_0 radius $R_0 = \sqrt{R_1 R}$. All points on C_0 transform into themselves, in particular M_0 on C_0 remains unchanged. Points in S_1, bounded by C_0 and C_1, go into points of S_2, bounded by C_0 and C, and vice versa. Hence the region S is unchanged. Let E denote the point of equilibrium. By symmetry it is on the same diameter as M_0. Now if E were in S_1, it would go by inversion into S_2 and vice versa. But this would cause a change in the region. Hence E must be on C_0 also.

To account for the physical fact, it appears then that $dr/dr_0 = 0$ or is very small, for $r = r_0 = \sqrt{R_1 R}$ where $M_0(r_0, \theta_0)$ is the pole and $M(r, \theta)$ the equilibrium point. Vaughn calculated the value as zero to seven significant figures; the question of a proof was still left open. It is the purpose of this paper to show that in spite of the appearances, dr/dr_0 is not zero, and to develop other facts about the theoretical relations, which seem interesting to a student in the theory of functions of a complex variable.

1. **The necessary and sufficient condition for an equilibrium point by Fourier series.*** Consider the region S defined above. Let $P_1(R_1, \varphi)$, $P(R, \varphi)$ be points on C_1 and C respectively. The Green's function is of the form

$$g(M_0, M) = \log \frac{1}{M_0 M} + u(r, \theta),$$

where $u(r, \theta)$ is harmonic and single valued in S and equal $\log M_0 M$ on the boundary. To determine such a function, consider two functions $U(\varphi)$ and $V(\varphi)$ continuous, of period 2π, and developable in Fourier series which are uniformly convergent,

$$(1.1) \qquad \begin{aligned} U(\varphi) &= \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\varphi + b_m \sin m\varphi), \\ V(\varphi) &= \frac{\alpha_0}{2} + \sum_{m=1}^{\infty} (\alpha_m \cos m\varphi + \beta_m \sin m\varphi). \end{aligned}$$

Choose the function $u(r, \theta)$ so that for points $P(R, \varphi)$, $P_1(R_1, \varphi)$ it reduces to $U(\varphi)$ and $V(\varphi)$ respectively. Take $u(r, \theta)$ in the form

* Method outlined in Goursat, Cours d'analyse, vol. 3, Paris (1915), p. 191.

$$u(r,\theta) = A_0 + B_0 \log r + \sum_1^\infty [(r^m A_m + r^{-m} B_m)\cos m\theta + (r^m C_m + r^{-m} D_m)\sin m\theta] \tag{1.2}$$

where A_m, B_m, C_m, D_m are constant coefficients to be chosen to satisfy (1.1). The function $u(r,\theta)$ given by (1.2) is harmonic. For $\theta = \varphi$ we must have then

$$\begin{aligned} &A_0 + B_0 \log R = \frac{a_0}{2},\ A_m R^m + B_m R^{-m} = a_m,\ C_m R^m + D_m R^{-m} = b_m, \\ &A_0 + B_0 \log R_1 = \frac{\alpha_0}{2},\ A_m R_1^m + B_m R_1^{-m} = \alpha_m,\ C_m R_1^m + D_m R_1^{-m} = \beta_m. \end{aligned} \tag{1.3}$$

To solve these equations for the constants we determine first the Fourier coefficients $a_m, b_m, \alpha_m, \beta_m$ from the usual formulae. The evaluation of these definite integrals by means of the boundary values $U(\varphi) = \log M_0 P$, $V(\varphi) = \log M_0 P_1$, where $\log M_0 P$ is written in a familiar development,

$$\log M_0 P = \log R - \frac{r}{R}\cos(\varphi - \theta_0) - \frac{1}{2}\left(\frac{r}{R}\right)^2 \cos 2(\varphi - \theta_0) - \cdots - \frac{1}{m}\left(\frac{r}{R}\right)^m \cos m(\varphi - \theta_0) - \cdots, \qquad \frac{r}{R} < 1,$$

gives

$$\begin{aligned} &a_0 = 2\log R,\ a_m = -\frac{1}{m}\left(\frac{r_0}{R}\right)^m \cos m\theta_0,\ b_m = -\frac{1}{m}\left(\frac{r_0}{R}\right)^m \sin m\theta_0, \\ &\alpha_0 = 2\log r_0,\ \alpha_m = -\frac{1}{m}\left(\frac{R_1}{r_0}\right)^m \cos m\theta_0,\ \beta_m = -\frac{1}{m}\left(\frac{R_1}{r_0}\right)^m \sin m\theta_0. \end{aligned} \tag{1.4}$$

Substituting these in (1.3) we obtain the desired constants

$$A_0 = \log R \frac{\log \frac{R_1}{r_0}}{\log \frac{R_1}{R}}, \qquad B_0 = \frac{\log \frac{R}{r_0}}{\log \frac{R}{R_1}},$$

$$\begin{aligned} A_m &= -\frac{1}{m}\frac{\left[r_0^m - \left(\frac{R_1}{r_0}\right)^m\right]}{(R^{2m} - R_1^{2m})}\cos m\theta_0,\ B_m = -\frac{1}{m}\frac{R_1^{2m}\left[\left(\frac{R^2}{r_0}\right)^m - r_0^m\right]}{(R^{2m} - R_1^{2m})}\cos m\theta_0, \\ C_m &= -\frac{1}{m}\frac{\left[r_0^m - \left(\frac{R_1}{r_0}\right)^m\right]}{(R^{2m} - R_1^{2m})}\sin m\theta_0,\ D_m = -\frac{1}{m}\frac{R_1^{2m}\left[\left(\frac{R}{r_0}\right)^m - r_0^m\right]}{(R^{2m} - R_1^{2m})}\sin m\theta_0. \end{aligned} \tag{1.5}$$

Then the Green's function for S may be written in the form

$$g(M_0, M) = \log\frac{1}{M_0 M} + \frac{1}{\log R_1/R}\left[\log R \log\frac{R_1}{r_0} + \log r \log\frac{r_0}{R}\right] - \sum_{m=1}^{\infty}\frac{\cos m(\theta-\theta_0)}{(R^{2m}-R_1^{2m})}\left\{r^m\left[r_0^m - \left(\frac{R_1}{r_0}\right)^m\right] + r^{-m}R_1^{2m}\left[\left(\frac{R^2}{r_0}\right)^m - r_0^m\right]\right\}. \tag{1.6}$$

The necessary and sufficient condition for an equilibrium point for the Green's function is that $\partial g/\partial\theta = 0$ and $\partial g/\partial r = 0$ for $\theta - \theta_0 = \pi$. The first of these conditions is easily verified; the second gives, for $R_1 = 1$,

$$\frac{\partial g}{\partial r} = -\frac{1}{r_0 + r} - \frac{1}{r}\frac{\log r_0/R}{\log R} - \sum_{m=1}^{\infty}\frac{(-1)^m}{R^{2m}-1}\left\{r^{m-1}[r_0^m - r_0^{-m}] - r^{-m-1}\left[\left(\frac{R^2}{r_0}\right)^m - r_0^m\right]\right\}. \tag{1.7}$$

This evidently becomes zero for the case where the pole and equilibrium point are on the geometric mean circle C_0. In general, however, let $F(r, r_0) = \partial g/\partial r$ where $\theta - \theta_0 = \pi$. Then the necessary and sufficient condition for an equilibrium point is that $F(r, r_0) = 0$.

What we wish to study is the function dr/dr_0. If it vanishes at the equilibrium point, then that point is instantaneously at rest when the pole is moved inward or outward from the geometric mean circle along a radius. From the relation $F(r, r_0) = 0$ at the equilibrium point, we obtain an expression for dr/dr_0 there,

$$\frac{dr}{dr_0} = -\frac{\partial F}{\partial r_0}\Big/\frac{\partial F}{\partial r}. \tag{1.8}$$

The vanishing of dr/dr_0 then depends on the function $\partial F/\partial r_0$, since if $dr/dr_0 = 0$, we must have $\partial F/\partial r_0 = 0$.

From (1.7) $\partial F/\partial r_0$ is given by

$$\frac{\partial F}{\partial r_0} = \frac{1}{(r+r_0)^2} - \frac{1}{r_0 r \log R} - \sum_{m=1}^{\infty}\frac{(-1)^m m}{R^{2m}-1}\{r^{m-1}[r_0^{m-1} + r_0^{-m-1}] + r^{-m-1}[R^{2m}r_0^{-m-1} + r_0^{m-1}]\}. \tag{1.9}$$

For a point for which $r = r_0 = \sqrt{R}$, equation (1.9) becomes

$$\left.\frac{\partial F}{\partial r_0}\right]_{\text{on } C_0} = -\frac{2}{R}\left[\frac{1}{2\log R} - \frac{1}{8} + \sum_{m=1}^{\infty}\frac{(-1)^m m}{R^m - 1}\right]. \tag{1.10}$$

In the particular case $R = e$ which W. S. Vaughn considered (1.10) gives

$$\frac{\partial F}{\partial r_0} = -\frac{2}{e}\left[\frac{3}{8} + \sum_{m=1}^{\infty} \frac{(-1)^m m}{e^m - 1}\right].$$

The quantity in brackets yielded zero to seven significant figures. The value of $\partial F/\partial r_0$, however, for any R is considered below in Section 3 with the result that it cannot vanish.

2. **The necessary and sufficient condition for an equilibrium point by means of doubly periodic functions.** Consider the region S in the z plane. Take $R_1 = 1$, $R > 1$. In putting $u = i \log z$, we transform the circle C_1 into the real axis, the circle C into a parallel straight line of ordinate length $\log R$, the region S into the region K included between these two straight lines. To a point of S there corresponds an infinite number of representations in K which have the same ordinate length, and abscissae forming an arithmetic series of constant difference 2π. Consider the system of periods $2\omega = 2\pi$, $2\omega' = 2i \log R$.

Let $\alpha + \beta i$ be a point in K in the period parallelogram $0, 2\omega, 2\omega', 2\omega + 2\omega'$. It has been shown* that the Green's function for K can be written in the form

$$g(M_0, M) = \text{Real part of } V(u), \tag{2.1}$$

where

$$V(u) = e^{\frac{\beta \log z}{\log R}} e^{\frac{2\beta\eta i u}{\pi}} \frac{\sigma(u - \alpha - \beta i)}{\sigma(u - \alpha + \beta i)}.$$

This function becomes logarithmically infinite at the point $u = \alpha + \beta i$. So the pole in the z plane is at the point $e^{\beta - \alpha i}$.

At the equilibrium point in the z plane we must have $\partial g/\partial z = 0$. But $\frac{\partial g}{\partial z} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial z}$, and since $\partial u/\partial z = i/z$, which does not vanish, then $\partial g/\partial u = 0$; that is, $\partial \log V/\partial u = 0$. This derivative is

$$\frac{\partial \log V}{\partial u} = \frac{\beta}{\omega'} + \frac{2\beta\eta i}{\omega} + \zeta(u - \alpha - \beta i) + \zeta(u - \alpha + \beta i). \tag{2.2}$$

Since the figure is symmetrical we may take $\alpha = 0$. For the conditions $u = \frac{1}{2}\omega' + \omega$, $\beta i = \omega'/2$, $\partial \log V/\partial u$ can be easily shown to vanish.

To see what happens as βi moves up or down on the imaginary axis away from $\omega'/2$ we form $du/d\beta$ at the point of equilibrium. When $\partial \log V/\partial u = 0$, $du/d\beta$ is given by

* Method outlined in Goursat, loc. cit., page 241, problem 14.

$$0 = \frac{1}{\omega'} + \frac{2\eta i}{\pi} + i[\wp(u-\beta i)+\wp(u+\beta i)] + \frac{du}{d\beta}[\wp(u+\beta i)-\wp(u-\beta i)]. \tag{2.3}$$

The necessary condition then that $du/d\beta$ vanish at the equilibrium point is that

$$0 = \frac{1}{\omega'} + \frac{2\eta i}{\pi} + i[\wp(\pi)+\wp(\pi+\omega')]. \tag{2.4}$$

By means of the relations $\wp(\pi)+\wp(\omega')+\wp(\pi+\omega') = 0$, and $\eta\omega'-\eta'\omega = \pi i/2$, this condition reduces to

$$2\zeta(\omega') = \omega'\wp(\omega'). \tag{2.5}$$

Condition (2.5) is also sufficient since $\wp(\omega+\omega')-\wp(\omega)$ does not vanish.

The two necessary conditions obtained from Sections 1 and 2 for an equilibrium point to be instantaneously fixed are equivalent for the case $R_1 = 1$, $R = e$; or $\alpha = 0$, $\beta i = \omega'/2$, $u = \omega+\omega'/2$. This fact can be shown by substituting for $\zeta(\omega')$, $\wp(\omega')$ their values in terms of infinite series, obtained from the theory of elliptic functions. The result is that (2.5) reduces to $-2i\left[\frac{3}{8}+\sum_{m=1}^{\infty}\frac{(-1)^m m}{e^m-1}\right]$.

According to a method of proof suggested by Professor E. T. Bell, it may be shown that the equation $2\zeta(\omega')-\omega'\wp(\omega') = 0$ is not an identity in ω'; in fact, this statement is verified by substituting for $\zeta(\omega')$ and $\wp(\omega')$ their values in terms of infinite series from elliptic functions and proving the coefficients of like terms of the variable are not equal. But it still remains to show that this relation is true for no values of ω, ω'.

The relation (2.5) is in fact homogeneous in ω and ω' and is therefore merely an equation in ω/ω'. In fact

$$\zeta(\omega') = \frac{1}{\omega'} + \sum_{m,m'}{}'\left[\frac{1}{\omega'(1-2m')-2m\omega} + \frac{1}{2m\omega+2m'\omega'} + \frac{\omega'}{(2m\omega+2m'\omega')^2}\right],$$

$$\omega'\wp(\omega') = \frac{1}{\omega'} + \sum_{m,m'}{}'\left[\frac{\omega'}{(\omega'(1-2m')-2m\omega)^2} - \frac{\omega'}{(2m\omega+2m'\omega')^2}\right],$$

$$m, m' = 0, \pm 1, \cdots, \qquad m^2+m'^2 \neq 0.$$

Hence if (2.5) were true for any values of ω, ω' the point of equilibrium would be instantaneously at rest, and therefore the equation $\partial F/\partial r_0 = 0$ would be satisfied for the corresponding value of R. But we shall show that $\partial F/\partial r_0 = 0$ for no value of R.

3. THEOREM 1. *For every real R, greater than 1, the expression*

$$\frac{1}{2\log R}-\frac{1}{8}+\sum_{n=1}^{\infty}\frac{(-1)^n n}{R^n-1}$$

is negative.

The proof of this statement depends on an application by Schlömilch* of the following theorem:

THEOREM. *If the value of an integral of the form*

$$B_n=\frac{2}{\pi}\int_0^{\infty} f(t)\sin nt\,dt \tag{3.1}$$

is known, then the development in series is possible

$$\sum_{n=1}^{\infty} B_n \sin nx = f(x)-\sum_{n=1}^{\infty}[f(2n\pi-x)-f(2n\pi+x)], \tag{3.2}$$

where $f(x)$ is a bounded, continuous function of x, monotonically decreasing to zero, for x in the interval $0<x<2\pi$.

To choose the function $f(x)$ Schlömilch considers the integral formula

$$\int_0^{\infty}\frac{\sin\lambda\varphi}{e^{2\pi\varphi}-1}d\varphi=\frac{1}{4}+\frac{1}{2}\left[\frac{1}{e^{\lambda}-1}-\frac{1}{\lambda}\right],\qquad \lambda\neq 0.$$

Then

$$\int_0^{\infty}\left[\frac{1}{e^{2\pi\varphi}-1}-\frac{1}{2\pi\varphi}\right]\sin\lambda\varphi\,d\varphi=\frac{1}{2}\left[\frac{1}{e^{\lambda}-1}-\frac{1}{\lambda}\right]. \tag{3.3}$$

With the substitutions $\lambda=2n\pi\varrho$, $\varphi=t/2\pi\varrho$, (3.3) becomes

$$\frac{2}{\pi}\int_0^{\infty}\left[\frac{1}{e^{t/\varrho}-1}-\frac{\varrho}{t}\right]\sin nt\,dt=2\varrho\left[\frac{1}{e^{2n\pi\varrho}-1}-\frac{1}{2n\pi\varrho}\right]. \tag{3.4}$$

Schlömilch chooses

$$\begin{aligned} f(t)&=\frac{1}{e^{t/\varrho}-1}-\frac{\varrho}{t},\\ B_n&=2\varrho\left[\frac{1}{e^{2n\pi\varrho}-1}-\frac{1}{2n\pi\varrho}\right]. \end{aligned} \tag{3.5}$$

The function $f(t)$ satisfies the conditions of the above theorem and so equation (3.2) for this function may be written in the form

* Über einige unendliche Reihen, Zeitschr. für Math. und Physik, vol. 23 (1878), p. 132. The suggestion that this reference might prove useful was given by Professor Bell.

$$2\varrho\sum_{n=1}^{\infty}\frac{\sin nx}{e^{2n\pi\varrho}-1}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$(3.6)\qquad =\frac{1}{e^{x/\varrho}-1}-\frac{\varrho}{x}-\sum_{n=1}^{\infty}\left[\frac{1}{2n\pi+x}-\frac{1}{2n\pi-x}\right]$$

$$-\sum_{n=1}^{\infty}\left[\frac{1}{e^{\frac{2n\pi-x}{\varrho}}-1}-\frac{1}{e^{\frac{2n\pi+x}{\varrho}}-1}\right].$$

This can be simplified by the relations

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{1}{2}(\pi-x),\qquad 0<x<2\pi,$$

and

$$\frac{1}{x}-\sum_{n=1}^{\infty}\left[\frac{1}{2n\pi-x}-\frac{1}{2n\pi+x}\right]=\frac{1}{2}\cot\frac{x}{2}.$$

The remaining series on the right may be rewritten

$$\sum_{n=1}^{\infty}\left[\frac{1}{e^{\frac{2n\pi-x}{\varrho}}-1}-\frac{1}{e^{\frac{2n\pi+x}{\varrho}}-1}\right]=\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(e^{rx/\varrho}-e^{-rx/\varrho})\,e^{-2nr\pi/\varrho}$$

$$=\sum_{n=1}^{\infty}\frac{e^{nx/\varrho}-e^{-nx/\varrho}}{e^{2n\pi/\varrho}-1}.$$

With these relations and the identity

$$\frac{1}{e^{x/\varrho}-1}=\frac{1}{2}\frac{e^{x/2\varrho}+e^{-x/2\varrho}}{e^{x/2\varrho}-e^{-x/2\varrho}}-\frac{1}{2},$$

equation (3.6) becomes

$$\frac{x}{2\varrho}+2\varrho\sum_{n=1}^{\infty}\frac{\sin nx}{e^{2n\pi\varrho}-1}$$

$$(3.7)\qquad =-\frac{\varrho}{2}\cot\frac{x}{2}+\frac{1}{2}\frac{e^{x/2\varrho}+e^{-x/2\varrho}}{e^{x/2\varrho}-e^{-x/2\varrho}}-\sum_{n=1}^{\infty}\frac{e^{nx/\varrho}-e^{-nx/\varrho}}{e^{2n\pi/\varrho}-1},$$

or using the relations $(e^u+e^{-u})/2=\cosh u$, $(e^u-e^{-u})/2=\sinh u$, dividing by $\sqrt{\varrho}$ and rearranging terms, we obtain

$$\frac{x}{4\pi\sqrt{\varrho}}+\frac{\sqrt{\varrho}}{2}\cot\frac{x}{2}+2\varrho\sum_{n=1}^{\infty}\frac{\sin nx}{e^{2n\pi\varrho}-1}$$

$$(3.8)\qquad =-\frac{x}{4\pi\sqrt{\varrho}}+\frac{1}{2\sqrt{\varrho}}\coth\frac{x}{2\varrho}-\frac{2}{\sqrt{\varrho}}\sum_{n=1}^{\infty}\frac{\sinh(nx/\varrho)}{e^{2n\pi/\varrho}-1}.$$

In this equation x and ϱ are variables. The equation can be differentiated with respect to x or ϱ. Differentiation with respect to x and multiplication by $\pi\sqrt{\varrho}$, gives

$$\frac{1}{4}-\frac{\pi\varrho}{4\sin^2(x/2)}+2\pi\varrho\sum_{n=1}^{\infty}\frac{n\cos nx}{e^{2n\pi\varrho}-1}$$

(3.9)

$$=-\frac{1}{4}-\frac{\pi}{4\varrho\sinh^2(x/2\varrho)}-\frac{2\pi}{\varrho}\sum_{n=1}^{\infty}\frac{n\cosh(nx/\varrho)}{e^{2n\pi/\varrho}-1}.$$

If the left side of this equation be denoted by $\psi(x, \varrho)$, (3.9) states

$$\psi(iz, \varrho) = -\psi\left(\frac{z}{\varrho}, \frac{1}{\varrho}\right).$$

Equation (3.9) *gives the result we use.* Suppose we take $x = \pi$ and choose ϱ to satisfy the condition,

$$e^{n\log R} = e^{2n\pi\varrho}$$

that is, $\varrho = (\log R)/2\pi$, $R>1$ and real. Then (3.9) becomes, on being simplified and divided by $\log R$

$$\frac{1}{2\log R}-\frac{1}{8}+\sum_{n=1}^{\infty}\frac{(-1)^n n}{e^{n\log R}-1}$$

(3.10)

$$=-\frac{4\pi^2}{(\log R)^2}\left[\frac{1}{8\sinh^2(\pi^2/\log R)}+\sum_{n=1}^{\infty}\frac{n\cosh(2n\pi^2/\log R)}{e^{4n\pi^2/\log R}-1}\right]$$

the right hand member of which is evidently not zero, since the bracket is essentially positive, R being greater than 1. Hence from (3.10), with $e^{n\log R} = R^n$,

$$\frac{1}{2\log R}-\frac{1}{8}+\sum_{n=1}^{\infty}\frac{(-1)^n n}{R^n-1}<0 \tag{3.11}$$

for every $R>1$ and real.

So the function $\partial F/\partial r_0$ for $r = r_0 = \sqrt{R}$ does not vanish for any $R>1$. For $R = e$ (3.11) gives $\frac{3}{8}+\sum_{n=1}^{\infty}\frac{(-1)^n n}{e^n-1}<0$.

The point of equilibrium therefore does not remain fixed when the pole is moved outward or inward from the geometric mean circle along a radius, however small the motion may be.

The investigation of the second derivative, however, from the σ-function treatment yields the following result which helps to account for the physical results of the problem.

THEOREM 2. *The second derivative $d^2u/d\beta^2$ vanishes identically at the point of equilibrium $\beta i = \omega'/2$, $u = \pi + \omega'/2$.*

From Section 2, equation (2.3) defines u as a function of β. This differentiated gives

$$0 = \wp'(u+\beta i)\left(\frac{du}{d\beta}+i\right)^2 - \wp'(u-\beta i)\left(\frac{du}{d\beta}-i\right)^2 + \frac{d^2u}{d\beta^2}\left[\wp(u+\beta i)-\wp(u-\beta i)\right]. \tag{3.12}$$

For the special values $\beta i = \omega'/2$, $u = \pi + \omega'/2$ (3.12) becomes

$$0 = \wp'(\pi+\omega')\left(\frac{du}{d\beta}+i\right)^2 - \wp'(\pi)\left(\frac{du}{d\beta}-i\right)^2 + \frac{d^2u}{d\beta^2}\left[\wp(\pi+\omega')-\wp(\pi)\right]. \tag{3.13}$$

Since $\wp'(\pi+\omega') = \wp'(\pi) = 0$ and $\wp(\pi+\omega') - \wp(\pi) \neq 0$, then at the equilibrium point

$$\frac{d^2u}{d\beta^2} \equiv 0.$$

4. An approximation for the magnitude of the shift of the equilibrium point may be obtained from equation (3.10). This gives

$$\left|\frac{1}{2\log R} - \frac{1}{8} + \sum_{m=1}^{\infty}\frac{(-1)^m m}{R^m - 1}\right| = \frac{4\pi^2}{(\log R)^2}\left[\frac{1}{8\sinh^2(\pi^2/\log R)} + \sum_{m=1}^{\infty}\frac{m\cosh(2m\pi^2/\log R)}{e^{4m\pi^2/\log R}-1}\right]. \tag{4.1}$$

The form of the second member may be changed, using the relations

$$\frac{1}{8\sinh^2(\pi^2/\log R)} = \frac{1}{2}\,\frac{e^{-2\pi^2/\log R}}{(1-e^{-2\pi^2/\log R})^2}$$

and

$$\sum_{m=1}^{\infty}\frac{m\cosh(2m\pi^2/\log R)}{e^{4m\pi^2/\log R}-1} = \frac{1}{2}\sum_{m=1}^{\infty}\frac{m(1+e^{-4m\pi^2/\log R})e^{-2m\pi^2/\log R}}{1-e^{-4m\pi^2/\log R}}.$$

The quantity on the right hand is surely less than the expression

$$\frac{1}{2}\,\frac{1+e^{-4\pi^2/\log R}}{1-e^{-4\pi^2/\log R}}\sum_{m=1}^{\infty} m e^{-2m\pi^2/\log R} = \frac{1}{2}\,\frac{1+e^{-4\pi^2/\log R}}{1-e^{-4\pi^2/\log R}}\,\frac{e^{-2\pi^2/\log R}}{(1-e^{-4\pi^2/\log R})^2}.$$

Then an upper bound for $|\partial F/\partial r_0|$ is given by

$$\frac{2\pi^2}{(\log R)^2}\,\frac{e^{-2\pi^2/\log R}}{(1-e^{-2\pi^2/\log R})^2}\,\frac{e^{-2\pi^2/\log R}}{(1-e^{-4\pi^2/\log R})}.$$

For the particular case $R_1 = 1$, $R = e$, this upper bound for $|\partial F/\partial r_0|$ may be written in a simpler form by putting $k = e^{-2\pi^2} = 2.7 \times 10^{-9}$.

$$\left|\frac{\partial F}{\partial r_0}\right| < \frac{4\pi^2 k}{(1-k)^3(1+k)} = 4\pi^2 k(1+3k)(1-k) = 4\pi^2 k \text{ approximately.}$$

Hence $|\partial F/\partial r_0| < 0.000\,000\,1066$.

In fact, for $R = e$, $\partial F/\partial r_0 = -\frac{2}{e}(0.000\,000\,1066)$ very nearly. To obtain a value for dr/dr_0 for $R = e$ we form $\frac{\partial F}{\partial r_0}\Big/\frac{\partial F}{\partial r}$. At the equilibrium point

$$\frac{\partial F}{\partial r} = -\frac{2}{R}\left[\frac{1}{8} - \sum_{m=1}^{\infty}\frac{(-1)^m m}{R^m+1}\right].$$

For $R = e$

$$\frac{\partial F}{\partial r} = -\frac{2}{e}\left[\frac{1}{8} - \sum_{m=1}^{\infty}\frac{(-1)^m m}{e^m+1}\right] = -\frac{2}{e}(0.001), \text{ nearly.}$$

The required value of dr/dr_0 is then approximately

$$\frac{dr}{dr_0} = 0.0001.$$

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ON A CLASS OF INTEGRAL FUNCTIONS.*

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1. Let $\varphi(z)$ be an integral function of z, where $z = \varrho e^{i\psi}$. Then, evidently, there is a class of integral functions for which the asymptotic equality

$$\limsup_{\varrho\to\infty} \frac{\log|\varphi(\varrho e^{i\psi})|}{\varrho} = A\cos\psi + B\sin\psi, \qquad (\text{for } |\psi| \leqq \pi), \tag{1.1}$$

where A and B are some real constants, is true. The simplest number of this class of functions is e^z. In this paper, I propose to establish certain results concerning this class of functions.

2. Theorem 1. *If $\varphi(z)$ is an integral function of z satisfying the relation*

$$\varphi(z) = O(e^{k\varrho}), \tag{2.1}$$

where k is some real constant, everywhere in the plane; the inequality

$$|\varphi(\varrho e^{i\psi})| < e^{\varrho(\lambda(\psi)+\varepsilon)}, \text{ where } \lambda(\psi) \equiv A\cos\psi + B\sin\psi, \tag{2.2}$$

is satisfied, for every positive ε and $\varrho \geqq \varrho_0(\varepsilon)$ on the three radii vectores $\psi = \psi_1$, ψ_2, and ψ_3, which are such that the angle between each consecutive pair is less than π; then

$$\limsup_{\varrho\to\infty} \frac{\log|\varphi(\varrho e^{i\psi})|}{\varrho} = A\cos\psi + B\sin\psi, \qquad (\text{for } |\psi| \leqq \pi), \tag{2.3}$$

unless $\varphi(z) \equiv 0$.

In the first place, we observe that, in virtue of the fundamental theorem of the classical memoir† of Phragmén and Lindelöf, the hypotheses (2.1) and (2.2) at once lead to the inequality

$$\frac{\log|\varphi(\varrho e^{i\psi})|}{\varrho} < A\cos\psi + B\sin\psi + \varepsilon, \qquad (\text{for } |\psi| \leqq \pi), \tag{2.4}$$

ε being an arbitrarily small positive number, and ϱ exceeding a certain finite limit depending upon ε, but independent of ψ.

Next, we proceed to show that the inequality (2.4) implies the asymptotic equality (2.3), unless $\varphi(z) \equiv 0$.

* Received September 21, 1928.

† "Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier", *Acta Math.*, vol. 31 (1908), pp. 381–406. The theorem referred to here is that of § 10.

Suppose, if possible, that the equality (2.3) is not true. Then there must be at least one value of ψ, say $\bar{\psi}$, in the interval $-\pi \leqq \psi \leqq \pi$, for which

$$\limsup_{\varrho\to\infty} \frac{\log|\varphi(\varrho e^{i\psi})|}{\varrho} = A\cos\psi + B\sin\psi - \delta, \tag{2.5}$$

where δ is some fixed positive number.

Now suppose $\varphi(z) = \sum_0^\infty c_\nu z^\nu/\nu! = O(e^{k\varrho})$, and let

$$\begin{aligned} I(s) &= \int_0^\infty \varphi(z) e^{-sz}\, dz, \\ &= \int_0^\infty e^{-sz} \sum_0^\infty \frac{c_\nu z^\nu}{\nu!}\, dz. \end{aligned} \qquad (s = \sigma + it), \tag{2.6}$$

By Cauchy's theorem, we have

$$c_\nu = O((k+\varepsilon)^\nu), \qquad \text{for every } \varepsilon > 0, \tag{2.7}$$

and hence it follows that

$$|e^{-sz}| \cdot \sum_0^\infty \left|\frac{c_\nu z^\nu}{\nu!}\right| < K e^{-(\sigma-k-\varepsilon)z},$$

where K is some constant, for positive values of z.

Hence, if $\sigma > k$, we can perform term-by-term integration in (2.6), from which it follows that

$$I(s) = \sum_{\nu=0}^\infty \frac{c_\nu}{s^{\nu+1}} = \int_0^\infty \varphi(z)\, e^{-sz}\, dz, \tag{2.8}$$

where the series $\sum_{\nu=0}^\infty c_\nu/s^{\nu+1}$ is absolutely and uniformly convergent for $|s| \geqq k' > k$, and vanishes at infinity.

Now let

$$I(s, \psi) = \int_0^{\infty(\psi)} \varphi(z)\, e^{-sz}\, dz,$$

where the integral is taken along a radius vector making an angle ψ, $(|\psi| \leqq \pi)$, with the positive real axis. Putting $s = Re^{i\theta}$, we have, by virtue of (2.4),

$$|\varphi(\varrho e^{i\psi})\, e^{-sz}| < e^{-\varrho\{R\cos(\theta+\psi) - A\cos\psi - B\sin\psi - \varepsilon\}}, \qquad (\varepsilon > 0),$$

so that $I(s, \psi)$ represents a regular function of s in the domain $D(\psi)$ given by

$$R\cos(\theta+\psi) > A\cos\psi + B\sin\psi + \varepsilon. \tag{2.9}$$

Now suppose we take two values of ψ, say ψ' and ψ'', sufficiently near to each other. Then the domains $D(\psi')$ and $D(\psi'')$ will have a common portion, and for a domain of values of s within this portion, the expressions $I(s, \psi')$ and $I(s, \psi'')$ will be equal. For

$$\left|\varrho e^{i\psi}\varphi(\varrho e^{i\psi})\, e^{-R\varrho e^{i(\theta+\psi)}}\right| \to 0, \quad \text{as } \varrho \to \infty,$$

if s lies in the region given by

$$R\cos(\theta+\psi) > k,$$

where ψ varies continuously from ψ' to ψ''.

Therefore, by the principle of analytic continuation, either of $I(s, \psi')$ and $I(s, \psi'')$ gives the analytic continuation of the other. Now introducing a set of values of ψ sufficiently near to each other between $-\pi$ and π, and comparing the expressions $I(s, \psi)$, two at a time, corresponding to these values of ψ, we arrive at the conclusion that $I(s, \psi)$, where ψ may have *any* value between $-\pi$ and π, gives the analytic continuation of the function $I(s)$ defined by (2.6).

It follows, therefore, that the function $I(s)$ is regular over the domain D obtained by varying ψ from $-\pi$ to π in $D(\psi)$ given by (2.9). That is, D is the region lying exterior to the envelope of the lines

$$R\cos(\theta+\psi) = A\cos\psi + B\sin\psi + \varepsilon, \qquad (|\psi| \leqq \pi). \tag{2.10}$$

Now this envelope is

$$(\sigma - A)^2 + (t+B)^2 = \varepsilon^2, \tag{2.11}$$

so that it follows that $I(s)$ is a regular function of s in the whole plane except at the point $s = A - iB$. Further, it can be easily shown that $I(s)$ is also a uniform function of s in the region where it is regular. Also $I(s)$ vanishes at infinity. Hence $I(s)$ is either an integral function of $1/(s-\alpha)$, where $\alpha = A - iB$, without a constant term, or is identically zero.

So far we have made no use of the equality (2.5). Now utilizing this and arguing as above, we can prove that $I(s)$ is an analytic function of s in the half-plane

$$R\cos(\theta+\bar{\psi}) > A\cos\bar{\psi} + B\sin\bar{\psi} + \varepsilon - \delta, \tag{2.12}$$

where ε is an arbitrarily small positive number.

But the half-plane (2.12) is easily seen to include within it the point $s = A - iB$, so that if (2.4) and (2.5) are both true then $I(s) \equiv 0$, and so is $\varphi(z)$.

It follows, therefore, that if $\varphi(z)$ is not identically zero, the inequality (2.4) implies the asymptotic equality (2.3).

COROLLARY. *If $\varphi(z)$ is an integral function of order 1 and type zero in Pringsheim's terminology*, then it follows from the preceding theorem that*

$$\limsup_{\varrho\to\infty}\frac{\log|\varphi(\varrho e^{i\psi})|}{\varrho}=0, \qquad (\text{for } |\psi|\leqq\pi), \tag{2.13}$$

and conversely.

The corresponding result for an integral function of order 1 and type k, $(k>0)$, is not necessarily true, as will be seen from a consideration of the function e^{kz}.

3. THEOREM 2†. *The necessary and sufficient condition that an integral function $\varphi(z)$ should satisfy the asymptotic equality*

$$\limsup_{\varrho\to\infty}\frac{\log|\varphi(\varrho e^{i\psi})|}{\varrho}=A\cos\psi+B\sin\psi, \quad (\text{for } |\psi|\leqq\pi), \tag{3.1}$$

is that $\varphi(z)$ should be of the form

$$\varphi(z)=\frac{1}{2\pi i}\int_C e^{uz}\,d(u)\,du, \tag{3.2}$$

where C is a simple contour enclosing the point $u=A-iB$, and $d(u)$ is an integral function of $\dfrac{1}{u-(A-iB)}$, not identically a constant.

In the first place we shall show that the condition is sufficient. Replace the contour by a circle whose centre is the point $u=A-iB$, and whose radius is ε.

Since $d(u)$ is an integral function of $1/(u-\alpha)$, $(\alpha=A-iB)$, not identically a constant, we can expand it in Laurent's series

$$d(u)=d_0+\frac{d_1}{(u-\alpha)}+\frac{d_2}{(u-\alpha)^2}+\cdots, \tag{3.3}$$

where all the coefficients of $1/(u-\alpha)^n$, $(n=1,2,3,\cdots)$, are not zero.

We, therefore, have, by a simple substitution,

$$\varphi(z)=\frac{e^{\alpha z}}{2\pi i}\int e^{u'z}\left\{d_0+\frac{d_1}{u'}+\frac{d_2}{u'^2}+\cdots\right\}du', \tag{3.4}$$

* A. Pringsheim, "Elementare Theorie der ganzen transcendenten Funktionen von endlicher Ordnung", *Math. Ann.*, vol. 58 (1904), pp. 257–342.

† Compare this with a lemma established by G. H. Hardy, *Acta Math.*, vol. 42, p. 333.

the path of integration being the circle of radius ϵ round the point $u' = 0$.

Now the series within the brackets in (3.4) being uniformly convergent on the path of integration, we can perform term-by-term integration, and thus have

$$\varphi(z) = \frac{e^{az}}{2\pi i} \sum_{\nu=0}^{\infty} d_\nu \int \frac{e^{u'z}}{u'^{\nu}}\, du',$$

$$= e^{az} \sum_{\nu=1}^{\infty} \frac{d_\nu\, z^{\nu-1}}{(\nu-1)!},$$

where the infinite series represents an integral function of z, not identically zero, and satisfies the relation $\sum_{\nu=1}^{\infty} d_\nu\, z^{\nu-1}/(\nu-1)! = O(e^{\epsilon\varrho})$, for every positive ϵ, in the whole plane.

It follows, therefore, that $\varphi(z)$ is not identically zero, and that it satisfies the inequality (2.4). Hence, by virtue of Theorem 1, it satisfies the asymptotic equality (3.1).

In the second place, we shall show that the condition is necessary. If $\varphi(z) = \sum_{0}^{\infty} d_\nu\, z^\nu/\nu!$ is an integral function of z satisfying the asymptotic equality (3.1), then $\varphi(z)$ is not identically zero, and it follows from § 2 that the function (3.6)

$$d(u) = \sum_{\nu=0}^{\infty} \frac{d_\nu}{u^{\nu+1}}$$

is an integral function of $1/(u-(A-iB))$, not identically a constant.

Now, by Cauchy's theorem, we have

$$\frac{z^\nu}{\nu!} = \frac{1}{2\pi i} \int \frac{e^{uz}}{u^{\nu+1}}\, du, \tag{3.7}$$

the contour of integration being a circle round the origin.

Multiplying both sides of (3.7) by d_ν, and summing under the sign of integration from 0 to ∞, we have

$$\varphi(z) = \sum_{\nu=0}^{\infty} \frac{d_\nu\, z^\nu}{\nu!} = \frac{1}{2\pi i} \int e^{uz} \sum_{\nu=0}^{\infty} \frac{d_\nu}{u^{\nu+1}}\, du, \tag{3.8}$$

$$= \frac{1}{2\pi i} \int e^{uz}\, d(u)\, du,$$

provided we take the radius of the contour of integration to be sufficiently large.

But now it is clear that the contour of integration in (3.8) may be replaced by any closed curve surrounding the point $u = A - iB$ in the u-plane.

Hence

$$\varphi(z) = \frac{1}{2\pi i}\int_C e^{uz}\, d(u)\, du. \tag{3.9}$$

4. Now we are in a position to establish the main theorem of this paper.

THEOREM 3. *The necessary and sufficient condition that the integral function* $\varphi(z) = \sum_0^\infty c_\nu z^\nu/\nu!$ *should satisfy the asymptotic equality*

$$\lambda(\psi) \equiv \limsup_{\varrho\to\infty} \frac{\log|\varphi(\varrho e^{i\psi})|}{\varrho} = A\cos\psi + B\sin\psi, \quad (\text{for } |\psi| \leqq \pi), \tag{4.1}$$

A and B being some real constants not simultaneously zero, is that there exists an integral function $c(z)$, which assumes the values $c_0, c_1, c_2, \cdots$ for $z = 1, 2, 3, \cdots$, and which satisfies the asymptotic equality

$$\begin{aligned} H(\psi) &\equiv \limsup_{\varrho\to\infty} \frac{\log|c(\varrho e^{i\psi})|}{\varrho} \\ &= \tan^{-1}\left(\frac{B}{A}\right)\sin\psi + \frac{1}{2}\cos\psi\log(A^2+B^2), \end{aligned} \tag{4.2}$$

for $|\psi| \leqq \pi$, $|\tan^{-1}(B/A)|$ being less than or equal to $\pi/2$.

In the first place we shall show that the condition is sufficient.

Let us consider the Taylor's series

$$h(w) = \sum_{\nu=0}^\infty c_\nu w^{\nu+1} = \sum_{\nu=0}^\infty c(\nu+1) w^{\nu+1} = \sum_{\nu=1}^\infty c(\nu) w^\nu, \tag{4.3}$$

where $c(z)$ satisfies (4.2).

The asymptotic equality (4.2) implies that $c(z)$ is not identically zero, and so $c_0, c_1, c_2, c_3, \cdots$ do not all vanish.* Hence $h(w)$ does not reduce to a constant. Further (i) $c(z) = O(e^{kz})$, where k is some constant throughout the plane; and (ii) $c(z) = O(e^{(\pi/2)\varrho})$, on the imaginary axis, so that following the method of Carlson,† we can prove that $h(w)$ is an analytic function of w all over the plane (including infinity) with the exception of the sole finite singularity at the point whose polar coördinates are

$$\left(\frac{1}{\sqrt{A^2+B^2}},\ \tan^{-1}\left(\frac{B}{A}\right)\right).$$

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* F. Carlson, *"Sur une classe de séries de Taylor"*, Thèse, Upsal (1914), p. 58.

† *Loc. cit.*, p. 18. In this connection see also G. H. Hardy, "On two theorems of F. Carlson and S. Wigert", *Acta Math.*, vol. 42, pp. 327–339; and, in particular, the theorem of p. 338.

Making the substitution $w = 1/u$, we find that

$$d(u) \equiv h(w) = \sum_{\nu=0}^{\infty} \frac{c_\nu}{u^{\nu+1}} \tag{4.4}$$

is an integral function of $1/(u-(A-iB))$, not identically a constant.

Now, proceeding as in the latter half of § 3, we have

$$\begin{aligned} \varphi(z) = \sum_{\nu=0}^{\infty} \frac{c_\nu z^\nu}{\nu!} &= \frac{1}{2\pi i} \int e^{uz} \sum_{\nu=0}^{\infty} \frac{c_\nu}{u^{\nu+1}} du, \\ &= \frac{1}{2\pi i} \int_C e^{uz} d(u)\, du, \end{aligned} \tag{4.5}$$

where C is a simple contour enclosing the point $u = A - iB$.

It follows, therefore, from Theorem 2 that $\varphi(z)$ satisfies the asymptotic equality (4.1).

In the second place we shall assume that there is an integral function $\varphi(z)$ satisfying the asymptotic equality (4.1), and then show that there exists an integral function $c(z)$ which assumes the values $c_0, c_1, c_2, \cdots$ for $z = 1, 2, 3, \cdots$, and which satisfies the asymptotic equality (4.2).

$\varphi(z)$ not being identically zero, the function $I(s)$, as defined by

$$I(s) = \sum_{\nu=0}^{\infty} \frac{c_\nu}{s^{\nu+1}} = \int_0^\infty \varphi(z) e^{-sz}\, dz, \tag{4.6}$$

may be shown, as in § 2, to be an integral function of $1/(s-(A-iB))$ without a constant term.

Making the substitution $s = 1/u$ we have $g(u) \equiv I(1/u)$ regular and uniform all over the plane (including infinity) with the exception of the sole finite singularity at $u = (A+iB)/(A^2+B^2)$.

Now consider the integral function

$$c(z) = \frac{-1}{2\pi i} \int \frac{g(u)}{u^{z+1}} du, \tag{4.7}$$

the contour of integration being a closed curve Γ enclosing the point $u = (A+iB)/(A^2+B^2)$, and u^{z+1} being interpreted to mean $\exp.\{(z+1)\log u\}$, where $\log u$ has its principal value.

It is evident that, if $R(z) \geqq 1$, we may deform this contour Γ into a circle Σ having its centre at the origin and radius less than $1/\sqrt{A^2+B^2}$. That is, if $R(z) \geq 1$,

$$c(z) = \frac{1}{2\pi i} \int_\Sigma \frac{g(u)}{u^{z+1}} du, \tag{4.8}$$

so that it follows that the integral function $c(z)$ assumes the values $c_0, c_1, c_2, \cdots$ for $z = 1, 2, 3, \cdots$.

In (4.7) let us now effect the transformation $u = e^{-\zeta}$, $(\zeta = \epsilon + i\eta)$, where that value of $\log u$ is chosen whose imaginary part lies between $-\pi$ and π. Plainly we may suppose the closed contour Γ enclosing the point $u = (A + iB)/(A^2 + B^2)$ to be of such a form that it becomes transformed into a circle C of arbitrarily small radius ϵ round the corresponding the point $\frac{1}{2}\log(A^2 + B^2) - i\tan^{-1}(B/A)$ in the ζ-plane. That is

$$c(z) = \frac{-1}{2\pi i}\int_C e^{z\zeta} g(e^{-\zeta})\, d\zeta. \tag{4.9}$$

Also $-g(e^{-\zeta})$ is regular except for $\zeta = \frac{1}{2}\log(A^2 + B^2) - i(\tan^{-1}B/A \pm 2k\pi)$, $(k = 0, 1, 2, 3, \cdots)$, and is therefore of the form

$$-g(e^{-\zeta}) = F(\zeta) + S(\zeta),$$

where $F(\zeta)$ is an integral function of $1/(\zeta - \beta)$, $(\beta = \frac{1}{2}\log(A^2 + B^2) - i\tan^{-1}B/A)$, and $S(\zeta)$ is a power series whose radius of convergence is 2π. Therefore,

$$c(z) = \frac{1}{2\pi i}\int_C e^{z\zeta} F(\zeta)\, d\zeta. \tag{4.10}$$

Now it follows from Theorem 2 that $c(z)$ satisfies the asymptotic equality (4.2).

In virtue of a theorem of Carlson already referred to, the function $c(z)$ is unique. Also $\varphi(z)$ and $c(z)$ belong to the same class of functions so far as the property (1.1) is concerned.

Particular cases. If $A = 1$, $B = 0$, then $\lambda(\psi) = \cos\psi$, and $H(\psi) = 0$. If $A = 0$, $B = 1$, then $\lambda(\psi) = \sin\psi$, and $H(\psi) = \pi/2 \sin\psi$.

Finally, we may remark that there exist pairs of integral functions $\varphi(z)$ and $c(z)$ for which $\lambda(\psi)$ and $H(\psi)$ are identically equivalent.

This will be the case if

$$B = \tan^{-1}\left(\frac{B}{A}\right), \tag{4.11}$$

and

$$A = \frac{1}{2}\log(A^2 + B^2), \tag{4.12}$$

which lead to

$$B\operatorname{cosec} B = e^{B\cot B}, \tag{4.13}$$

which has real solutions.

5. Wigert* was the first to prove the following theorem.

The necessary and sufficient condition that

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots \tag{5.1}$$

should be an integral function of $1/(1-z)$, *not regular for* $z = 1$, *is that there should be an integral function* $c(z)$ *of order* 1 *and type zero, which takes the values* $c_1, c_2, c_3, \cdots$ *for* $z = 1, 2, 3, \cdots$.

Now from Theorem 3 we know that the necessary and sufficient condition that there should exist an integral function $c(z)$ of order 1 and type zero, taking the values $c_1, c_2, c_3, \cdots$ for $z = 1, 2, 3, \cdots$ is that the integral function $\varphi(z) = \sum_{\nu=0}^{\infty} c_{\nu+1} z^\nu/\nu!$ should satisfy the asymptotic equality

$$\limsup_{\varrho\to\infty} \frac{\log|\varphi(\varrho e^{i\psi})|}{\varrho} = \cos\psi, \qquad (\text{for } |\psi| \leqq \pi). \tag{5.2}$$

Hence we have the following theorem.

THEOREM 4. *The necessary and sufficient condition that* (5.1) *should be an integral function of* $1/(1-z)$, *not regular for* $z = 1$, *is that the integral function* $\varphi(z) = \sum_{\nu=0}^{\infty} c_{\nu+1} z^\nu/\nu!$ *should satisfy the asymptotic equality* (5.2).

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* "Sur les fonctions entières", *Öfversikt af K. Vet.-Ak. Förhandlingar*, Årg. 57 (1900), pp. 1001–1011. See also G. H. Hardy, "On two theorems of F. Carlson and S. Wigert", *loc. cit.*

ALLAHABAD,
June 30, 1928.

THE JACOBI CONDITION FOR A PROBLEM OF MAYER WITH VARIABLE END POINTS*.

By E. L. Mackie.

Introduction. The study carried out in this paper was suggested by the problem considered in a thesis by G. Senftner at Rostock in 1905, entitled *Ein Mechanisches Problem aus der Variationsrechnung.* Briefly stated, his problem is to find the shape of an inextensible wire of given length, suspended at its ends from two fixed points on a horizontal line and coplanar with the line, in order that the time of oscillation of the wire through a given angle about the horizontal line as axis shall be a maximum or a minimum.

The integral that he gets to be made an extreme is

$$L = \frac{\int_0^a y^2 \sqrt{1+y'^2}\, dx}{\int_0^a y \sqrt{1+y'^2}\, dx} \tag{1}$$

the x-axis being taken in the horizontal line of the ends of the wire, the y-axis vertically downward, and a being the distance along the x-axis between the points from which the two ends of the wire are suspended. The length of the wire is given by the integral

$$l = \int_0^a \sqrt{1+y'^2}\, dx \tag{2}$$

and so his problem is to find among all admissible arcs joining the fixed points $(0, 0)$ and $(a, 0)$, and giving (2) a fixed value l, one which minimizes or maximizes the quotient (1). The unusual feature of this problem is the type of quantity, the quotient of two integrals, which is to be made an extreme, and his application of Jacobi's condition does not seem to be justified, as he fails to obtain any transversality condition such as those upon which the determination of focal points for analogous problems is known to depend.

In the following pages a generalization of the problem of Senftner is considered, in which arbitrary functions of x, y, y' replace the integrands of (1) and (2). The problem here considered is that of finding among the

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sets of functions $y(x)$, $u(x)$, $v(x)$, $w(x)$ $(x_1 \leqq x \leqq x_2)$ which satisfy the differential equations,

$$u' - f(x, y, y') = 0, \quad v' - g(x, y, y') = 0, \quad w' - h(x, y, y') = 0,$$

and the end conditions at 1 and 2,

$$\begin{aligned} x_1 - \alpha_1 &= 0, & x_2 - \alpha_2 &= 0, \\ y(x_1) - \beta_1 &= 0, & y(x_2) - \beta_2 &= 0, \\ u(x_1) = v(x_1) = w(x_1) &= 0, & w(x_2) - l &= 0, \end{aligned}$$

one which minimizes the quotient $u(x_2)/v(x_2)$. It is readily seen that the problem includes that of Senftner, when

$$f(x, y, y') = y^2\sqrt{1+y'^2}, \qquad g(x, y, y') = y\sqrt{1+y'^2},$$
$$h(x, y, y') = \sqrt{1+y'^2}.$$

The theory developed applies equally to a maximizing or a minimizing problem, but for simplicity we shall use the terminology of the latter in this paper.

This generalized problem is itself a special case of the problem of Mayer with variable end points studied by Bliss*. Using his theory, we set up the differential equations and obtain the transversality condition at the second end point for our problem, and then an analogue of Jacobi's Condition is found. The last-named contribution is the important part of the paper, as it is believed that this is the first time that such a condition has been justified for the type of problem studied by Bliss.

The following discussion is divided into five sections. Section 1 contains a more detailed statement of the problem to be considered. In Section 2 the Euler-Lagrange differential equations are deduced as a first necessary condition on the minimizing arc E_{12}, and their solution yielding a seven-parameter family of extremals is discussed. Also, a transversality condition for the minimizing arc at the second end point is found. In Section 3 by applying the transversality and other end conditions at the second end point, the class of arcs within which E_{12} lies is reduced from the seven-parameter family of extremals to a four-parameter family satisfying the end conditions at the second end point. Section 4 gives a method of selecting from this four-parameter family a one-parameter family of extremals containing E_{12}, in such a way that the family has an enveloping curve D lying on an enveloping four-space. It also gives a method of determining

* Trans. Amer. Math. Soc., vol. 19 (1918), p. 305.

this same one-parameter family directly from the seven-parameter family of extremals. In Section 5 it is shown that an envelope theorem and a necessary condition analogous to that of Jacobi apply to this problem just as analogous conditions apply to simpler problems in the Calculus of Variations.

1. **Statement of the problem.** The problem proposed is to find in the class of sets of functions,

$$y(x), \quad u(x), \quad v(x), \quad w(x) \qquad (x_1 \leqq x \leqq x_2), \tag{3}$$

satisfying the differential equations,

$$u' - f(x, y, y') = 0, \quad v' - g(x, y, y') = 0, \quad w' - h(x, y, y') = 0, \tag{4}$$

and the end conditions at 1 and 2,

$$\begin{aligned} x_1 - \alpha_1 &= 0, & x_2 - \alpha_2 &= 0, \\ y(x_1) - \beta_1 &= 0, & y(x_2) - \beta_2 &= 0, \\ u(x_1) = v(x_1) = w(x_1) &= 0, & w(x_2) - l &= 0, \end{aligned} \tag{5}$$

one which minimizes $u(x_2)/v(x_2)$. Let the functions f_i $(i = 1, \cdots, 9)$ of the end values of x, y, u, v, w be defined by the equations

$$\begin{aligned} f_1 &= u_2/v_2, & f_4 &= u_1, & f_7 &= x_2 - \alpha_2, \\ f_2 &= x_1 - \alpha_1, & f_5 &= v_1, & f_8 &= y_2 - \beta_2, \\ f_3 &= y_1 - \beta_1, & f_6 &= w_1, & f_9 &= w_2 - l, \end{aligned} \tag{6}$$

where y_i, u_i, v_i, w_i, $(i = 1, 2)$ are the values of y, u, v, w, respectively, at x_1 and x_2, and consider a minimizing arc E_{12},

$$y = y(x), \quad u = u(x), \quad v = v(x), \quad w = w(x) \qquad (x_1 \leqq x \leqq x_2),$$

which fulfills the above requirements—that is, which minimizes f_1 in the class of sets (3) which satisfy (4) and make $f_2 = f_3 = \cdots = f_9 = 0$. It is assumed that E_{12} is of class C″,* that the admissible arcs among which a minimum is sought are of class C′, and that there is a neighborhood R of the values x, y, y', about E_{12} such that in R f, g, and h are of class C^{IV}.

The admissible curves which satisfy (4) and (5) all pass through the fixed point 1, whose coördinates in the five-space are $(x, y, u, v, w) = (\alpha_1, \beta_1, 0, 0, 0)$, and have their second end points in the two-dimensional set of points (x, y, u, v, w) which satisfy the equations

$$x = \alpha_2, \qquad y = \beta_2, \qquad w = l.$$

* See Bolza, Vorlesungen über Variationsrechnung, pp. 13, 63.

2. The differential equations for a minimizing arc, and the transversality condition. The problem of the preceding Section, as was stated in the Introduction, is a special case of the Mayer problem with variable end points studied by Bliss. By applying his more general theory here, it is known that if Ω denotes the function

$$(7)\quad \Omega(x, y, y', u', v', w', \lambda_1, \lambda_2, \lambda_3) = \lambda_1(u' - f) + \lambda_2(v' - g) + \lambda_3(w' - h),$$

then a necessary condition that E_{12} shall be a minimizing arc is that there shall exist functions $\lambda_i = \lambda_i(x)$ $(i = 1, 2, 3)$ not all identically zero on the interval x_1, x_2, which with the functions $y(x)$, $u(x)$, $v(x)$, $w(x)$, defining the minimizing arc satisfy the following conditions:

They are solutions of the differential equations (4) and the equations

$$(8)\quad \begin{aligned} &\Omega_y - \frac{d}{dx}\Omega_{y'} = 0, \qquad \Omega_v - \frac{d}{dx}\Omega_{v'} = 0, \\ &\Omega_u - \frac{d}{dx}\Omega_{u'} = 0, \qquad \Omega_w - \frac{d}{dx}\Omega_{w'} = 0, \end{aligned}$$

the subscripts denoting partial derivatives and the primes denoting derivatives with respect to x, as they shall throughout this paper, except when otherwise specifically stated. Furthermore, they make the following determinant vanish:

$$(9)\quad \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial w_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial v_2} & \frac{\partial f_1}{\partial w_2} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & & & & & & & \frac{\partial f_2}{\partial w_2} \\ \cdots & & & & & & & & & \cdots \\ \frac{\partial f_9}{\partial x_1} & \cdots & & & & & & & & \frac{\partial f_9}{\partial w_2} \\ a_1 & b_1 & \Omega_{u'}(x_1) & \Omega_{v'}(x_1) & \Omega_{w'}(x_1) & -a_2 & -b_2 & -\Omega_{u'}(x_2) & -\Omega_{v'}(x_2) & -\Omega_{w'}(x_2) \end{vmatrix}$$

where a_1, b_1, a_2, b_2 are the values at x_1 and x_2 of the expressions

$$\begin{aligned} a_i &= \Omega(x_i) - \Omega_{y'}(x_i)\, y' - \Omega_{u'}(x_i)\, u' - \Omega_{v'}(x_i)\, v' - \Omega_{w'}(x_i)\, w', \\ b_i &= \Omega_{y'}(x_i) \qquad\qquad (i = 1, 2). \end{aligned}$$

On account of the special character of the function Ω, the equations (8) take the form

$$(10)\quad \Omega_y - \frac{d}{dx}\Omega_{y'} = 0, \quad -\frac{d}{dx}\lambda_1 = 0, \quad -\frac{d}{dx}\lambda_2 = 0, \quad -\frac{d}{dx}\lambda_3 = 0,$$

the last three of these equations having as their solutions,

$$\lambda_1 = \text{constant}, \qquad \lambda_2 = \text{constant}, \qquad \lambda_3 = \text{constant}.$$

The first equation of (10) is a second order differential equation in x and y, which can be written in the form

$$-\lambda_1 f_y - \lambda_2 g_y - \lambda_3 h_y + \frac{d}{dx}(\lambda_1 f_{y'} + \lambda_2 g_{y'} + \lambda_3 h_{y'}) = 0, \tag{11}$$

or

$$\Omega_{y'y'}\, y'' + \Omega_{y'y}\, y' + \Omega_{y'x} = \Omega_y. \tag{12}$$

Assuming that

$$\Omega_{y'y'} = -\lambda_1 f_{y'y'} - \lambda_2 g_{y'y'} - \lambda_3 h_{y'y'} \neq 0, \tag{13}$$

in a neighborhood of E_{12}, we find that the solution of (12) for which $\lambda_1 \neq 0$ has the form

$$y = y(x, c_1, c_2, c_3, c_7), \tag{14}$$

where $c_2 = \lambda_2/\lambda_1$, $c_3 = \lambda_3/\lambda_1$, and c_1, c_7 are constants of integration. On account of the continuity properties assumed for f, g, and h, this function and its derivatives with respect to x are defined and of class C'' in a neighborhood of the values (x, c_1, c_2, c_3, c_7) defining the special minimizing arc E_{12} which we are studying.

The other differential equations (4) determine u, v, and w in terms of (14), so that the whole family of extremals is defined by the functions

$$\begin{aligned} y &= y(x, c_1, c_2, c_3, c_7), \\ u &= u(x, c_1, c_2, c_3, c_5, c_7) = \int_{x_1}^{x} f(x, y, y')\,dx + c_5, \\ v &= v(x, c_1, c_2, c_3, c_4, c_7) = \int_{x_1}^{x} g(x, y, y')\,dx + c_4, \\ w &= w(x, c_1, c_2, c_3, c_6, c_7) = \int_{x_1}^{x} h(x, y, y')\,dx + c_6. \end{aligned} \tag{15}$$

The tenth order determinant (9), when the indicated operations are performed, is seen to reduce at once to a third order determinant which when expanded is found to be

$$\lambda_2/v_2 + \lambda_1 u_2/v_2^2,$$

and so the condition that the determinant vanishes reduces to the equation

$$\lambda_1 u_2 + \lambda_2 v_2 = 0. \tag{16}$$

Now it is known that not all the λ_i $(i = 1, 2, 3)$ are zero, but since the rôles played by u, v, and w are not interchangeable, there might

obviously, from (7), be loss of generality in arbitrarily assuming that a particular one of the three is different from zero. We can prove, however, that $\lambda_1 \neq 0$ for a minimizing arc. For suppose $\lambda_1 = 0$. Then (16) becomes

$$\lambda_2 v_2 = 0.$$

But for $f_1 = u_2/v_2$ to be a minimum, obviously $v_2 \neq 0$, and so $\lambda_2 = 0$. For $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$, as they must be in the case under consideration, E_{12} would be an extremal arc for the integral

$$w_2 = \int_{x_1}^{x_2} h(x, y, y')\, dx$$

as seen from (7), a case which will be excluded by hypothesis. For if the arc E_{12} were a minimizing arc for the integral w_2, there would in general be no other arcs near it giving w_2 the required value $w_2 = l$, and the minimizing problem of this paper would be trivial.*

Hence we may assume that $\lambda_1 \neq 0$. From this and the fact that the λ_i enter homogeneously in Ω it is seen that the solution of (8) contains only the constants $c_2 = \lambda_2/\lambda_1$ and $c_3 = \lambda_3/\lambda_1$ in addition to the constants of integration, as has been heretofore indicated. In terms of c_2, equation (16) becomes

$$u_2 + c_2 v_2 = 0, \tag{17}$$

which is the transversality condition at the second end point.

We may summarize the results of this Section, by making the following statement:

A minimizing arc E_{12} for a problem fulfilling the requirements imposed in Section 1 *must satisfy the differential equations*

$$\Omega_y - \frac{d}{dx}\Omega_{y'} = 0, \quad u' - f = 0, \quad v' - g = 0, \quad w' - h = 0, \tag{18}$$

and the transversality condition

$$\lambda_1 u(x_2) + \lambda_2 v(x_2) = 0,$$

where

$$\Omega = \lambda_1 (u' - f) + \lambda_2 (v' - g) + \lambda_3 (w' - h).$$

The solutions of the differential equations (18) *have the form*

$$y = y(x, c_1, c_2, c_3, c_7),$$
$$u = u(x, c_1, c_2, c_3, c_5, c_7) = \int_{x_1}^{x} f(x, y, y')\, dx + c_5,$$

* Cf. Bolza, loc. cit., p. 460, footnote.

$$v = v(x, c_1, c_2, c_3, c_4, c_7) = \int_{x_1}^{x} g(x, y, y')\,dx + c_4,$$

$$w = w(x, c_1, c_2, c_3, c_6, c_7) = \int_{x_1}^{x} h(x, y, y')\,dx + c_6,$$

where $c_2 = \lambda_2/\lambda_1$ *and* $c_3 = \lambda_3/\lambda_1$.

Thus, we have a seven-parameter family of extremals in the $xyuvw$-space, and we shall henceforth suppose that this family contains E_{12} for the particular set of parametric values

$$c_i = c_{i0} \qquad (i = 1, \cdots, 7). \tag{19}$$

The end conditions for the determination of these seven constants are that $f_3 = f_4 = f_5 = f_6 = f_8 = f_9 = 0$ and the transversality condition (17), seven in all, as there should be.

3. A special four-parameter family of extremals. Out of the seven-parameter family of extremals just obtained, let us select those which satisfy the end conditions at 2. Leaving off the restrictions at 1, we find that these conditions are expressed by the equations

$$\begin{gathered} x_2 = \alpha_2, \quad y(x_2, c_1, c_2, c_3, c_7) = \beta_2, \quad w(x_2, c_1, c_2, c_3, c_6, c_7) = l, \\ u(x_2, c_1, c_2, c_3, c_5, c_7) + c_2\, v(x_2, c_1, c_2, c_3, c_4, c_7) = 0. \end{gathered} \tag{20}$$

The end values of E_{12} at 2 satisfy the conditions (20) applying at the point 2, and so the constants c_{i0} $(i = 1, \cdots, 7)$ from (19) defining E_{12} must satisfy (20).

As a consequence of the assumption (13), it follows* that the determinant

$$\begin{vmatrix} y_{c_1} & y_{c_7} \\ y'_{c_1} & y'_{c_7} \end{vmatrix} \tag{21}$$

is different from zero along E_{12} and in particulier at the second end point of E_{12}. In a neighborhood of the values $c_{10}, \cdots, c_{70}$ at least one of the quantities y_{c_1} and y_{c_7} is different from zero at 2, as otherwise the determinant (21) would vanish. We suppose the notation so chosen that $y_{c_7} \neq 0$. Then it is possible to solve the second equation of (20) for c_7 in terms of c_1, c_2, and c_3. The third determines c_6 as a function of the same three parameters, and the fourth equation then determines c_5 as a function of c_1, c_2, c_3, c_4. These functions have the form

$$\begin{gathered} c_5 = -c_2\, c_4 - c(c_1, c_2, c_3) = c_5(c_1, c_2, c_3, c_4), \\ c_6 = c_6(c_1, c_2, c_3), \qquad c_7 = c_7(c_1, c_2, c_3). \end{gathered}$$

* Cf. Bolza, loc. cit., p. 176; Bliss, Calculus of Variations, p. 147.

Substituting the values so determined of these constants c_5, c_6, c_7, in (15), we obtain a four-parameter family of extremals

$$
\begin{aligned}
y &= Y(x, c_1, c_2, c_3), \\
u &= U(x, c_1, c_2, c_3, c_4) = \int_{x_1}^{x} f(x, Y, Y')\,dx + c_5(c_1, c_2, c_3, c_4), \\
v &= V(x, c_1, c_2, c_3, c_4) = \int_{x_1}^{x} g(x, Y, Y')\,dx + c_4, \\
w &= W(x, c_1, c_2, c_3) \quad = \int_{x_1}^{x} h(x, Y, Y')\,dx + c_6(c_1, c_2, c_3).
\end{aligned} \tag{22}
$$

This four-parameter family of curves in the space of the variables x, y, u, v, w is the part of the original seven-parameter family (15) that satisfies the end conditions at the second end point, or comprises all the extremals that satisfy the conditions (20) which apply at 2. Hence E_{12} is also contained in the family (22) and is defined in that family by the parametric values c_{i0} $(i = 1, \cdots, 4)$.

4. A one-parameter family of extremals with an envelope. In addition to the necessary conditions imposed on a minimizing arc E_{12} that were established in Section 2, let us consider the possibility of applying to this problem an analogue of Jacobi's condition for simpler problems in the calculus of variations.

For this purpose let a one-parameter family containing E_{12} be selected from the four-parameter family of extremals (22) by determining the c_i as functions of a single parameter t, given by the notation,

$$c_i = c_i(t) \qquad (i = 1, \cdots, 4), \tag{23}$$

in such a way that for a definite value $t = t_0$ we have $c_{i0} = c_i(t_0)$ $(i = 1, \cdots, 4)$. Furthermore, the functions $c_i(t)$ are to be so determined that this one-parameter family of extremals shall touch an enveloping curve D at the points defined by a suitably chosen function $x = x(t)$.

The equations of the one-parameter family sought are

$$
\begin{aligned}
y &= Y(x, c_1(t), c_2(t), c_3(t)), \quad & u &= U(x, c_1(t), c_2(t), c_3(t), c_4(t)), \\
w &= W(x, c_1(t), c_2(t), c_3(t)), \quad & v &= V(x, c_1(t), c_2(t), c_3(t), c_4(t)),
\end{aligned} \tag{24}
$$

and those defining D are

$$
\begin{aligned}
x &= x(t), \\
y &= Y(x(t), c_1(t), c_2(t), c_3(t)), \\
u &= U(x(t), c_1(t), c_2(t), c_3(t), c_4(t)), \\
v &= V(x(t), c_1(t), c_2(t), c_3(t), c_4(t)), \\
w &= W(x(t), c_1(t), c_2(t), c_3(t)),
\end{aligned} \tag{25}
$$

the $c_i(t)$ and $x(t)$ to be found. Since the slope of D at every point is to be the same as that of the curve of (24) through the point of contact, the corresponding derivatives of (24) with respect to x and of (25) with respect to t must be proportional. Calling the proportionality factor $\varrho = \varrho(t)$, we see that this condition is expressed by the following equations:

$$
\begin{aligned}
&\varrho \cdot 1 && = x_t,\\
&\varrho\, Y'(x, c_1(t), c_2(t), c_3(t)) && = Y' x_t + Y_{c_1} c_{1_t} + Y_{c_2} c_{2_t} + Y_{c_3} c_{3_t},\\
&\varrho\, U'(x, c_1(t), c_2(t), c_3(t), c_4(t)) && = U' x_t + U_{c_1} c_{1_t} + U_{c_2} c_{2_t} + U_{c_3} c_{3_t} + U_{c_4} c_{4_t},\\
&\varrho\, V'(x, c_1(t), c_2(t), c_3(t), c_4(t)) && = V' x_t + V_{c_1} c_{1_t} + V_{c_2} c_{2_t} + V_{c_3} c_{3_t} + V_{c_4} c_{4_t},\\
&\varrho\, W'(x, c_1(t), c_2(t), c_3(t)) && = W' x_t + W_{c_1} c_{1_t} + W_{c_2} c_{2_t} + W_{c_3} c_{3_t},
\end{aligned}
\tag{26}
$$

where the arguments of Y', Y_{c_1}, etc., on the right hand side are the same as those on the left. From the first equation, $\varrho = x_t$. When this value is substituted, equations (26) become

$$
\begin{aligned}
&Y_{c_1} c_{1_t} + Y_{c_2} c_{2_t} + Y_{c_3} c_{3_t} = 0, && U_{c_1} c_{1_t} + U_{c_2} c_{2_t} + U_{c_3} c_{3_t} + U_{c_4} c_{4_t} = 0,\\
&W_{c_1} c_{1_t} + W_{c_2} c_{2_t} + W_{c_3} c_{3_t} = 0, && V_{c_1} c_{1_t} + V_{c_2} c_{2_t} + V_{c_3} c_{3_t} + V_{c_4} c_{4_t} = 0,
\end{aligned}
\tag{27}
$$

which hold at each point on the enveloping curve D. Hence if the derivatives c_{i_t} $(i = 1, \cdots, 4)$ are not all zero, a case ruled out from the nature of the problem, as these are the variables for which we are solving, it follows that at every point along D the determinant of the coefficients,

$$\Delta(x, c_1, c_2, c_3, c_4) = \frac{\partial(Y, U, V, W)}{\partial(c_1, c_2, c_3, c_4)},$$

or, as this becomes for the family (24),

$$
\Delta(x, c_1, c_2, c_3, c_4) = \begin{vmatrix} Y_{c_1} & Y_{c_2} & Y_{c_3} & 0 \\ U_{c_1} & U_{c_2} & U_{c_3} & U_{c_4} \\ V_{c_1} & V_{c_2} & V_{c_3} & V_{c_4} \\ W_{c_1} & W_{c_2} & W_{c_3} & 0 \end{vmatrix}
\tag{28}
$$

vanishes identically in t when $x(t)$ and $c_i(t)$ $(i = 1, \cdots, 4)$ are substituted.*

The determinant $\Delta(x, c_{10}, c_{20}, c_{30}, c_{40})$ will vanish at every focal point on E_{12},—that is, at every point of contact of E_{12} with D. Suppose that this vanishes at points for which $x < x_2$. Let the last such focal point before 2 occur at a point 3, at which $x = x_3 < x_2$, and assume that

$$\Delta_x(x_3, c_{10}, c_{20}, c_{30}, c_{40}) \neq 0. \tag{29}$$

* Cf. Darboux, Théorie Générale des Surfaces, t. 2 (1915), p. 7; Bliss, Lectures on the Lagrange Problem, p. 52; Bolza, loc. cit., p. 611.

Consider the equation obtained by equating (28) to zero. From theorems on implicit functions,* it is known that there exists a unique solution of this in the form

$$x = \xi(c_1, c_2, c_3, c_4) \tag{30}$$

which holds in a neighborhood of the values $x = x_3$, $c_i = c_{i0}$ $(i = 1, \cdots, 4)$ and satisfies the initial condition

$$x_3 = \xi(c_{10}, c_{20}, c_{30}, c_{40}).$$

When (30) is substituted in (24), the locus of all the focal points of the four-parameter family (22) in a neighborhood of the point 3 is obtained, an enveloping four-space, the portion of which lying in the neighborhood of 3 shall be designated as R_4. The equations defining R_4, obtained by making the above substitution, are

$$\begin{aligned}
x &= \xi(c_1, c_2, c_3, c_4), \\
y &= Y(\xi(c_1, c_2, c_3, c_4), c_1, c_2, c_3) = \eta(c_1, c_2, c_3, c_4), \\
u &= U(\xi(c_1, c_2, c_3, c_4), c_1, c_2, c_3, c_4) = \zeta(c_1, c_2, c_3, c_4), \\
v &= V(\xi(c_1, c_2, c_3, c_4), c_1, c_2, c_3, c_4) = \varphi(c_1, c_2, c_3, c_4), \\
w &= W(\xi(c_1, c_2, c_3, c_4), c_1, c_2, c_3) = \psi(c_1, c_2, c_3, c_4),
\end{aligned} \tag{31}$$

the c_i lying in a neighborhood of c_{i0} $(i = 1, \cdots, 4)$.

Now the determinant (28) vanishes at every point on R_4, and in particular at 3, given on R_4 by the parametric values c_{i0} $(i = 1, \cdots, 4)$. The rank of the determinant cannot be less than three, however, by virtue of (29). For if so,—that is, if all third order minors vanished—each of the four determinants obtained by differentiating a column of (28) would vanish, when expanded in terms of the elements of the differentiated column, and so the sum would vanish, thus contradicting (29). Hence the rank of the determinant is three. For the three rows in which there is at least one non-vanishing minor, let the minors proportional to c_{1_i}, c_{2_i}, c_{3_i}, c_{4_i} be denoted by Δ_1, Δ_2, Δ_3, Δ_4, respectively, where

$$\Delta_i = \Delta_i(c_1, c_2, c_3, c_4) \qquad (i = 1, \cdots, 4),$$

and let the proportionality factor be taken to be one. Then these equations follow:

$$c_{i_i} = \Delta_i(c_1, c_2, c_3, c_4) \qquad (i = 1, \cdots, 4). \tag{32}$$

For initial conditions, we take

$$c_i(t_0) = c_{i0} \qquad (i = 1, \cdots, 4). \tag{33}$$

* Goursat-Hedrick, Mathematical Analysis, vol. 1, p. 35.

Then existence theorems for a system of ordinary differential equations* tell us that there exists a solution of (32) in the form

$$c_i = c_i(t) \qquad (i = 1, \cdots, 4) \tag{34}$$

which for the value t_0 of t satisfies the given initial conditions (33). Putting the functions (34) in the first equation of (31), we find the function

$$x(t) = \xi(c_1(t),\ c_2(t),\ c_3(t),\ c_4(t)),$$

it and the functions (34) having been so chosen that the one-parameter family (24) contains the arc E_{12} for the value t_0 of t.

This one-parameter family of extremals could also have been obtained directly from the seven-parameter family (15) by imposing the restrictions on the $c_i(t)$ $(i = 1, \cdots, 7)$ that they satisfy the end conditions (20) at 2. The method is similar to the one just completed, and so merely an outline of it will be given.

The equations of the one-parameter family sought are

$$\begin{aligned} &y = y(x, c_1(t), c_2(t), c_3(t), c_7(t)), && v = v(x, c_1(t), c_2(t), c_3(t), c_4(t), c_7(t)), \\ &u = u(x, c_1(t), c_2(t), c_3(t), c_5(t), c_7(t)), && w = w(x, c_1(t), c_2(t), c_3(t), c_6(t), c_7(t)), \end{aligned} \tag{35}$$

the $c_i(t)$ having to also satisfy the equations

$$\begin{aligned} &y(x_2, c_1(t), c_2(t), c_3(t), c_7(t)) = \beta_2, \qquad w(x_2, c_1(t), c_2(t), c_3(t), c_6(t), c_7(t)) = l, \\ &u(x_2, c_1(t), c_2(t), c_3(t), c_5(t), c_7(t)) + c_2(t)v(x_2, c_1(t), c_2(t), c_3(t), c_4(t), c_7(t)) = 0. \end{aligned} \tag{36}$$

We wish to determine these functions $c_i(t)$ $(i = 1, \cdots, 7)$ in such a way that for $t = t_0$ we have $c_i(t_0) = c_{i0}$ and such that the one-parameter family of extremals (35) shall have an enveloping curve D defined by the equations

$$\begin{aligned} &x = x(t), \\ &y = y(x(t), c_1(t), c_2(t), c_3(t), c_7(t)), && v = v(x(t), c_1(t), c_2(t), c_3(t), c_4(t), c_7(t)), \\ &u = u(x(t), c_1(t), c_2(t), c_3(t), c_5(t), c_7(t)), && w = w(x(t), c_1(t), c_2(t), c_3(t), c_6(t), c_7(t)), \end{aligned} \tag{37}$$

the function $x(t)$ also to be found. Imposing the condition that the slope of the one-parameter family (35) shall be the same as that of the curve (37) at the point of contact, we find that at every point on the curve D the following equations must hold, where primes here denote derivatives with respect to t:

* Bliss, Princeton Colloquium Lectures, p. 86; Bolza, loc. cit., p. 168.

$$y_{c_1}c_1'+y_{c_2}c_2'+y_{c_3}c_3'+y_{c_7}c_7' = 0, \quad v_{c_1}c_1'+v_{c_2}c_2'+v_{c_3}c_3'+v_{c_4}c_4'+v_{c_7}c_7' = 0,$$
$$u_{c_1}c_1'+u_{c_2}c_2'+u_{c_3}c_3'+u_{c_5}c_5'+u_{c_7}c_7' = 0, \quad w_{c_1}c_1'+w_{c_2}c_2'+w_{c_3}c_3'+w_{c_6}c_6'+w_{c_7}c_7' = 0.$$

By virtue of equations (36), the functions c_i' $(i = 1, \cdots, 7)$ must also satisfy

$$y_{c_1}(x_2)\cdot c_1'+y_{c_2}(x_2)\cdot c_2'+y_{c_3}(x_2)\cdot c_3'+y_{c_7}(x_2)\cdot c_7' = 0,$$
$$w_{c_1}(x_2)\cdot c_1'+w_{c_2}(x_2)\cdot c_2'+w_{c_3}(x_2)\cdot c_3'+w_{c_6}(x_2)\cdot c_6'+w_{c_7}(x_2)\cdot c_7' = 0,$$
$$[u_{c_1}(x_2)+c_2v_{c_1}(x_2)]c_1'+[u_{c_2}(x_2)+c_2v_{c_2}(x_2)+v(x_2)]c_2'+[u_{c_3}(x_2)+c_2v_{c_3}(x_2)]c_3'$$
$$+c_2v_{c_4}(x_2)\cdot c_4'+u_{c_5}(x_5)\cdot c_5'+[u_{c_7}(x_2)+c_2v_{c_7}(x_2)]c_7' = 0,$$

and so there are seven homogeneous equations which the seven quantities c_i' must satisfy. Hence unless all the c_i' are zero, a case ruled out as before as being irrelevant, at every point along D the determinant of the coefficients must vanish identically in t when $x(t)$ and $c_i(t)$ $(i = 1, \cdots, 7)$ are substituted. This determinant is

$$(38)\qquad \begin{vmatrix} y_{c_1} & y_{c_2} & y_{c_3} & 0 & 0 & 0 & y_{c_7} \\ u_{c_1} & u_{c_2} & u_{c_3} & 0 & u_{c_5} & 0 & u_{c_7} \\ v_{c_1} & v_{c_2} & v_{c_3} & v_{c_4} & 0 & 0 & v_{c_7} \\ w_{c_1} & w_{c_2} & w_{c_3} & 0 & 0 & w_{c_6} & w_{c_7} \\ y_{c_1}(x_2) & y_{c_2}(x_2) & y_{c_3}(x_2) & 0 & 0 & 0 & y_{c_7}(x_2) \\ w_{c_1}(x_2) & w_{c_2}(x_2) & w_{c_3}(x_2) & 0 & 0 & w_{c_6}(x_2) & w_{c_7}(x_2) \\ \gamma_1 & \gamma_2 & \gamma_3 & c_2v_{c_4}(x_2) & u_{c_5}(x_2) & 0 & \gamma_7 \end{vmatrix}$$

where

$$\gamma_k = u_{c_k}(x_2)+c_2v_{c_k}(x_2) \qquad (k = 1, 3, 7),$$
$$\gamma_2 = u_{c_2}(x_2)+c_2v_{c_2}(x_2).$$

By repeating the argument of the previous case concerning the point 3 and assuming that

$$(39)\qquad D_x(x_3, x_2, c_{10}, \cdots, c_{70}) \neq 0,$$

we find a unique solution

$$(40)\qquad x = \Xi(c_1, \cdots, c_7)$$

of the equation obtained by equating (38) to zero, which holds in a neighborhood of the values $x = x_3$, $c_i = c_{i0}$ $(i = 1, \cdots, 7)$ and satisfies the initial condition

$$x_3 = \Xi(c_{10}, \cdots, c_{70}).$$

Putting the value (40) in (15), we get the equations defining the enveloping hyperspace to be

$$
\begin{aligned}
x &= \Xi(c_1, \cdots, c_7), \\
y &= y(\Xi(c_1, \cdots, c_7), c_1, c_2, c_3, c_7), \quad = H(c_1, \cdots, c_7), \\
u &= u(\Xi(c_1, \cdots, c_7), c_1, c_2, c_3, c_5, c_7) = Z(c_1, \cdots, c_7), \\
v &= v(\Xi(c_1, \cdots, c_7), c_1, c_2, c_3, c_4, c_7) = \Phi(c_1, \cdots, c_7), \\
w &= w(\Xi(c_1, \cdots, c_7), c_1, c_2, c_3, c_6, c_7) = \Psi(c_1, \cdots, c_7),
\end{aligned} \tag{41}
$$

the equations holding for values c_i near c_{i0} $(i = 1, \cdots, 7)$. The seven constants are, of course, not all independent, the c_5, c_6, c_7 being defined by equations (36) as functions of c_1, c_2, c_3 and c_4, so that (41) above represents a portion of a four-space as before.

The rank of the determinant (38) at 3 is six by the previous reasoning applied to hypothesis (39). Call D_i, where

$$D_i = D_i(c_1, \cdots, c_7) \qquad (i = 1, \cdots, 7),$$

the minors proportional to c_i' in the six rows of (38) in which a non-vanishing minor lies and take the proportionality factor to be one. Then

$$c_i' = D_i(c_1, \cdots, c_7) \qquad (i = 1, \cdots, 7).$$

For this set of ordinary differential equations, we know as before that there exists a solution

$$c_i = c_i(t) \qquad (i = 1, \cdots, 7)$$

which for $t = t_0$ satisfies the initial conditions

$$c_i(t_0) = c_{i0} \qquad (i = 1, \cdots, 7)$$

and so the one-parameter family (35) contains E_{12}. The function $x(t)$ is given now by

$$x(t) = \Xi(c_1(t), \cdots, c_7(t)) = \xi(c_1(t), \cdots, c_4(t)).$$

In Sections 3 *and* 4 *we have selected out of the seven-parameter family of extremals* (15) *those satisfying the transversality and other end conditions at the second end point, given by equations* (20). *The curves selected comprise a four-parameter family,*

$$
\begin{aligned}
y &= Y(x, c_1, c_2, c_3), \\
u &= U(x, c_1, c_2, c_3, c_4) = \int_{x_1}^{x} f(x, Y, Y')\, dx + c_5(c_1, c_2, c_3, c_4),
\end{aligned}
$$

$$v = V(x, c_1, c_2, c_3, c_4) = \int_{x_1}^{x} g(x, Y, Y')\,dx + c_4,$$
$$w = W(x, c_1, c_2, c_3) \quad = \int_{x_1}^{x} h(x, Y, Y')\,dx + c_6(c_1, c_2, c_3).$$

A one-parameter family has been selected by determining the c_i as functions $c_i(t)$ in two ways: first, from the four-parameter family of extremals above, and second, from the original seven-parameter family (15). *In each case this has been done in such a way that the one-parameter family contains E_{12} for the value t_0 of t and has an enveloping curve D lying on an enveloping four-space.*

5. The envelope theorem and Jacobi condition. In the preceding Section it was assumed that E_{12} touches the enveloping curve D at a point or points for which $x < x_2$, the last such point being at $x = x_3$, as shown in Figure 1. Let the point of contact of D with another of the one-parameter family, an arbitrary arc E_{54} lying in a neighborhood of E_{12}, be a point 5, defined by the value t_5 of t. Consider the composite path $E_{13}+D_{35}+E_{54}$ as shown in Figure 1, the equations of whose constituent arcs E_{13}, D_{35}, and E_{54}, respectively, are

$$(42)\quad \begin{aligned} y &= Y(x, c_1(t_0), c_2(t_0), c_3(t_0)), \quad u = U(x, c_1(t_0), c_2(t_0), c_3(t_0), c_4(t_0)), \\ w &= W(x, c_1(t_0), c_2(t_0), c_3(t_0)), \quad v = V(x, c_1(t_0), c_2(t_0), c_3(t_0), c_4(t_0)), \end{aligned}$$

for $x_1 \leqq x \leqq x_3$;

$$(43)\quad \begin{aligned} x &= x_5(t), \\ y &= Y(x_5(t), c_1(t), c_2(t), c_3(t)) &&= y_5(t), \\ u &= U(x_5(t), c_1(t), c_2(t), c_3(t), c_4(t)) &&= u_5(t), \\ v &= V(x_5(t), c_1(t), c_2(t), c_3(t), c_4(t)) &&= v_5(t), \\ w &= W(x_5(t), c_1(t), c_2(t), c_3(t)) &&= w_5(t), \end{aligned}$$

for $t_3 \leqq t \leqq t_5$; and

$$(44)\quad \begin{aligned} y &= Y(x, c_1(t_5), c_2(t_5), c_3(t_5)), \\ u &= U(x, c_1(t_5), c_2(t_5), c_3(t_5), c_4(t_5)), \\ v &= V(x, c_1(t_5), c_2(t_5), c_3(t_5), c_4(t_5)), \\ w &= W(x, c_1(t_5), c_2(t_5), c_3(t_5)), \end{aligned}$$

for $x_5 \leqq x \leqq x_4$.

Since 1 is an end point for this composite arc as well as for E_{12} and since by hypothesis the end values at 1 are satisfied for E_{12}, they are likewise satisfied for the composite arc.

Along E_{12} and E_{54} which are extremals and so are solutions of (18), the differential equations (4) are obviously satisfied, since they are included in (18). Also, the slope of D at every point is the slope of one of the

one-parameter family of extremals (24), and so D satisfies (4). Hence the whole composite arc at every point satisfies the differential equations (4), just as E_{12} does.

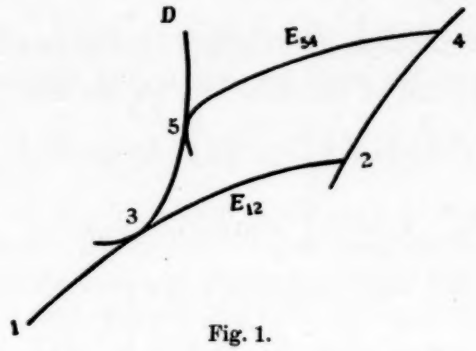

Fig. 1.

The functions (22) defining the four-parameter family were determined, as is seen from (20), so that at 2

$$\begin{aligned} &Y(x_2, c_1, c_2, c_3) = \beta_2, \quad W(x_2, c_1, c_2, c_3) = l, \\ &U(x_2, c_1, c_2, c_3, c_4) + c_2 V(x_2, c_1, c_2, c_3, c_4) = 0, \end{aligned} \tag{45}$$

identically in c_1, c_2, c_3, c_4. Hence the transversality condition

$$u_2 + c_2 v_2 = 0$$

is satisfied, and the required values are obtained for y and w at 2 along the composite path.

Thus the arcs $E_{13}+D_{35}+E_{54}$, for all positions of 5 near 3 on D, have been shown to satisfy the original differential equations and to cause all the functions (6) to vanish except the function $f_1 = u_2/v_2$. It can now be shown that f_1 has the same value on all of the composite arcs $E_{13}+D_{35}+E_{54}$ for any position of the point 5 near 3 on D.

Since $E_{13}+D_{35}+E_{54}$ satisfies (4), the values of u_2, v_2, w_2 along it are, respectively,

$$\begin{aligned} &U(x_2, c_1, c_2, c_3, c_4) \\ &= \int_{x_1}^{x_3} f(x, Y, Y')\,dx + \int_{t_3}^{t_5} f(x_5, y_5, y_{5_t}/x_{5_t})\,x_{5_t}\,dt + \int_{x_5(t)}^{x_4} f(x, Y, Y')\,dx, \\ &V(x_2, c_1, c_2, c_3, c_4) \\ &= \int_{x_1}^{x_3} g(x, Y, Y')\,dx + \int_{t_3}^{t_5} g(x_5, y_5, y_{5_t}/x_{5_t})\,x_{5_t}\,dt + \int_{x_5(t)}^{x_4} g(x, Y, Y')\,dx, \\ &W(x_2, c_1, c_2, c_3) \\ &= \int_{x_1}^{x_3} h(x, Y, Y')\,dx + \int_{t_3}^{t_5} h(x_5, y_5, y_{5_t}/x_{5_t})\,x_{5_t}\,dt + \int_{x_5(t)}^{x_4} h(x, Y, Y')\,dx, \end{aligned} \tag{46}$$

the constants of integration c_4, c_5, c_6 being zero since 1 is the initial point of these arcs and at this point $u_1 = v_1 = w_1 = 0$ from (5). From the last equation of (46) and the second of equations (45), it is seen that

$$(47)\quad \int_{x_1}^{x_3} h(x, Y, Y')\,dx + \int_{t_3}^{t_5} h(x_5, y_5, y_{5_t}/x_{5_t})\,x_{5_t}\,dt + \int_{x_5(t)}^{x_4} h(x, Y, Y')\,dx = l.$$

From (46) and the last of equations (45) we see that

$$(48)\quad \begin{aligned} &\int_{x_1}^{x_3} f(x, Y, Y')\,dx + \int_{t_3}^{t_5} f(x_5, y_5, y_{5_t}/x_{5_t})\,x_{5_t}\,dt + \int_{x_5(t)}^{x_4} f(x, Y, Y')\,dx \\ &+ c_2(t)\left(\int_{x_1}^{x_3} g(x, Y, Y')dx + \int_{t_3}^{t_5} g(x_5, y_5, y_{5_t}/x_{5_t})x_{5_t}dt + \int_{x_5(t)}^{x_4} g(x, Y, Y')dx\right) = 0. \end{aligned}$$

Differentiating (47) with respect to t, we get

$$(49)\quad h(x_5, y_5, y_{5_t}/x_{5_t})\,x_{5_t} - h(x_5, Y_5, Y_5')\,x_{5_t} + \int_{x_5(t)}^{x_4} (h_y\,Y_{t_5} + h_{y'}\,Y_{t_5}')\,dx = 0.$$

There is no term in x_{4_t} since $x_4 = x_2$ on all the composite arcs on account of the conditions (20). The first two terms of (49) cancel each other, since D and E_{54} are tangent to each other at the point 5, and so (49) reduces to

$$\int_{x_5(t)}^{x_4} (h_y\,Y_t + h_{y'}\,Y_t')\,dx = 0,$$

the derivatives with respect to t being taken at $t = t_5$. Integrating this by parts, we get

$$[h_{y'}\,Y_t]_{x_5(t)}^{x_4} + \int_{x_5(t)}^{x_4} Y_t\left(h_y - \frac{d}{dx}h_{y'}\right)dx = 0,$$

or

$$(50)\quad [h_{y'}\,(Y_{c_1}\,c_{1_t} + Y_{c_2}\,c_{2_t} + Y_{c_3}\,c_{3_t})]_{x_5(t)}^{x_4} + \int_{x_5(t)}^{x_4} Y_t\left(h_y - \frac{d}{dx}h_{y'}\right)dx = 0.$$

From the first of equations (45), it follows that

$$(51)\quad Y_{c_1}\,c_{1_t} + Y_{c_2}\,c_{2_t} + Y_{c_3}\,c_{3_t} = 0$$

at $x_4 = x_2$ for all values t_5 of t in a neighborhood of t_0, and (51) holds at the lower limit by means of the first equation of (27), since this point of the arc E_{54} is on the envelope D. Therefore, equation (50) may be written

$$(52)\quad \int_{x_5(t)}^{x_4} Y_t\left(h_y - \frac{d}{dx}h_{y'}\right)dx = 0.$$

Differentiating (48) with respect to t_5, we get similarly, where again the derivatives are to be taken for $t = t_5$,

$$\int_{x_5(t)}^{x_4} Y_t\left(f_y - \frac{d}{dx} f_{y'}\right) dx + c_2 \int_{x_5(t)}^{x_4} Y_t\left(g_y - \frac{d}{dx} g_{y'}\right) dx$$

(53)
$$+ \frac{dc_2}{dt}\left(\int_{x_1}^{x_3} g(x, Y, Y')dx + \int_{t_3}^{t_5} g(x_5, y_5, y_{5_t}/x_{5_t})x_{5_t}dt + \int_{x_5(t)}^{x_4} g(x, Y, Y')dx\right) = 0.$$

Now from the Euler-Lagrange equation (11), replacing the λ_i $(i = 1, 2, 3)$ by their values in terms of c_2 and c_3, we know that the equation

$$f_y - \frac{d}{dx} f_{y'} + c_2\left(g_y - \frac{d}{dx} g_{y'}\right) + c_3\left(h_y - \frac{d}{dx} h_{y'}\right) = 0 \tag{54}$$

holds at every point on each extremal arc, and so certainly along the arc E_{54}. Multiply (54) through by Y_t and integrate between the limits $x_5(t)$ and x_4. Since the integrand vanishes at every point of the interval, the integral will be zero, yielding the following equation:

$$\int_{x_5(t)}^{x_4} Y_t\left(f_y - \frac{d}{dx} f_{y'}\right) dx + c_2 \int_{x_5(t)}^{x_4} Y_t\left(g_y - \frac{d}{dx} g_{y'}\right) dx$$

(55)
$$+ c_3 \int_{x_5(t)}^{x_4} Y_t\left(h_y - \frac{d}{dx} h_{y'}\right) dx = 0.$$

But from (52) the last integral is zero, and (55) becomes

$$\int_{x_5(t)}^{x_4} Y_t\left(f_y - \frac{d}{dx} f_{y'}\right) dx + c_2 \int_{x_5(t)}^{x_4} Y_t\left(g_y - \frac{d}{dx} g_{y'}\right) dx = 0.$$

By means of this equation, (53) reduces to

$$\frac{dc_2}{dt}\left(\int_{x_1}^{x_3} g(x, Y, Y')dx + \int_{t_3}^{t_5} g(x_5, y_5, y_{5_t}/x_{5_t})x_{5_t}dt + \int_{x_5(t)}^{x_4} g(x, Y, Y')dx\right) = 0.$$

Since the second factor is the value of v_2 along the composite path, and since from continuity properties v_2 must be different from zero along all admissible arcs in a neighborhood of E_{12} in order that u_2/v_2 be a minimum, it follows that

$$dc_2/dt = 0,$$

or $c_2(t) =$ constant for the family of arcs $E_{13} + D_{35} + E_{54}$. But from (17), $u_2/v_2 = -c_2(t)$, and so

$$f_1 = u_2/v_2 = \text{constant}.$$

This quotient has therefore the same value as when taken along E_{12}, when taken over any member of the family of composite arcs for t_5 in a neighborhood of t_0, and the envelope theorem holds.

By existence theorems for differential equations, we know that there is one and but one solution of the equations (18) through each point 5 in

the direction of D. This is the extremal E_{54}, and it is evident therefore that D cannot be an extremal of the family. If it were, it would have to coincide with each extremal E_{54} with which it has a point of contact, each of which in turn would have to coincide with E_{12}, since D touches E_{12} at 3, thus contradicting our hypothesis of the existence of a one-parameter family (24).* Then $E_{13} + D_{35} + E_{54}$ can be replaced in a neighborhood of E_{12} by an arc giving the quotient f_1 a smaller value. Hence no composite arc of the family considered can minimize f_1 if the point 3 is such that $x_3 \geqq x_1$. Therefore E_{12} cannot minimize it, and an analogue of Jacobi's condition is established.

We may state this condition in the

THEOREM. *Let D be an envelope of a one-parameter family of extremal arcs satisfying equations* (18), *containing E_{12}, and satisfying the end conditions* (20), *which apply at the point* 2. *Let* 3 *be the last contact point of D with E_{12} before* 2, *given by the value $x = x_3 < x_2$. If E_{12} is a minimizing arc for the problem of Section* 1, *then it can have on it no such point* 3 *for which $x_3 \geqq x_1$.*

Analytically, the points 3 are determined by the zeros of the determinant (28), where Y, U, V, W are the functions in equations (24), or they may be determined by the zeros of the determinant (38), where y, u, v, w are the functions given in (15).

* Cf. Bliss, Calculus of Variations, p. 141.

ON FINDING IDEALS.*

By N. R. Wilson.

1. Introduction and notation. The present paper is a continuation of a former paper on finding the basis of an algebraic number field.† In the three sections following the introduction, standardized bases for ideals similar to that for the field are set up. Necessary and, unless p^2 divides the discriminant of the field, sufficient conditions that $[p]$ should be prime are then obtained: and, if composite, the factors of $[p]$ in terms of the irreducible factors of $f(x)$, $\mod p$, where $f(x) = 0$ generates the field. Methods when p^2 divides the discriminant are discussed and the whole applied to the general cubic.

To save explanation we use the following notation, terms and results from the preceding paper. Latin letters except x denote rational integers, p being a rational prime. Greek letters and x denote algebraic numbers and German letters ideals. Divisibility is denoted thus, $a|b$: the symbol a/b denoting division. The irreducible equation generating the field is $f(x) \equiv x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, reduced to its normal form: i. e., there is no prime p such that $p^i | a_{n-i}$ for all values of i. A simple integer‡ of degree $(i-1)$ in x is a homogeneous linear form in $1, x, x^2, \cdots, x^{n-1}$. If integers of degree $(i-1)$ in x are not entirely simple, integers occur in the field of the form $\mu_i(p) = (x^{i-1} + a^{(i)}_{i-2}x^{i-2} + \cdots + a^{(i)}_0)/p_j^{u_{ij}}$, called single-prime elements‡ in p of degree $(i-1)$ in x. The a's satisfy certain congruences (Theorem IVa); $u_{ij} \geqq u_{i-1,j}$: and the square of the product $\prod p_j^{u_{ij}}$ for all values of i and j is a factor of Δ, the discriminant of $f(x)$. The singe-prime elements are uniquely determined if we select the set for which $\frac{1}{2}p_j^{u_{ij}} < a^{(i)}_k \leqq \frac{1}{2}p_j^{u_{ij}}$ and all integers of the field are homogeneous linear forms of such. There is a least degree $r > 1$ for which such single-prime elements occur. For the basis of the field we take $\omega_i = x^{i-1}$ for $1 \leqq i < r$ and for $r \leqq i \leqq n$, we take

$$\omega_i = (x^{i-1} + b^{(i)}_{i-2}x^{i-2} + \cdots + b^{(i)}_0)/P_i = \sum k_{ij}\mu_{ij}(p_j)$$

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* Received August 25, 1928; presented to the American Mathematical Society September 6, 1928.

† Trans. Amer. Math. Soc., vol. 29 (1927), pp. 111–126.

‡ The terms *simple integer* and *single-prime element* have been suggested and are used here in place of *ordinary integer* and *maximal reduced integer* of the former paper.

where P_i is the product of $p_j^{u_{ij}}$ for all values of p_j and the k_{ij} are solutions of $\sum k_{ij} P_i/p_j^{u_{ij}} = 1$. This basis can be made unique as for the set of single-prime elements. The discriminant of the field is Δ_1 where $P_r^2 P_{r+1}^2 \cdots P_n^2 \Delta_1 = \Delta$.

2. **Simple integers of an ideal.** The familiar argument that every ideal $\mathfrak{a}$ contains a least positive integer m, of which every rational integer in $\mathfrak{a}$ is a multiple, can be extended to the simple integers of $\mathfrak{a}$. For at least one simple integer of each degree $(i-1)$ in x occurs in $\mathfrak{a}$, viz., mx^{i-1}. Hence $\mathfrak{a}$ contains a simple integer of the form

$$\sigma_i = m_i x^{i-1} + b_{i-2}^{(i)} x^{i-2} + \cdots + b_0^{(i)}$$

for which m_i has the least positive value in $\mathfrak{a}$. If any other simple integer of degree $(i-1)$ in x occurs in $\mathfrak{a}$, $\alpha = cx^{i-1} + \cdots$, we have $c = m_i q + r$, $0 \leqq 1 < m_i$; whence $\alpha - q\sigma_i$ is of the form $rx^{i-1} + \cdots$: so that $r = 0$, since $r < m_i$. Hence $m_i | c$. In particular $m_i | m_{i-1}$ the latter being the leading coefficient in $x\sigma_{i-1}$ of degree $(i-1)$ in x: and similarly $m_2 | m$. Further if $c_{i-1} = q m_i$, $\alpha_i - q\sigma_i$ is of degree $< i-1$ in x. It follows in the usual way (cf. Thm. IIa, former paper) that every simple integer of $\mathfrak{a}$ of degree $(i-1)$ in x is a homogeneous linear form in $\sigma_i, \sigma_{i-1}, \cdots, \sigma_1 = m$.

We prove by induction that every coefficient of σ_i is a multiple of its leading coefficient. We have $\sigma_2 = m_2 x + b_0^{(2)}$. If $m = q m_2$, $q\sigma_2 - mx = q b_0^{(2)}$ is in $\mathfrak{a}$: whence $m | q b_0^{(2)}$ or $m_2 | b_0^{(2)}$. Assuming the statement true for $i = 1, 2, \cdots, r$, since $m_r | m_{r-1} | \cdots | m_2 | m$, coefficients of powers of x in every number α of degree $< r-1$ in x are multiples of m_r. If $m_r = q m_{r+1}$, $q\sigma_{r+1} - x\sigma_r$ is of such degree. Since the coefficients of $x\sigma_r$ are divisible by $m_r = q m_{r+1}$, those of σ_{r+1} must be divisible by m_{r+1}, completing the induction.

Simple integers of $\mathfrak{a}$ may evidently be represented as linear forms of any set, $\sigma_i + d_{i-1}\sigma_{i-1} + \cdots + d_1\sigma_1$, $1 \leqq i \leqq n$, as well as of σ_i. Hence we may reduce the coefficients of σ_i so that $-\frac{1}{2} m_j < m_i c_{j-1}^{(i)} \leqq \frac{1}{2} m_i$ for $1 \leqq j < i$, where $m_i c_{j-1}^{(i)}$ is the coefficient of x^{j-1}. The determination is then unique: as, if there were two such, the coefficients of x^{j-1}, the highest non-vanishing term, $1 \leqq i < j$, in their difference would be $< m_j$.

THEOREM I. *If m denote the least positive integer in an ideal $\mathfrak{a}$, there exists in $\mathfrak{a}$ a set of numbers $\sigma_n, \sigma_{n-1}, \cdots, \sigma_1 = m$ of the form $\sigma_i = m_i (x^{i-1} + c_{i-2}^{i} x^{i-2} + \cdots + c_0^{(i)})$, where each m_i is positive and $m_n | m_{n-1} | \cdots | m_2 | m_1 = m$, such that every simple integer of $\mathfrak{a}$ of degree $(i-1)$ in x is a homogeneous linear form in $\sigma_i, \sigma_{i-1}, \cdots, \sigma_1$ and its leading coefficient is divisible by m_i.*

Any number of $\mathfrak{a}$ *whose highest coefficient is* m_i *will serve for* σ_i *and the set may be chosen uniquely so that* $-\frac{1}{2}m_j < m_i c_{j-1}^{(i)} \leqq \frac{1}{2}m_j$ *for* $j<i$.

For convenience of reference, we call such a set a *simple-integer basis* of $\mathfrak{a}$. If $\mathfrak{a}$ contains only simple integers, it is the basis of $\mathfrak{a}$. Thus for the field $x^3+x^2-2x-1=0$, basis $1, x, x^2$,* where m is the prime p, possible bases are of the forms $(p, x-a, x^2+ax+b)$, (p, px, x^2+ax+b), and $(p, px, px^2) = [p]$, where $-\frac{1}{2}p<a$, $b \leqq \frac{1}{2}p$. In the first case, $x\sigma_2-\sigma_3-a\sigma_2 = -b-a^2$ and $x\sigma_3+\sigma_3-(b+2)\sigma_2 = b+1+a(b+2)$: whence $a^3+a^2-2a-1 \equiv 0$, $b+1+a(b+2) \equiv 0$, mod p. The second leads to an equivalent congruence. This congruence has solutions, if, and only if, p is 7 or of the forms $7m \pm 1$,† so that $[p]$ is prime except in these cases. Also the given sets of forms are closed under multiplication by $1, x, x^2$, so that they are ideals and factors of $[p]$. For $p=7$, the factors have bases $(7, x-3, x^2+3)$ and $(7, 7x, x^2+3x-3)$. These are expressed in the form of the present section, the principal ideals $[1+x+x^2]$ and $[9-2x-3x^2]$; the expressions within brackets being factors of 7 in this field.

3. A basis for ideals: first form. We consider next all numbers of $\mathfrak{a}$ when the latter contains other than simple integers. Let $(r-1)$ and $(s-1)$, be the lowest degrees in x for which such occur in the field and in $\mathfrak{a}$. Then since the field contains $\mathfrak{a}$, $r \leqq s$. For each degree $(i-1)$ in x, $r \leqq i \leqq n$, there exists in $\mathfrak{a}$ a number, $\Omega_i = m_i' x^{i-1}/Q_i + \cdots$, the leading coefficient having the least positive value. Since $1/P_i$ is the leading coefficient of ω_i, $1/P_i \mid m_i'/Q_i$: and, since we may suppose the latter in its simplest terms, $Q_i \mid P_i$. We may therefore write Ω_i in the form $m_i R_i x^{i-1}/P_i + \cdots$ where $R_i \mid P_i$ and $R_i = P_i$ for $r \leqq i < s$ since $Q_i = 1$. As in Theorem I $m_i' R_i/P_i$ divides the leading coefficient of every number of $\mathfrak{a}$ of degree $(i-1)$ in x, any such number is a homogeneous linear form in $\Omega_i, \Omega_{i-1}, \cdots, \Omega_r, \sigma_{r-1}, \cdots, \sigma_1$: also $m'R/P_i \mid m_{i-1}' R_{i-1}/P_{i-1}$, including $m_r' R_r/P_r \mid m_{r-1}$. Since $m\omega_i$, leading coefficient m/P_i, occurs in $\mathfrak{a}$, $m_i' R_i \mid m$. Every coefficient of Ω_i is a multiple of its leading coefficient. For, being true of $\sigma_{r-1}, \cdots, \sigma_1$, it follows exactly as in the proof of Theorem I that it is true of Ω_r and thence of Ω_i, $r<i \leqq n$. From this we can prove that $m_i' = m_i$ of Theorem I. For $Q_i \Omega_i$ is a simple integer, leading coefficient m_i'. By Theorem I, $m_i \mid m_i'$. Also, since σ_i is in $\mathfrak{a}$, $m_i'/Q_i \mid m_i$, the leading coefficient of σ_i. As we have already supposed m_i/Q_i in its lowest terms, $m_i' \mid m_i$. Also, as before, taking $\Omega_i = m_i R_i (x^{i-1} + d_{i-2}^{(i)} x^{i-2} + \cdots + d_0^{(i)})/P_i$, we may obtain a unique set Ω_i such that $-\frac{1}{2}m_j R_j/P_j < m_i R_i/P_i d_{j-1}^{(i)} \leqq \frac{1}{2}m_j R_j/P_j$, $r \leqq j < i$ and between $\pm\frac{1}{2}m_j$ for $j<r$. Since every integer of the field

* Bases throughout are read from the table at the end of the former paper.

† Cailler, Interm. de Math., vol. 16 (1909), pp. 185–7.

can be reduced to the form $(a_i x^{i-1}/P_i + \cdots)$, the number of classes, mod $\mathfrak{a}$, of degree $(i-1)$ is $m_i R_i$, so that $\prod_1^n m_i R_i = N\mathfrak{a}$.

THEOREM II. *Let m_i, $1 \leqq i \leqq n$, denote the leading coefficients of a simple integer basis σ_i for an ideal $\mathfrak{a}$, $m_1 = m$ denoting the least rational integer in $\mathfrak{a}$; and let $(r-1)$ and $(s-1)$ be the lowest degrees in x for which integers other than simple integers occur in the field and in $\mathfrak{a}$ respectively. Then $s \geqq r > 1$ and there exists in $\mathfrak{a}$ a set of numbers $\Omega_n, \Omega_{n-1}, \cdots, \Omega_r$, of the form $\Omega_i = m_i R_i (x^{i-1} + d_{i-2}^{(i)} x^{i-2} + \cdots + d_0^{(i)})/P_i$, where (i) $m_n R_n/P_n | m_{n-1} R_{n-1}/P_{n-1} | \cdots | m_r R_r/P_r | m_{r-1} | \cdots | m_1 = m$, the least positive rational integer in $\mathfrak{a}$: and* (ii) *$R_i | P_i$, $m_i R_i | m$ and m_i is prime to P_i/R_i:* (iii) *$\prod_1^n m_i R_i$ is the norm of $\mathfrak{a}$; such that every number of $\mathfrak{a}$ of degree $(i-1)$ in x, $r \leqq i \leqq n$, is a homogeneous linear form in $\Omega_i, \Omega_{i-1}, \cdots, \Omega_r, \sigma_{r-1}, \cdots, \sigma_1$, its leading coefficient being divisible by $m_i R_i/P_i$. Any number of $\mathfrak{a}$ of degree $(i-1)$ in x whose highest coefficient is $m_i R_i/P_i$ will serve for Ω_i and the set may be determined uniquely so that $-\frac{1}{2} m_j R_j/P_j < m_i R_i d_{j-1}^{(i)}/P_i \leqq \frac{1}{2} m_j R^j/P$ where $R_j = P_j$ for $j < s$. If $r < s$, $\Omega_r, \Omega_{r+1}, \cdots, \Omega_{s-1}$ are identical with $\sigma_r \cdots \sigma_{s-1}$.*

We refer to such a *canonical basis* for $\mathfrak{a}$ and frequently use Ω_i for any member of the set, Ω_i being identical with σ_i for $i < s$. We call $m_i R_i$ the *leading numerator* (including m_i for $i < r$ when $R_i = 1$). The theorem that the number of ideals containing a given rational integer is finite is an immediate consequence since such must be of the type of Theorem II, m being a divisor of the rational integer. We may also prove as in the corresponding corollaries in the next section that: (1) $R_i(x^{i-1} + d_{i-2}^{(i)} x^{i-2} + \cdots + d_0^{(i)})/P_i$ is an integer; (2) $s = r$ unless $m_r P_r | m$; (3) if m is prime to P_i, $R_i = 1$ and $\Omega_i = m_i(\omega_i + c_{i-1}\omega_{i-1} + \cdots + c_1 \omega_1)$; (4) Ω_i is of the form $R_i(\sigma_i + c_{i-1}\sigma_{i-1} + \cdots + c_1 \sigma_1)/P_i$; (5) $m_i/P_j Q_i | m_{i+j-1}/Q_{i+j-1}$ for $i + j \leqq n$.

The determinant whose square is the discriminant of $\mathfrak{a}$ is the same as that in Theorem III of the former paper, except for the common factor m_i in rows $1 \cdots (r-1)$, and $m_i R_i/P_i$ instead of $1/P_i$ in rows $r, \cdots, n$. Hence $\Delta_2 = \prod_1^n (m_i R_i)^2 \Delta_1 = \prod_1^n (m_i R_i/P_i)^2 \Delta$; Δ_2, Δ_1, and Δ being the discriminants of $\mathfrak{a}$ of the field and of the generating equation and $R_i = 1$ for $1 < i < r$. We have therefore the familiar theorem $(N\mathfrak{a})^2 \Delta_1 = \Delta_2$.

These with the restrictions $P_{i-1} | P_i$,* $P_r^2 P_{r+1}^2 \cdots P_n^2 | \Delta$† and the expressions given in Theorem IV a of the former paper which must be rational

* Trans. Amer. Math. Soc., l. c., Cor. p. 115.

† Theorem I.

integers are sufficient for numerical cases and the simpler literal cases; including the field by taking $m = 1$. For example for the quadratic field, $x^2+ax+b=0$, we have $P_2^{(2)}|\Delta = a^2-4b$; or, since the equation is in its normal form,* $P_2^{(2)}|4$ only if a is even. The basis of the field is therefore 1, $\frac{1}{2}(x+1)$, if, and only if, a is even, $a \equiv b+1$, mod 4.† Since the norm must be p, possible ideals other than $[p]$ are of the form $[p, x-c]$ or $[p, \frac{1}{2}(x-c)]$, c odd in the latter case. To be closed under multiplication respectively by 1, x, and 1, $\frac{1}{2}(x+1)$, requires only that $c^2+ca+b \equiv 0$, mod p, be solvable; except in the latter case where also we must have $a \equiv b+1$, mod 8. Hence except in these cases $[p]$ is prime and in these cases, its factors are the given ideals.

4. **A basis for ideals: second form.** Except in simple numerical cases ideals may be obtained more readily from the single-prime elements of the basis of the field. Let $\Omega_i = m_i(x^{i-1}+d_{i-2}^{(i)}x^{i-2}+\cdots+d_0^{(i)})/Q_i$, $Q_i>1$. Then m_i is prime to Q_i (Theorem II, (ii)). If $Q_i = p_j^{v_{ij}}S$, p_j prime to S, then $S\Omega_i$ is a single-prime integer in p_j of degree $(i-1)$ in x. Hence $\mu_i(p_j)$ (See Introduction) occurs in the field and there exists in $\mathfrak{a}$ a single-prime integer in p_j of degree $(i-1)$ in x, $\sigma_i(p_j) = m'x^{i-1}/p_j^v+\cdots$, whose leading coefficient has a least positive value. As before, m'/p_j^v divides the highest coefficient of any single-prime integer in p_j of degree $(i-1)$ in x contained in $\mathfrak{a}$. Applied to SQ_i, $m'/p_j^v|m_i/p_j^{v_{ij}}$. Supposing the former in its lowest terms and remembering that m_i is prime to Q_i and therefore to p_j, $m'|m_i$ and $v \geqq v_{ij}$. But, by Theorem II, $m_i/Q_i|m'/p_j^v$, the leading coefficient of $\sigma_i(p_j)$, or $m_j|m'$, $v_{ij}>v$. Hence $m' = m_i$ and $v = v_{ij}$. Also $\sigma_i(p_j) = m_ix^{i-1}/p_j^{v_{ij}}+\cdots$ where $p_j^{v_{ij}}$ is the highest power of p_j in Q_i. As the numbers $x^w\sigma_i(p_j)$ for $w = 1, 2, \cdots, n-1$, occur in $\mathfrak{a}$, such elements in p_j occur for all higher degrees in x, and, since $m_i/p_j^{v_{ij}}$ must divide the highest coefficient of $x\sigma_{i-1}(p_j)$, we have $v_{ij} \geqq v_{i-1,j}$. Since Q_i is the product of all $p_j^{v_{ij}}$, $Q_{i-1}|Q_i$. We may prove as before that the leading coefficient of $\sigma_i(p_j)$ divides the remaining coefficients.

We prove that every number of $\mathfrak{a}$ of degree $(i-1)$ in x, is a homogeneous linear form in $\sigma_i(p_j)\cdots\sigma_s(p_j)$, $\sigma_{s-1}, \cdots, \sigma_1$ where p_j in $\sigma_k(p_j)$ runs through all primes in Q_k, $Q_k>1$, by proving that $\Omega_i, \Omega_{i-1}, \cdots, \Omega_s$, are such. For we can determine w_j so that $Q_k\sum w_j/p^{v_{kj}} = 1$. The coefficient of x^{k-1} in $\Omega_k\sum_j w_j\sigma_k(p_j)$ is 0 and hence this difference is a linear form in $\Omega_{k-1}, \cdots, \Omega_s, \sigma_{s-1}, \cdots, \sigma_1$ (Theorem II); whence the result follows as usual. As before any single-prime number of $\mathfrak{a}$ with highest coefficient $m_i/p_j^{v_{ij}}$ will serve for $\sigma_i(p_j)$ in this representation and the determination

* p. 111.

† Theorem IVa.

may be made unique. Further, if $\mu_i(p_j) = (x^{i-1}+\cdots)/p_j^{\mu_{ij}}$ is the corresponding single-prime element to $\sigma_i(p_j)$, $m\mu_i(p_j)$ occurs in $\mathfrak{a}$ and hence $m_i/p_j^{v_{ij}} | m/p^{u_{ij}}$. Since m_i is prime to p_j, $p^{u_{ij}-v_{ij}} | m$.

THEOREM III. *With the notation of Theorem* II, *there exists in* $\mathfrak{a}$ *a set of numbers* $\sigma_i(p_j)$, $s \leqq i \leqq n$, *of the form* $\sigma_i(p_j) = m_i(x^{i-1}+e_{i-2}^{(i)}x^{i-2}+e_0^{(i)})/p_j^{v_{ij}}$; *one for each distinct prime* p_j *in* $Q_i = P_i/R_i$; *where*

(i) $p_j^{v_{ij}}$ *is the highest power of* p_j *in* Q_i; *and* p_j *runs at least through all distinct primes in the denominator of* m/P_i *in its lowest terms, but* m_i *contains none of the primes* p_j;

(ii) $v_{ij} \geqq v_{i-1,j}$;

(iii) $\mu_{ij} \geqq v_{ij}$, $p^{u_{ij}-v_{ij}} | m$;

μ_{ij} *being the index of* p_j *in the denominator of the corresponding single-prime element* $\mu_i(p_j)$ *in* p_j; *such that every number of* $\mathfrak{a}$ *of degree* $(i-1)$ *in* x *is a homogeneous linear form in* $\sigma_k(p_j)$, $s \leqq k < i$, *and* $\sigma_{s-1}, \cdots, \sigma_1$. *Any single-prime integer of* $\mathfrak{a}$ *of degree* $(i-1)$ *in* x *whose highest coefficient is* $m_i/p_j^{v_{ij}}$ *will serve for* $\sigma_i(p_j)$ *and the set may be determined uniquely so that* $-\frac{1}{2}m_k/p_j^{v_{kj}} < m_i e_{k-1}^{(i)}/p_j^{v_{ij}} < \frac{1}{2}m_k/p_j^{v_{kj}}$ *where* $v_{kj} = 0$ *for* $k < s$. *If* $\sigma_i(p_j)$ *occurs for any degree* i, *then* $\sigma_k(p_j)$ *occurs for* $i < k \leqq n$. *Also, if* $Q_i \sum w_j/p_j^{v_{ij}} = 1$, *then* $\Omega_i = \sum w_j \sigma_i(p_j)$.

COROLLARY 0. If $\Omega_i = m_i(x^{i-1}+\cdots)/Q_i$, *then* $Q_{i-1} | Q_i$.

COROLLARY 1. $(x^{i-1}+e_{i-2}^{(i)}x^{i-2}+\cdots+e_0^{(i)})/p_j^{v_{ij}}$ *is an integer.* For, since m_i and p_j are relatively prime, we may determine u and v so that $um_i+vp_j = 1$: whence $(x^{i-1}+\cdots+e_0^{(i)})/p_j^{v_{ij}} = um_i(x^{i-1}+\cdots+e_0^{(i)})/p_j^{v_{ij}} v(x^{i-1}+\cdots+e_0^{(i)})$, both of which are integers.

COROLLARY 2. *If* r_{j-1} *denote the lowest degree in* x *for which single-prime elements in* p_j *occur in the field, then* $\sigma_{r_j}(p_j)$ *occurs in* $\mathfrak{a}$ *unless* $p_j^{u_{r_j}} | m$. For $\mathfrak{a}$ contains $m\mu_{r_j}(p_j) = m(x^{r_j-1}+\cdots)/p_j^{u_{r_j}}$.

COROLLARY 3. *If* p_j *is prime to* m, *then* $u_{ij} = v_{ij}$ and $\sigma_i(p_j) = m_i\{\mu_i(p_j)+c_{i-1}\mu_{i-1}(p_j)+\cdots+c_{r_j}\mu_{r_j}(p_j)+c_{r_j-1}x^{r_j-1}+\cdots+c_1\}$ *where* c_k *may be taken as* 0 *if* $m_k = 1$. From Corollary 1, $(x^{i-1}+\cdots)/p_j^{v_{ij}}$ is an integer whence $u_{ij} \geqq v_{ij}$. Also $m\mu_i(p_j) = m(x^{i-1}+\cdots)/p_j^{u_{ij}}$ occurs in $\mathfrak{a}$, whence $m_i/p_j^{v_{ij}} | m/p_j^{u_{ij}}$. Since m and m_i are prime to p_j, $v_{ij} \geqq u_{ij}$. Hence $v_{ij} = u_{ij}$ and, being an integer, $(x^{i-1}+\cdots)/p_j^{v_{ij}}$ has the required form. If $m_k = 1$, $\sigma_k(p_j)$ being an integer is of the form $\mu_k(p_j)+\cdots$, and c_k may be reduced to 0 by subtracting $c_k\sigma_k(p)_j$ or $c_k\sigma_k$.

COROLLARY 4. $\sigma_i(p_j)$ *is of the form* $(\sigma_i+c_{i-1}\sigma_{i-1}+\cdots+c_1\sigma_1)/p_j^{v_{ij}}$. For $p_j^{v_{ij}}\sigma_i(p_j)$ is a simple integer in $\mathfrak{a}$.

COROLLARY 5. $v_{i+k-1,\,j} \geqq \mu_{ij} + v_{kj}$ for $i+k-1 \leqq n$. For $\mu_i(p_j)\sigma_k(p_j)$ must occur in $\mathfrak{a}$ and the coefficient of its leading term $m_k x^{i+k-2}/p_j^{u_{ij}+v_{kj}}$ must be divisible by $m_{i+k-1}/p_j^{v_{i+k-1}}$.

THEOREM IV. *Let m be the prime p, k be the least value of i for which $m_i = 1$, and $(r-1)$ the least degree in x for which single-prime elements in p exist in the field. Then $k \leqq r$, $i<k$, $m_i = p$ and for $i \geqq k$, $m_i = 1$. When $p_j \neq p$, $p\mu_i(p_j)$ may be taken for $\sigma_i(p_j)$ for $i<k$. When $p_j = p$, $\sigma_i(p)$ does not occur for $i<k$. For $i \leqq k$, $R_i = p$ or 1. In the former case $p\mu_i(p)$ may be taken for $\sigma_i(p)$ and in the latter $\mu_i(p) + \sum c_j \mu_j(p) + c_{k-1}x^{k-1} + \cdots + c_1$, the summation extending only over elements for which $R_j = p$.*

For, since $p\mu_r(p)$ occurs in $\mathfrak{a}$, the simple integer $p\mu_r(p)$ occurs with $m_r = 1$, so that $k \leqq r$. The second result follows from $m_i | m_{i-1}$. For $p_j \neq p$, $p\mu_i(p_j)$ occurs and has the required highest coefficient. Hence it may be used for $\sigma_i(p)$, (Theorem III). For $p_j = p$, $p\mu_i(p)$ occurs and leads to a simple integer with $m_i = 1$. Hence it cannot occur for $i<k$. Since $m_i R_i | p, k_i$ (Theorem III), $R_i = p$ or 1. If p, $p\mu_i(p)$ occurs for $i \geqq k$ and has the required highest coefficient. It may therefore be used for $\sigma_i(p)$. If $R_i = 1$, $\sigma_i(p)$ must be of the form $\mu_i(p) + \sum_r c_j \mu_j(p) + \sum^{r-1} c_j x^j$, which can be reduced to the form stated by subtracting $c_j \sigma_j(p)$ wherever $R_j = 1$ and multiples of σ_j for $r > j \geqq k$.

We refer to the elements in this section as the *single-prime elements* of $\mathfrak{a}$. Given the norm of $\mathfrak{a}$, the leading numerators are determinate to a small number of choices and thence the leading terms of the σ's, which we have seen in the former paper to be the chief difficulty in determining such integers. Results similar to those in Section V of that paper may also be obtained.

5. **Ideals as homogeneous linear forms.** For the set of all rational integral homogeneous linear forms (more briefly "linear forms") in $\alpha, \beta, \gamma, \cdots$, we use the notation $(\alpha, \beta, \gamma, \cdots)$ square brackets being used for ideals. The ideal, $[\alpha_1, \alpha_2, \alpha_3, \cdots]$ is a linear form in $\alpha_i \omega_j$. The reduction to the canonical form is facilitated by the result $m_i | m_{i-1}$, as the example $(5x-3, 5x^2+1, 4x^2+8x+3)$ to $(7, x-2, x^2+3)$ in the field $x^3+x^2-2x-1=0$ will indicate. A necessary and sufficient condition that $S = (\alpha_1, \alpha_2, \cdots)$ constitute an ideal in a field $(\beta_1, \beta_2, \cdots)$ is obviously that each $\beta_i \alpha_j$ is contained in S. If any β is a rational integral function of other β's the test need not be applied to it: e. g. in Case 4b of the former paper, where $\omega_2 = (x \pm 1)/3$, $\omega_3 = (x^2+2x+1)/9$, we need

apply only to ω_2. In particular in determining a simple integer basis, we need apply only to x.

If $f(x) = 0$ of degree n be the generating equation, *reduction* of any number α by some or all of $f(x)$, Ω_n, $\Omega_{n-1}, \cdots$, means that, from the remainder on ordinary division of α by $f(x)$, rational integral multiples of Ω_n, $\Omega_{n-1}, \cdots$, are subtracted successively so as to give the coefficients $x^{n-1}, x^{n-2}, \cdots$, the least non-negative values. A necessary and sufficient condition that $\mathfrak{a}|\mathfrak{a}'$ where $\mathfrak{a}$ and $\mathfrak{a}'$ are written in the canonical form is that each Ω_i on reduction by the set Ω_j' leaves the remainder 0. Hence we can reduce the finding of ideals to those in which $m_n = 1$. For, if $m_n \neq 1$, since $m_n | m_i$, $i < n$, we can divide through by $[m_n]$. Also if $\mathfrak{a}|\mathfrak{a}'$, then each $m_i | m_i'$. For $\mathfrak{a}$ contains $\mathfrak{a}'$ and therefore the simple integer system of $\mathfrak{a}$ contains that of $\mathfrak{a}'$. The algorithm for finding the G. C. D. of $\mathfrak{a}$ and $\mathfrak{a}'$ is similar in form and proof with that for ordinary numbers each Ω_i or σ_i' of $\mathfrak{a}'$ being divided successively by each of the elements of $\mathfrak{a}$ each element of $\mathfrak{a}$ by the remainders and so on; e. g. the G. C. D. of $(91, 7x+21, 7x^2+28)$ and $(182, 2x-46, 2x^2+34)$ is $(91, x-23, x^2+17)$ independently of the field. Hence if m and m' are relatively prime, ideals with these as least rational integers are relatively prime, since their G. C. D. is 1.

Since we can make $m_n = 1$, the simple integer basis of a factor of $[p]$ must take the form $(p, px, \cdots, px^{i-1}, x^i + \cdots, \cdots)$ where σ_{i+k} may be taken as $x^k \sigma_i$. The same method may be used to determine the remaining coefficients of σ_i. For, if $f(x) = 0$ be the generating equation, $f(x)$ occurs in every set of simple integers. Since the quotient on division of $f(x)$ by σ_i will be a simple integer, the remainder is a simple integer of degree $< i$ in x. Hence all its coefficients are congruent to 0, mod p.

THEOREM V. *The simple integer basis of any factor of* $[p]$ *is of the form* $p, px, \cdots, px^{i-2}, x^{i-1} + d_{i-2}^{(i)} x^{i-2} + \cdots + d_0^{(i)} = \sigma_i, x\sigma_i, \cdots, x^{n-1}\sigma_i$. *If such a factor exists, the congruences obtained by making congruent to* 0, *mod* p, *the coefficients of the remainder on dividing* $f(x)$, *where* $f(x) = 0$ *is the generating equation, by* σ_i *must be solvable and the d's are such solutions.* Thus in the field $x^3 + x^2 - 2x - 1 = 0$, the existence of a simple integer basis of the form $p, px, x^2 + ax + b$ requires that $a^2 - a - b - 2 \equiv 0$, $ab - b - 1 \equiv 0$, mod p. For $p = 11$, there are no solutions or [11] is prime and for $p = 13$, there are three solutions, $x^2 - 5x + 2$, $x^2 - 2x + 4$, $x^2 - 4x + 5$, all of which yield ideal factors of [13].

In a simple integer field, to prove the sufficiency of these conditions, it is only necessary to show that $x\sigma_n$ is in the set. If

$$f(x) \equiv (x^{n-i+1} + a x^{n-i} + b x^{n-i-1} + \cdots + k)\sigma_i \bmod p,$$

since $x\sigma_n = x^{n-i+1}\sigma_i$, we have

$$f(x) = x\sigma_n + a\sigma_n + b\sigma_{n-1} + k\sigma_i + m\sigma_{i-1} + \cdots + = 0.$$

Hence,

COROLLARY. *For a field containing only simple integers, the conditions of the theorem are sufficient for the existence of such ideal factor.* This is the most convenient form for this theorem though it can be generalized, $f(x)$ being replaced by any multiple of $f(x)$, σ_i by Ω_i and the simple integer quotient by any integer in the field.

THEOREM VI. *If* $\mathfrak{a}$, $\mathfrak{a}'$ *and* $\mathfrak{a}\mathfrak{a}'$, *in the canonical form have leading numerators* $m_i R_i$, $m_i' R_i'$ *and* $m_i'' R_i''$, $1 \leqq i \leqq n$, *then* $m_i'' R_i''$ *is a multiple of* $m_i R_i$ *and of* $m_i' R_i'$ *and a submultiple of* $(m_j R_j)(m_{i-j+1}' R_{i-j+1}')$ *for* $1 \leqq j < i$. For $\mathfrak{a}$ contains $\mathfrak{a}\mathfrak{a}'$, whence (Theorem II) $m_i R_i | m_i'' R_i''$; and similarly for $\mathfrak{a}'$. The third statement follows from the fact that $\mathfrak{a}\mathfrak{a}'$ contains $\Omega_i \Omega_{i-j+1}$ of degree $(i-1)$ in x.

COROLLARY. *If* m *and* m', *the least rational integers in* $\mathfrak{a}$ *and* $\mathfrak{a}'$, *are relatively prime, then* $m_i'' R_i'' = (m_i R_i)(m_i' R_i')$.

For $m_i R_i | m$ and $m_i' R_i' | m'$ (Theorem II), whence $m_i R_i$ and $m_i' R_i'$ are relatively prime. Putting $j = 1$ and i in Theorem VI, $m_i'' R_i'' | (m_i R_i) m'$ and $m_i'' R_i'' | m(m_i'' R_i'')$. That the theorem may or may not be true when m and m' are not relatively prime follows from the examples $(p, x, x^2/p)^2 = (p^2, x, x^2/p)$ and $(p, x, (x^2+Q)/p) = (p, x, x^2)$ in the field $x^3 + Qx + R = 0$, basis $1, x, x^2/p$ where $Q \equiv 0 \bmod p$, $Q \not\equiv 0$, $R \equiv 0 \bmod p^2$ (Section 7).

COROLLARY. *If* mm', *where* m *and* m' *are relatively prime, is the least rational integer in* $\mathfrak{a}$, *then* $\mathfrak{a}$ *is the product of two ideals, whose leading numerators* $m_i R_i$ *and* $m_i' R_i'$, $1 \leqq i \leqq n$, *are respectively the factors of* $m_i'' R_i''$, *the corresponding leading numerator of* $\mathfrak{a}$, *which divide* m *and* m' *respectively.*

This follows easily from the preceding corollary since $\mathfrak{a} | [mm']$. With this information as to leading numerators, we may multiply readily by the L. C. M. process when the least rational integers are relatively prime, or factor until these integers are powers of a prime. For example, $(7, x-2, x^2+3)(26, 2x+6, 2x^2+8) = (182, 2x-46, 2x^2+34)$, the leading numerators being 182, 2, 2 (Corollary 2), and Ω_2'', Ω_2'', being obtained by combining $26(x-2)$, $7(2x+6)$, and $26(x^2+3)$, $7(2x^2+8)$ accordingly. Reversing the process, we may take 7, 1, 1, and 26, 2, 2 or 14, 2, 2 and 13, 1, 1 for leading numerators. Taking the former, since the factor must

be of the form $(7, x+a, x^2+b)$, we have $2x-46 \equiv 2(x+a)$, mod 7, $-\frac{7}{2} < a \leqq \frac{7}{2}$; whence $a = -2$. Similarly $b = 3$.

When the two factors are not relatively prime or are relatively prime with least rational integers powers of p, to perform multiplication requires information as to how prime factors occur in $[p]$: obtained in part in the next section, though the actual values are unnecessary. Thus, in the same field, $[7] = \mathfrak{p}^3$. Hence, from the norm, $(7, 7x, x^2+3x-3)^2 = (\mathfrak{p}^2)^2 = [7](7, x+a, x^2+b)$, where $a = -2$ and $b = 3$ since $(x+a)^2 \equiv x^2+3x-3$, $(\text{mod } 7, 7x) \equiv x^2+b \ (\text{mod } x+a, 7)$. Again $[13] = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3$, distinct primes. If $\mathfrak{a} = (169, x+29, x^2+4)$ and $\mathfrak{b} = (13, 13x, x^2-5x+2)$, having the G. C. D. $(13, x+3, x^2+4)$, these are of the form $\mathfrak{p}_1^2$ and $\mathfrak{p}_1 \mathfrak{p}_2$ whence $\mathfrak{ab} = \mathfrak{p}_1^3 \mathfrak{p}_2$. The leading numerators are therefore 13^3, 13, 1. By inspection, $\Omega_2 = 13(x+29)$ and Ω_3 can be got by reducing $(x+29) \times (x^2-5x+2)$ or observing that since $\mathfrak{a}/\mathfrak{ab}$, $\Omega_3 = x^2-5x+b$ whence $b \equiv 2 \ (\text{mod } 13)$, $\mathfrak{b}$ being found to be 704 as in Theorem IV.

6. **Factors of $[p]$.** Besides that given in preceding sections, we use the following symbolism in discussing the factors of $[p]$;

$$\begin{aligned} \omega_i &= \varphi_i(x)/P_i, \\ \mu_i(p_j) &= \psi_{ij}(x)/p_j^{u_{ij}} \text{ for } p_j \neq p, \\ &= \psi_i(x)/p^{u_i} \text{ for } p_j = p, \\ \sigma_i(p) &= \chi_i(x)/p^{v_i}. \end{aligned}$$

The simple integer system of any factor of $[p]$ must be of the form $(p, px, \cdots, px^{i-2}, g(x) \cdots)$, where $g(x)$ is of the form $x^{i-1}+\cdots$; spoken of as of degree $(i-1)$ in p. A simple integer system of degree 1 in p is of the form $(p, x-c, x^2-cx, \cdots, x^{n-1}-cx^{n-2})$; equivalent to $(p, x-c, x^2-c^2, \cdots, x^{n-1}-c^{n-1})$; S, say. The rational remainder on reducing any simple integer $F(x)$ to a rational number by subtracting multiples of $x-c, \cdots, x^{n-1}-c^{n-1}$, is $F(c)$ if $F(x)$ is of degree $<n$ in x. If of degree $\geqq n$, since $F(x)-q(x)f(x)$, $q(x)$ the quotient on dividing $F(x)$ by $f(x)$, is of degree $<n$, the rational remainder on reduction is congruent to $F(c)$, mod p^w where p^w is the highest power of p dividing $f(c)$. If S is closed under multiplication by x, $x^n-c^{n-1}x = x^n-f(x)-c^{n-1}x$ of degree $<n$ must be in S: or $f(c) \equiv 0$, mod p. This condition is evidently sufficient. Hence if $F(x)$ is in S, $F(c) \equiv 0$, mod p and conversely. Also, if the field contain only simple integers S is an ideal. (Section 5).

The condition $f(c) \equiv 0$, mod p is also a necessary and sufficient condition for the existence of a factor of $[p]$ whose simple integer basis is of degree 1 in p unless $p \mid P_n$. The necessity follows from the preceding paragraph. Since Ω_i may be taken of the form $\omega_i + k_i$ (Theorem III Cor. 3)

and $P_i\,\Omega_i$ is in S, $\varphi_i(c)+P_i\,k_i \equiv 0$, mod p. On reduction, the product $\varphi_j(x)\,\{\varphi_i(x)+k_i\}$ will take the form $\sum a_m\{\varphi_m(x)+P_m\,k_m\}+a_1$ where, since the leading coefficient of every φ is 1, and the a's are rational integers, since this product is in S, $p\,|\,a_1$. Dividing $P_j\,P_i$, we have $\omega_j\,\Omega_i = \sum b_m\,\Omega_m + b_1$, where $b_m\,P_i\,P_j = a_m$. The Ω's differ from the ω's only in their rational terms and may therefore equally be used as a basis of the field. The b's are therefore rational integers. Since $b_1\,P_i\,P_j = a_1 \equiv 0$, mod p, and p is prime to P_n and therefore to P_i, P_j, $b_1 \equiv 0$, mod p. Hence S, with Ω_i replacing σ_i, is an ideal. Similarly we may obtain single prime elements for the same ideal, σ_i being replaced by the set, $\sigma_i(p_j)$ for each p_j, where $\sigma_i(p_j) = \mu_i(p_j)+k_{ij}$ and $\psi_{ij}(c+p_j^{u_{ij}}\,k_{ij}) \equiv 0$, mod p.

The set $S' = (p, px, \cdots, px^{n-2}, \sigma_n)$ where σ_n is defined in the theorem below, is closed under multiplication by x. In a simple integer field it is therefore an ideal. Unless $p\,|\,P_n$ we can determine $k_n(x)$ of degree $(n-2)$ in x such that $P_n\{\omega_n+k_n(x)\}$ is congruent to σ_n mod p. It follows that if we replace σ_n in S' by $\Omega_n = \omega_n+k_n(x)$, $(p, px, \cdots, \Omega_n)$ is an ideal. Since $(x-c)\,\sigma_n = f(x) = 0$, $SS' = [p]$. Similarly for a single prime basis.

THEOREM VIIa: *With the notation given above, if* $[p]$ *has a factor whose simple integer basis is of degree* 1 *in* p, $f(x) \equiv 0$ *mod* p, *must have a solution* $x = c$ *and the simple integer basis is* $S = p, x-c, x^2-c^2, \cdots, x^{n-1}-c^{n-1}$. *Unless* $p\,|\,P_n$, *this condition is sufficient. In a simple integer field this factor is* S. *Otherwise, unless* $p\,|\,P_n$, *the factor is obtained from* S *by replacing* $x^{i-1}-c^{i-1}$ *for* $P_i > 1$ *by* ω_i+k_i *where* $\varphi_i(c)+P_i k_i \equiv 0$, *mod* p; *or by the set* $\mu_i(p_j)+k_{ij}$ *for each* p_j *in* P_i, *where* $\psi_{ij}(c)+p_j^{u_{ij}}k_{ij} \equiv 0$, *mod* p. *The quotient of dividing* $[p]$ *by this ideal is* $S' = (p, px, \cdots, px^{n-2}, \sigma_n)$ *in a simple integer field, where* $\sigma_n = x^{n-1}+(c+a_{n-1})x^{n-2}+(c^2+a_{n-1}\,c+a_{n-2})x^{n-3}+\cdots+(c^{n-1}+a_{n-1}\,c^{n-2}+\cdots+a_1)$. *Otherwise unless* $p\,|\,P_n$, *the quotient is obtained from* S' *by replacing* σ_n *by* $\omega_n+k_n(x)$ *where* $k_n(x)$ *is of degree* $(n-2)$ *in* x *and such that* $\varphi_n(x)+P_n k_n(x) \equiv \sigma_n$, *mod* p; *or by the set* $\mu_n(p_j)+k_{nj}(x)$ *for each* p_j *in* P_n, *where* $\psi_{nj}(x)+p_j^{u_{nj}}k_{nj}(x) \equiv 0$, *mod* p.

Again, if $p\,|\,P_n$ and the simple integer basis is of the type S, the same reasoning shows that, for $p_j \nmid p$, the single prime elements $\sigma_i(p_j)$ replacing σ_i of S in the ideal $\mathfrak{a}$ are as in the preceding theorem and that $\mu_k(p_m)\,\sigma_i(p_j)$, $p_m \nmid p$, is in the new set. If (i) is contains $x\,\sigma_i(p)$, the product of $\sigma_i(p)$ by any simple integer and thence any $\mu_k(p_j)\,\sigma_i(p)$ is also in the set. If (ii) it contains $(x-c)\,\mu_i(p)$, since the numerator of any $\sigma_k(p_j)$ is a polynomial in $(x-c)$, it follows similarly that $\mu_i(p)\,\sigma_k(p)$ is in the set for $p_j \nmid p$. We must also have (iii) $\mu_i(p)\,\sigma_k(p)$ in the set. Of these, if $\sigma_i(p)$ is of the

form $\mu_i(p)+k_i$, the first two are equivalent since $(x-c)\,\mu_i(p) = x\sigma_i(p) - c\sigma_i(p)+k_i(x-c)$, the last two in the set: and we need test the third only for $i \leqq k$, since $\mu_i(p)\,\sigma_k(p) = \sigma_i(p)\,\mu_k(p)+k_k\,\sigma_i(p)+k_i\,\sigma_k(p)$.

Consider the set $S' = (p, x-c, \cdots, x^{r-2}-c^{r-2}, \chi_r(x)\cdots\chi_n(x))$ where $r-1$ is the lowest degree in x for which single-prime elements in p occur. (Notation at beginning of section); evidently equivalent to S. If $N\mathfrak{a}$ is p, the leading numerators are $p, 1, 1, \cdots$, (Theorem III). When $u_i = u_{i-1}$, since $(x-c)\,\mu_i(p)$ must occur in the ideal, we may take $\chi_i(x) = (x-c)\,\psi_i(x)$. Otherwise, as before $\sigma_i(p) = \mu_i(p)+k_i$ or $\chi_i(x) = \psi_i(x)+p^{u_i}k_i$. The condition that $\chi_i(c)$ is in S requires that $p \mid \psi_i(c)$. The condition that $x\sigma_i(p)$ is in the ideal is equivalent to the condition that $x\chi_i(x)$, reduced in S' is divisible by p^{u_i+1}. Let

$$(1) \qquad x\psi_i(x) = \psi_{i+1}(x)+\sum b_j^{(i)}\,\psi_j(x)+g_i(x), \qquad n \geqq i \geqq j \geqq r,$$

where $\psi_{n+1}(x) = f(x)$, and $g_i(x)$ is of degree $< r-1$.* On substituting and reducing $x\chi_i(x)$ we obtain the conditions

$$(2) \qquad p^{u_i}(c-b_i^{(i)})\,k_i - \left\{\sum_r^{i-1} b_j^{(i)}\,p^{u_j}k_j - g_i(c)\right\} \equiv 0, \mod p^{u_i+1}$$

the term from ψ_{i+1} disappearing from the congruence if $u_{i+1} > u_i$ and reducing to 0 if $u_{i+1} = u_i$. The condition for solvability is that

$$(3) \qquad \sum_r^{i-1} b_j^{(i)}\,p^{u_j}k_j \equiv g_i(c) \mod p^{u_i+1} \text{ or } \mod p^{u_i}$$

according as $c \equiv b_i^{(i)}$ or $c \not\equiv b_i^{(i)}$, mod p. In the former case k_i is undetermined.

For (iii), that $\mu_i(p)\,\sigma_m(p)$ should be in $\mathfrak{a}$ requires that $\psi_i(x)\,\chi_m(x)$, reduced to a rational in S', is divisible by $p^{u_i+u_m+1}$. Let

$$(4) \qquad \psi_i(x)\,\psi_m(x) = e(x)\,f(x)+\sum e_j^{(im)}\,\psi_j(x)+g_{im}(x)$$

as in (i): where $g_{im}(x)$ is of degree $< r-1$. Evaluating the rational remainder as before, we have

$$(5) \qquad p^{u_i+u_m}k_i\,k_m - \sum e_j^{(im)}\,p^{u_j}k_j - g_{im}(c) \equiv 0, \mod p^{u_i+u_m+1}$$

As remarked the cases $i \leqq k$ are sufficient: also the finding of (4) is simplified as in the preceding footnote.

* As will be seen in the examples in the next section, computation is simplified if we use the equivalent series for $(x-c)\,\psi_i(x)$ and write $f(x)$ in the form $F(x)+F_1(x)$, the latter containing the factors of $f(x)$, mod p.

THEOREM VIIb. *If* $[p]$ *has a factor* $\mathfrak{a}$ *whose simple integer basis is of degree* 1 *in* p *and* $p \mid P_n$, $f(x) \equiv 0$, *mod* p, *must have a solution,* $x = c$, *and the single prime elements of* $\mathfrak{a}$ *for* $p_j \nmid p$ *be as determined in Theorem* VIIa. *If* $N\mathfrak{a} = p$, $\chi_i(x) = (x - c)\,\psi_{i-1}(x)$ *where* $u_i = u_{i-1}$; *where* $u_i > u_{i-1}$, $\chi_i(x) = \psi_i(x) + p^{u_i} k^i$, *where* k_i *is determined from equations* (2) *and* (5): *that these equations should be compatible and* $p \mid \psi_i(c)$ *being necessary and sufficient conditions. In general, conditions* (i), (ii) *and* (iii) *are necessary and sufficient.*

COROLLARY 1. *If also* $p^{u_i} \mid \psi_i(c)$ *for* $i \leqq t \leqq n$ *and* $c \not\equiv b_i^{(i)}$, *mod* p, *for* $i \leqq z$, $\sigma_i(p) = \mu_i(p) - \psi_i(c)/p^{u_i}$ *for* $i \leqq t$ *and* $i < z$: *if* $z < t$ *then* $\sigma_i(p) = \mu_i(p) - \psi_i(c)/p^{u_i} + k_i$ *where* $p^{u_i} k_i (c - b_i^{(i)}) = \sum b_j^{(i)} k_j$, $z < j < i$. *If* $t = z = n$, $\sigma_i(p) = \mu_i(p) - \psi_i(c)/p^{u_i}$ *for* $i < n$ *and* $\sigma_n(p) = \mu_n(p) - \psi_n(c)/p^{u_n} + k$ *where* $p^{u_n}(c - b_n^{(n)})\,k \equiv f(c)$, *mod* p^{u_n+1}, $p^{2u_n} k^2 - e_n^{(nn)} p^{u_n} k + e(c) f(c) \equiv 0$, *mod* p^{2u_n+1}, *sufficient conditions being that these are compatible.*

For $\mu_i(p) - \psi_i(c)/p^{u_i}$ is an integer and may be used as an element of the field. Using such a basis, these results follow on evaluating $g_i(c)$ and $g_{im}(c)$ from (1) and (4). As an example from the general cubic field, $x^3 + Qx + R = 0$, consider Case 1, former paper, where $p \mid Q$ and $p^2 \mid R$, but not both $p^2 \mid Q$ and $p^3 \mid R$; field $(1, x, x^2/p)$. Here $c = 0$, $b_3^{(3)} = 0$, $e_3^{(33)} = -Q$, $e(x) = x$. The first condition is satisfied for all values of k and the second gives $p^2 k^2 - Qpk = 0$, mod p^2. If $p^2 \mid Q$, we have one solution, $k = 0$, but, if not, also $k = Q/p$. Hence we have $\mathfrak{p}_1 = (p, x, x^2/p)$ and unless $p^2 \mid Q$, $\mathfrak{p}_1' = (1, x, (x^2 + Q)/p))$ ideal factors of $[p]$; omitting elements $\sigma_i(p_j)$ for $p_j \nmid p$, as always in the sequel.

When $N\mathfrak{a} = p^s$, information as to s may be obtained from the next theorem. If the leading numerator of $\sigma_i(p)$ is p, then $\sigma_i(p) = p\,\mu_i(p)$; if 1, $\chi_i(x)$ is of the form $\psi_i(x) + \sum' k_j^{(c)} p^{u_i - u_j} \psi_j(x) + p^{u_i} k_i$, $\sum'$ extending only over the cases where $\sigma_i(p)$ has the leading numerator p (Theorem IV). If the leading numerator of $\sigma_k(p)$ is p, since $\mu_i(p)\,\mu_k(p)$ is an integer, $\mu_i(p)\,\sigma_k(p) = p\,\mu_i(p)\,\mu_k(p)$ is in the set and (iii) may be omitted. Equations similar to (2) and (5) may thus be obtained, omitting these cases. Single-prime integers in p of degree $(n-1)$ in x only will occur except for special multiple factors of $f(x)$, mod p. In this case applying (ii), we find $b_n^{(n)} = c$, mod p, and $g_n(c) = 0$, mod p^{u_n+1}. If so (i) is satisfied and (iii) is covered above. Adding the condition that $\chi_n(x)$ is in S,

COROLLARY 2. *When single-prime elements in* p *of degree* $(n-1)$ *in* x *only occur in the field, and the simple integer basis is of degree* 1 *in* p,

$N\mathfrak{a} = p$ *or* p^2. *In the latter case* $\sigma_n(p) = p\,\mu_n(p)$, *and necessary and sufficient conditions are that* $\psi_n(c) \equiv 0$, $b_n^{(n)} \equiv c$, *mod* p *and* $g_n(c) \equiv 0$, *mod* p^{u_n+1}. For the field cited above, these reduce to $R \equiv 0$, mod p^2: so that (p, x, x^2) is a factor of $[p]$ of norm p^2.

More generally if the simple integer system of $\mathfrak{a}$, a factor of $[p]$ is of the form $S = (p, px, \cdots, px^{s-2}, \sigma_s, \cdots)$, where $\sigma_s = w(x) = x^{s-1} + \cdots$, and if $h(x)$ denote the remainder on dividing $f(x)$ by $w(x)$, then $h(x) \equiv 0$, mod p (Theorem V). Also (Theorem III), $\Omega_i = p\,\omega_i$ for $i < s$, and if $p \mid p_n$, there is no single-prime element in p of degree $< s-1$ (Theorem IV). For $i > s$, unless $p \mid P_n$, we may take $\Omega_i = \omega_i + k_i(x)$ or $\sigma_i(p_j) = \mu_i(p_j) + k_{ij}(x)$, where $k_i(x)$ and $k_{ij}(x)$ are simple integers of degree $< s-1$. If $h_i(x)$ and $h_{ij}(x)$ denote the remainders on dividing $\varphi_i(x)$ and $\psi_{ij}(x)$ by $w(x)$ since $P_i\,\Omega_i$ and $p_j^{u_{ij}}\,\sigma_i(p_j)$ must be in S

$$(6) \qquad h_i(x) + P_i\,k_i(x) \equiv 0 \quad \text{or} \quad h_{ij}(x) + p_j^{u_{ij}}\,k_{ij}(x) \equiv 0, \text{ mod } p.$$

The former can be solved when p is prime to P_i and the latter unless $p_j = p$. For these cases, the sufficiency follows as in the preceding case, $k_i(x)$ and $k_{ij}(x)$ replacing k_i and k_{ij} and the remainder on division by $w(x)$ replacing substitution of c. Existence of a factor $w(x)$ of $f(x)$, mod p, is therefore necessary and sufficient, and the basis is obtained by substituting $p\,\omega_i$ for px^{i-1} for $i < s$ and $\omega_i + k_i(x)$ or the set $\mu_i(p_j) + k_{ij}(x)$ for σ_i for $i > s$. If $p \mid P_n$, $\sigma_i(p)$ does not occur for $i < s$ (Theorem IV). Single-prime elements in p_j, $p_j \nmid p$, to replace σ_i are determined by the preceding paragraph as in Theorem VII b. Likewise, (i) if $x\sigma_i(p)$ is in the set, so also is any $\mu_k(p_m)\,\sigma_i(p)$, $p_m \nmid p$: and (ii) if $w(x)\,\mu_i(p)$, also any $\mu_i(p)\,\sigma_k(p_j)$ since the numerator of the latter is of the form $\varphi(x)\,w(x)$, $\varphi(x)$ a simple integer. If the norm is p^{s-1} we may write $\sigma_i(p) = \mu_i(p) + k^{(i)}(x) = \mu_i(p) + \sum k_j^{(i)} x^j$. In this case (i) includes (ii). We require also that (iii) $\mu_i(p)\,\sigma_k(p)$ should be in the set. Applying (i) to equation (1) and reducing by $w(x)$, we obtain

$$(7) \quad p^{u_i}\left\{(x - b_i^{(i)})\,k^{(i)}(x) - k_{s-2}^{(i)}\,w(x)\right\} - \left\{\sum^{i-1} b_j^{(i)}\,k^{(j)}(x) - g_i'(x)\right\} \equiv 0, \text{ mod } p^{u_i}$$

if every $u_i > u_{i-1}$, where $g_i'(x)$ denotes the remainder on reducing $g_i(x)$ by $w(x)$. Hence $p^{u_i}\,k^{(i)}(x)$ is the quotient obtained on dividing by $(x - b_i^{(i)})$ the last three terms with sign changed; determining the remaining coefficients in terms of $k_{s-2}^{(i)}$. Writing this as k_i, since the quotient will not be changed by the added terms,

$$(8) \qquad k^{(i)}(x) = k_i\,\frac{w(x) - w(b_i^{(i)})}{x - b_i^{(i)}} + \frac{A_i(x) - A_i(b_i^{(i)})}{p^{u_i}(x - b_i^{(i)})},$$

where

$$A_i(x) = w(x)\sum^{i-1} b_j^{(i)} k_j - g_i'(x), \tag{9}$$

coefficients reduced mod p. Also, equating coefficients in (7) and eliminating,

$$p^{u_i} k_i\, w(b_i^{(i)}) + A_i(b_i^{(i)}) \equiv 0, \mod p. \tag{10}$$

If the last congruence is solvable, the coefficients of the last term of (8) are rational integers and k_i is determined unless $w(b_i^{(i)}) \equiv 0, \mod p$. Applying (iii) to (4), we obtain

$$p^{u_m}\psi_i(x)k^{(m)}(x) - \sum e_j^{(im)} p^{u_j} k^{(j)}(x) + g_{im}(x) \equiv 0, \mod p^{u_i+u_m+1} \tag{11}$$

the left side reduced in S. Because of relations between $f(x)$ and the ψ's as the next section will show, this is in practice simpler than the explicit formula below, in terms of the coefficients of elements of the field, obtained on simplification:

$$\psi_i(x)k^{(m)}(x) = \sum_{r-1}^{n} b_r d_j^{(v)} p^{u_j} k^{(j)}(x) + \sum_{0}^{r-2} B_r d_j^{(r)} x^j, \tag{11'}$$

where $f(x) = \sum a_j x^j$; $\psi_i(x) = \sum c_j^{(i)} x^j$; $x^i = \sum d_j^{(i)} \psi_j(x) + \sum d_j^{(i)} x^j$; and $B_r = \sum\sum k_j^{(m)} (c_{r-j}^{i} - c_{r+n-k-j}^{(i)} a_k)$ for all non-negative values of the subscripts.

THEOREM VIII. *If* $[p]$ *has a factor* $\mathfrak{a}$ *whose simple integer system is of the form* $S = (p, px, \cdots, px^{s-2}, \sigma_s = w(x), \cdots)$ *where* $w(x) = x^{s-1} + \cdots$, *then* $w(x) | f(x), \mod p$. *Unless* $p | P_n$ *this condition is sufficient and a simple integer field* $\mathfrak{a} = S$. *Otherwise, unless* $p | P_n$, $\Omega_i = p\omega_i$ *for* $i < s$: *for* $i \geqq s$, $\Omega_i = \omega_i + k_i(x)$ or $\sigma_i(p) = \mu_i(p) + k_{ij}(x)$ *where* $k_i(x)$ *and* $k_{ij}(x)$ *are determined by* (6).

If $p | P_n$, $\sigma_i(p)$ *for* $p_j \nmid p$ *is determined likewise. For* $i < s$, $\sigma_i(p)$ *does not occur. For* $i \geqq s$, *if* $N\mathfrak{a} = p^{s-1}$ *and each* $u_i > u_{i-1}$, $\sigma_i(p) = \mu_i(p) + k^{(i)}(x)$, *determind to one parameter by* (8). *The latter is determined by* (10) *unless* $w(b_i^{(i)}) = 0, \mod p$: *otherwise by* (11). *Necessary und sufficient conditions are that* $w(x) | \psi_i(x), \mod p$ *and* (10) *and* (11) *should be compatible.*

Corollaries similar to those under the preceding theorem may easily be written down. As an example, in the field cited above, $k^{(3)}(x) = k_1 x + Q/p$, $(k_1 pQ + R)x + pk_1 R \equiv 0, \mod p^3$, yielding $(px, (Qx^2 - Rx + Q^2)/p^2)$ when $Q \not\equiv 0, \mod p^2$.

7. Prime factorization of $[p]$. Prime factorization of $[p]$ is made to depend upon prime factorization of $f(x), \mod p$, by the following theorem.

THEOREM IX. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *have simple integer systems of the form* $(p, px, \cdots, px^{s-2}, w_s(x)\cdots)$ *and* $\mathfrak{b} = (p, \cdots, \cdots, px^{t-2}, w_t(x)\cdots)$ *where* $w_s(x) = x^{s-1} + \cdots$, $w_t(x) = x^{t-1} + \cdots$. *If* $\mathfrak{a}|\mathfrak{b}$, *then* $s \leqq t$ *and* $w_s(x)|w_t(x)$, *mod* p: *and conversely unless* $p|P_n$.

For $\mathfrak{a}$ contains $\mathfrak{b}$ and therefore the simple integer system of $\mathfrak{a}$ contains that of $\mathfrak{b}$. Hence $s \leqq t$. If $w(x) = w_s(x)q(x) + r(x)$, $r(x)$ of degree $< s-1$ in x, $r(x) \equiv 0$, mod p. In the converse case, $\mathfrak{a}$ will contain $\mathfrak{b}$ by the preceding theorem, whence $\mathfrak{a}|\mathfrak{b}$.

When $p|P_n$, and the factors of $[p]$ with various simple integer bases have been determined, their relations as to relative primeness can easily be determined by the G. C. D. process described in Section 5. Thus for the field cited before, for the simple integer basis (p, px, x^2), we have the factors $\mathfrak{p}_1 = (p, x, x^2/p)$, and if $Q \not\equiv 0$, mod p^2, $\mathfrak{p}_2 = (p, x, (x^2+Q)/p)$; with the simple integer basis (p, px, x^2), $\mathfrak{a} = (p, px, (Qx^2 - Rx + Q)/p^2)$ only if $Q \not\equiv 0$, mod p^2. By the G. C. D. process, $\mathfrak{p}_1$, is prime to $\mathfrak{a}$ and $\mathfrak{p}_2|\mathfrak{a}$, the quotient being $\mathfrak{p}_2$ since its norm is p. Since these form the complete sets of factors of $[p]$, we have $[p] = \mathfrak{p}_1^3$, when $Q \equiv 0$, mod p^2 and $[p] = \mathfrak{p}_1\mathfrak{p}_2^2$ when $Q \not\equiv 0$, mod p^2. The following theorem, however, renders unnecessary the determination of $\mathfrak{a}$.

THEOREM X. *If* $\mathfrak{p}$, *where* $\mathfrak{p}|[p]$ *and* $N\mathfrak{p} = p$, *is a simple factor of* $[p]$ *then the congruences of Theorem* VIIa *are satisfied to a modulus of degree in* p *one higher than that of those given*; *and conversely.*

For if $\mathfrak{p}$ is a simple factor of $[p]$, $\mathfrak{p}^2$ has p^2 for its least rational integer and a simple integer system of the form $(p^2, x - c^1, \cdots)$. Repeating the reasoning of that theorem, except that the rational remainder must be a multiple of p^2 instead of p, we have the result stated. Conversely, if these conditions are satisfied, an ideal $\mathfrak{a}$ with the above simple integer basis exists and $N\mathfrak{a} = p^2$. Taking the G. C. D. of $\mathfrak{a}$ and $[p]$, one of type $\mathfrak{p}$ exists and $\mathfrak{p}|\mathfrak{a}$; the quotient being $\mathfrak{p}$ since there can be only one of this type and norm p. Hence $\mathfrak{a} = \mathfrak{p}^2$, so that $\mathfrak{p}$ is a simple factor of $[p]$.

Thus in the example just cited if $p^2|Q$, $\mathfrak{p}$ cannot satisfy the conditions given after Theorem VIIa unless $p^3|R$; shown to be impossible in this case. Hence $\mathfrak{p}$ is a multiple factor of $[p]$; or since it is the only one of norm p, $[p] = \mathfrak{p}_1^3$. If $Q \not\equiv 0$, mod p^2, $\mathfrak{p}_2$ is similarly shown to be a multiple and $\mathfrak{p}_1$ a simple factor of $[p]$.

8. **Application to cubic fields.** We suppose the cubic reduced to the normal form $x^3 + Qx + R = 0$. (See Introduction). Fields are classified as on page 126, former paper. We deal first with cases in which $p|P_n$, omitting $\sigma_i(p_j)$ for $p_j \neq p$.

Case 1: $(p \mid Q, p^2 \mid R)$. Prime factors, $\mathfrak{p}_1 = (1, x, x^2/p)$ and, when $Q \not\equiv 0$, $\bmod p^2$, $\mathfrak{p}_2 = (1, x, (x^3+Q)/p)$: $[p] = \mathfrak{p}_1^3$ if $Q \equiv 0$, $\bmod p^2$, and $\mathfrak{p}_1 \mathfrak{p}_2^2$ if $Q \not\equiv 0$, $\bmod p^2$. The details are given in the preceding section.

Case 2. $(6QR \not\equiv 0$, *mod* p, $\Delta \equiv 0$, *mod* p^{2n} *but* $\Delta \not\equiv 0$, *mod* $p^{2n+2})$. We have $4Q^3(x^3+Qx+R) = (2Qx+3R)^2(Qx-3R)+\Delta(Qx+R)$. If c_1 and c_2 denote the values of c, $2Qc_1+3R \equiv 0$, $\bmod p$, and $Qc_2-3R \equiv 0$, $\bmod p$. Taking the solutions, $\bmod p^{2n+2}$, $\psi_3(x) = (x-c_1)(x-c_2)$ and $4Q^3(x^3+Qx+R) \equiv 4Q^3(x-c_1)^2(x-c_2)+\Delta(Qx+R)$, $\bmod p^{2n+2}$. Applying Theorem VII a, $p^n k(c-c_1)-\Delta(Qc+R) \equiv 0$, $\bmod p^{n+1}$, and $\Delta R(c-c_2)-8Q^3p^{2n}k^2 \equiv 0$, $\bmod p^{2n+1}$ where $c = c_1$ or c_2. For $c = c_2$ we have the factor $\mathfrak{p}_2 = (p, x-c_2, \psi_3/p^n)$; a simple factor. For $c = c_1$, writing $\Delta = p^{2n}\Delta'$, we have factors $\mathfrak{p}_1, \mathfrak{p}_1' = (p, x-c_1, (\psi_3+p^nk)/p^n)$ only if $-\Delta'$ is a quadratic residue, $\bmod p$: distinct if $\Delta \not\equiv 0, \bmod p^{2n+1}$. In any case $\mathfrak{p}_1'' = (p, x-c_1, \psi_3/p^{n-1})$ is a factor (Cor. 2 Theorem VII b); prime if $\mathfrak{p}_1$ does not exist. Hence if $-\Delta'$ is a quadratic residue, $\bmod p$, $[p] = \mathfrak{p}_1^2\mathfrak{p}_2$ or $\mathfrak{p}_1\mathfrak{p}_1'\mathfrak{p}_2$ according as $\Delta \equiv 0$, $\bmod p^{2n+1}$ or not. If $-\Delta'$ is not a quadratic residue, $\bmod p$, $[p] = \mathfrak{p}_1''\mathfrak{p}_2$.

Case 3a. $(Q \equiv 1$, *mod* 2, $R \equiv Q+1$, *mod* 4, *and* (3b) *does not hold)*. Here $c = 0, 1$; $\psi_3(x) = x(x+1)$; $x^3+Qx+R = x(x^2-1)+\{(Q+1)x+R\}$. When $c = 0$, we have from Theorem VII a, $\mathfrak{p}_1 = (2, x, (x^2+x)/2)$ or $(2, x, (x^2+x+2)/2)$ according as we have $R \equiv 0$, or 2, $\bmod 4$; a simple factor. When $c = 1$, $\mathfrak{p}_2 = (2, x-1, (x^2+x)/2)$ exists if $R \equiv Q+1$, $\bmod 8$ and $\mathfrak{p}_3 = (2, x-1, (x^2+x+2)/2)$ if $R \equiv -(Q+1)$, $\bmod 8$; simple if both occur. If neither occur, we have $\mathfrak{p}_4 = (2, x+1, x^2+1)$. Hence $[2] = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$, $\mathfrak{p}_1\mathfrak{p}_2$, $\mathfrak{p}_1\mathfrak{p}_3$ or $\mathfrak{p}_1\mathfrak{p}_4$ according as both, the first only, the second only, or neither of $R = \pm(Q+1)$ $\bmod 8$, occur.

Case 3b. $(Q \equiv 1$, $R \equiv 4$, *mod* 4, $2^4 \mid \Delta)$. There are two subcases: (a) when $\Delta \equiv 0$, $\bmod 2^k$, $2n+2 = k \leqq 2n+4$, Δ not of the form $2^{2k}(4s+1)$; (b) when Δ is of the form $2^{2n}(4s+1)$. Choose c_1, c_2, so that $2Qc_1+3R \equiv 0$, $Qc_2-3R \equiv 0$, $\bmod 2^{2n+4}$. Then $4Q^3(x^3+Qx+R) \equiv 4Q^3(x-c_1)^2(x-c_2)$ $+\Delta(Qx+R)$, $\bmod 2^{2n+4}$: $\psi_3(x) = (x-c_1)(x-c_2)$. For (a), Theorem VII a yields $\mathfrak{p}_2 = (2, x, \psi_3/2)$, a simple factor; and $\mathfrak{p}_1 = (2, x+1, \chi_3/2^n)$ where $\chi_3 = \psi_3(x)$ or $\psi_3(x)+2^n$ according as $2^{2n+3} \mid \Delta$ or not. Also $(2, x+1, \psi_3(x)/2^{n-1})$ exists, of norm 4, and $= \mathfrak{p}_1^2$. Hence, for (a), $[2] = \mathfrak{p}_1^2\mathfrak{p}_2$. In case (b), the congruences have no solution so that [2] is prime.

Case 4a. $(Q \equiv 0$, *mod* 3; $R = \pm(Q+1)$, *mod* 9, *but not* (4b)). Here $c = \mp 1$: and $\mathfrak{p} = (3, x \pm 1, (x^2 \mp x+1)/3)$ exists, a multiple factor of $[p]$ and the only one of norm p. Hence $[p] = \mathfrak{p}^3$.

Case 4b. $(Q \equiv -3$, *mod* 9; $R = \pm(Q+1)$, *mod* 27, *but not* (4c)). Here $c = \mp 1$: $x^3+Qx+R = (x \pm 1)^2(x \mp 2)+\{\pm(Q+1)x+R\}$. We obtain $\mathfrak{p} = (3, (x \pm 1)/3, (x^2 \mp x-2)/9)$ only if $R \equiv \pm(Q+1)$, $\bmod 81$,

and in this case, $(3, (x \pm 1), (x^2 \mp x - 9)/9)$, of norm 9 and therefore equal to $\mathfrak{p}^2$, is an ideal. Hence $[3] = \mathfrak{p}^3$ or is prime according as $R = \pm(Q+1)$, mod 81 or not.

Case 4c. ($Q = -3$, *mod* 9, $R = \pm(Q+1)$, *mod* 27, $3^{2n+2} \mid \Delta$, $n > 2$). Choose c_1 and c_2 so that $2Qc_1 + 3R \equiv 0$, $Qc_2 - 3R \equiv 0$, mod 3^{2n+5}; $4Q^3(x^3 + Qx + R) \equiv 4Q^3(x-c_1)^3\,(x-c_2)\,\Delta(Qx+R)$, mod 3^{2n+5}. $\psi_2(x) = x - c_1$: $\psi_3(x) = (x-c_1)\,(x-c_2'')$. From Theorem VIIb, we obtain $\mathfrak{p}_2 = (3, (x \mp 2)/3, \psi_3(x)/3^n)$, and $\mathfrak{p}_1 = (3, (x \pm 1)/3, (\psi_3(x) \pm 3^n)/3^n)$ only if $\Delta = -3^{2n+2}$, mod 3^{2n+3}. Also $(3, (x \pm 1)/3, \psi_3(x)/3^{n-1})$, is an ideal: a prime $\mathfrak{p}''$ or equal to $\mathfrak{p}_1\mathfrak{p}_1'$. Hence $[3] = \mathfrak{p}_1\mathfrak{p}_1'$ or $\mathfrak{p}_1''\mathfrak{p}_2$ according as $\Delta = -3^{2n+2}$, mod 3^{2n+3}, or not.

Finally, if $x^3 + Qx + R \equiv 0$, mod p, has no solution, $[p]$ is prime. If it has a solution c, and p is not one of the primes in the above special cases $[p] = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ or $\mathfrak{p}\mathfrak{p}''$ according as $x^3 + Qx + R$ has three linear or one linear and an irreducible quadratic factor, mod p; the bases being obtained by Theorems VIIa and VIII. If p is one of the primes listed above, the elements $\sigma_2(p)$ and $\sigma_3(p)$ of the factors of p are as given there and the single-prime elements in p_j, $p_j \nmid p$ are obtained by Theorems VIIa or VIII. These may be combined into the ordinary basis by Theorem III.

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AN INTERPRETATION OF CERTAIN DECOMPOSABLE ALGEBRAIC FORMS AS FUNCTIONS OF DIVISORS.*

By E. T. Bell.

1. Introduction. A form (homogeneous polynomial) of degree ω in ω indeterminates is called *decomposable* in a given field if it is the product of ω linear forms whose coefficients are in the field. When the field is algebraic of degree ω, certain of these forms are well known in the theory of ideals.

It is not necessary that the coefficients in the linear factors and the indeterminates of the form be considered as elements of the same field. If they are in different fields, the special case of the theory of an algebra over a field in which the algebra also is a field, is applicable.

We discuss certain decomposable forms in ω indeterminates, where ω is an arbitrary constant integer >0, such that the coefficients in the ω linear factors of a given form are algebraic numbers (but not necessarily in a field of degree ω), and the indeterminates are in the irregular field of extended numerical functions. The latter field is defined in § 5, after we have developed a few purely algebraic properties of the forms with indeterminates in any field.

As we operate simultaneously in two distinct fields, each of the final formulas has a dual interpretation. One interpretation refers to the abstract algebraic properties of the forms, the other to their properties as functions of divisors of positive rational integers. The forms have the remarkable property of repetition under each of the two kinds of multiplication considered. They are expressible as special determinants which, in the very simple case when the relevant algebraic number field is that defined by a primitive ωth root of unity, degenerate to circulants of order ω in either interpretation. The properties of the forms are intimately connected with those of ω sequences of numbers

$$A_k(n) \qquad (k = 0, 1, \cdots, \omega - 1;\ n = 0, 1, 2, \cdots)$$

defined by

$$a_\omega A_k(n+\omega) + a_{\omega-1} A_k(n+\omega-1) + \cdots + a_0 A_k(n) = 0,$$

where $a_\omega, a_{\omega-1}, \cdots, a_0$ are any constant rational numbers and $a_\omega \neq 0$, $a_0 \neq 0$. The ω sequences mentioned may be any $A_k(n)$ $(n = 0, 1, \cdots)$, such that

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$A_k(n)$ $(k = 0, 1, \cdots, \omega - 1)$ are ω linearly independent solutions of the given difference equation. Conversely, the forms arise naturally in the study of these sequences.

2. **The fundamental sequences.** Let a_j $(j = 0, 1, \cdots, \omega)$, $a_\omega \neq 0$, $a_0 \neq 0$, be rational numbers, and let Ω be any root of

$$\varphi(\Omega) \equiv \sum_{j=0}^{\omega} a_{\omega-j}\, \Omega^{\omega-j} = 0. \tag{1}$$

Let $P(x)$ be any polynomial in x with rational number coefficients. Then if $R(x)$ is the remainder on ordinary algebraic division of $P(x)$ by $\varphi(x)$, the polynomial $P(\Omega)$ has a unique canonical reduction to the form $R(\Omega)$. In particular, if n is a rational integer $\geqq 0$, we shall write the unique *canonical reduction* of Ω^n in the form

$$\Omega^n = A_{\omega-1}(n)\, \Omega^{\omega-1} + A_{\omega-2}(n)\, \Omega^{\omega-2} + \cdots + A_0(n). \tag{2}$$

Multiply (2) throughout by $a_j \Omega^j$, and apply (2) to the result. Then

$$a_j \Omega^{n+j} = a_j A_{\omega-1}(n+j)\, \Omega^{\omega-1} + \cdots + a_j A_0(n+j).$$

Take $j = 0, 1, \cdots, \omega$, add the results, observe that the left of the new result vanishes by (1) and that the right is a canonical reduction. Hence

$$\sum_{j=0}^{\omega} a_j A_k(n+j) = 0 \qquad (k = 0, 1, \cdots, \omega - 1) \tag{3}$$

for all integers $n \geqq 0$. Taking $n = 0, 1, \cdots, \omega - 1$ in (2) we see that

$$A_k(s) = 0, \quad s \neq k, \quad 0 \leqq s \leqq \omega - 1, \quad A_k(k) = 1 \quad (k = 0, 1, \cdots, \omega - 1). \tag{4}$$

That is, $A_k(n)$ $(n = 0, 1, \cdots)$ is the kth *fundamental recurring series*, or *sequence*, defined by

$$\sum_{j=0}^{\omega} a_j X(n+j) = 0 \tag{5}$$

and the ω initial values of $X(n) \equiv A_k(n)$ as given by (4).

3. **Derived sequences.** Let $X(n)$ be any solution of (5). Then

$$X(n) = \sum_{j=0}^{\omega-1} X(j)\, A_j(n) \qquad (n = 0, 1, \cdots). \tag{6}$$

Thus the nth term of the sequence $X(n)$ $(n = 0, 1, \cdots)$ is expressed as a linear homogeneous function of the nth terms of the ω fundamental sequences, the constants being the first ω terms $X(j)$ $(j = 0, 1, \cdots, \omega - 1)$ of the given solution. If $X(j) \neq A_k(j)$ for any pair j, k such that $0 \leqq j \leqq \omega - 1$, $0 \leqq k \leqq \omega - 1$, the sequence $X(n)$ $(n = 0, 1, \cdots)$ is called *derived*. We shall require several such.

Let y denote the one-rowed matrix $(y_0, y_1, \cdots, y_{\omega-1})$, whose elements $y_0, y_1, \cdots, y_{\omega-1}$ belong to an abstract field, and similarly for $z \equiv (z_0, \cdots, z_{\omega-1}), \cdots, u \equiv (u_0, \cdots, u_{\omega-1}), v \equiv (v_0, \cdots, v_{\omega-1})$. Define ω functions $B_k(n; y)$ of y by

$$(7) \quad B_k(n; y) \equiv \sum_{j=0}^{\omega-1} y_j A_k(n+j) \quad (k = 0, 1, \cdots, \omega-1; \ n = 0, 1, \cdots).$$

Then from (3), (7) it is seen that $B_k(n; y)$ is a derived solution of (5), and hence, by (6),

$$(8) \quad B_k(n; y) = \sum_{j=0}^{\omega-1} B_k(j; y) A_j(n).$$

Either from this or from (7), we find the ω initial values

$$B_k(0; y) = y_k;$$

$$(9) \quad B_k(j; y) = y_{k-j} + \sum_{i=1}^{j} y_{\omega-i} A_k(\omega+j-i) \quad (j = 1, \cdots, k);$$

$$B_k(j; y) = \sum_{i=1}^{j} y_{\omega-i} A_k(\omega+j-i) \quad (j = k+1, \cdots, \omega-1);$$
$$(k = 0, 1, \cdots, \omega-1).$$

The values of $B_k(n; y)$, $n > \omega-1$, are calculated from (8) or from (5) directly with $X(n)$ $(n = 0, 1, \cdots)$ replaced therein by $B_k(n; y)$.

Next define the ω functions $B(y, z)$ of the two matrices y, z by

$$(10) \quad B(y, z) \equiv \sum_{j=0}^{\omega-1} z_j B(j; y) \quad (k = 0, 1, \cdots, \omega-1).$$

Then from (7) we have

$$(11) \quad B_k(y, z) = \sum_{i=0}^{\omega-1} \sum_{j=0}^{\omega-1} y_i z_j A_k(i+j);$$

whence $B_k(y, z)$ is symmetric in y, z. From (4), (11) we obtain the following explicit form for (10). Write

$$(12) \quad \begin{aligned} p_n(y, z) &\equiv \sum_{j=0}^{n} y_j z_{n-j}, \quad 0 \leqq n \leqq \omega-1; \\ p_n(y, z) &\equiv \sum_{j=0}^{2\omega-n-1} y_{n-\omega+j} z_{\omega-j}, \quad \omega \leqq n \leqq 2\omega-2. \end{aligned}$$

Then

$$(13) \quad B_k(y, z) = \sum_{n=0}^{2\omega-2} p_n(y, z) A_k(n).$$

The foregoing can be iterated to any number of one-rowed matrices $y, z, \cdots, u, v$ of order ω. Write as definitions of the functions on the left, (see also (24)),

$$(14)\qquad B_k(y, z, \cdots, u, v) \equiv \sum_{j=0}^{\omega-1} v_j\, B_k(j; y, z, \cdots, u),$$

$$(15)\qquad B_k(n; B(y, z, \cdots, u, v)) \equiv \sum_{j=0}^{\omega-1} B_j(y, z, \cdots, u, v)\, A_k(n+j),$$
$$(k = 0, 1, \cdots, \omega-1;\ n = 0, 1, \cdots).$$

Then from (3), (15) we have

$$(16)\qquad \sum_{j=0}^{\omega} a_j\, B_k(n+j; B(y, z, \cdots, u, v)) = 0,$$
$$(k = 0, 1, \cdots, \omega-1;\ n = 0, 1, \cdots).$$

That is, $B_k(n; B(y, z, \cdots, u, v))$ $(k = 0, 1, \cdots, \omega-1)$ are ω linearly independent (obviously) solutions of (5); the initial values are given by $n = 0, 1, \cdots, \omega-1$. The explicit form of (14) when more than 2 arguments (matrices) $y, z, \cdots$ occur, corresponding to (13), is too complicated to be of interest, and it will not be needed. Both (14), (15) are symmetric in $y, z, \cdots, u, v$. From (6), (16) we have

$$(17)\qquad B_k(n; B(y, z, \cdots, u, v)) = \sum_{j=0}^{\omega-1} B_k(j; B(y, z, \cdots, u, v))\, A_j(n),$$
$$(k = 0, 1, \cdots, \omega-1;\ n = 0, 1, \cdots),$$

the generalization to any number of arguments of (8).

4. **Generalized circulants.** With $y, z, \cdots, u, v$ as in § 3, write

$$(18)\qquad Y \equiv \sum_{j=0}^{\omega-1} y_j\, \Omega^j, \cdots, \qquad V \equiv \sum_{j=0}^{\omega-1} v_j\, \Omega^j.$$

Then we have, for example, where j is an integer $\geqq 0$,

$$\Omega^j\, Y = \sum_{i=0}^{\omega-1} B_i(j; y)\, \Omega^i,$$

by (2), (7). In this put $j = 0, 1, \cdots, \omega-1$, and get ω homogeneous linear equations in $1, \Omega, \cdots, \Omega^{\omega-1}$ from which these can be eliminated. The result is the characteristic equation for Y, namely

$$\begin{vmatrix} B_0(0; y) - Y & B_1(0; y) & \cdots & B_{\omega-1}(0; y) \\ B_0(1; y) & B_1(1; y) - Y & \cdots & B_{\omega-1}(1; y) \\ \cdots & \cdots & \cdots & \cdots \\ B_0(\omega-1; y) & B_1(\omega-1; y) & \cdots & B_{\omega-1}(\omega-1; y) - Y \end{vmatrix} = 0.$$

Denote the absolute term of this equation, obtained by suppressing Y in the determinant, by $N(Y)$. Then, by considering the product of the ω roots we see that

$$(19)\quad N(Y) \equiv \begin{vmatrix} B_0(0;y) & B_1(0;y) & \cdots & B_{\omega-1}(0;y) \\ B_0(1;y) & B_1(1;y) & \cdots & B_{\omega-1}(1;y) \\ \cdots & \cdots & \cdots & \cdots \\ B_0(\omega-1;y) & B_1(\omega-1;y) & \cdots & B_{\omega-1}(\omega-1;y) \end{vmatrix}$$

has the value

$$(20)\qquad N(Y) = \prod_{j=0}^{\omega-1} (y_0 + y_1\,\Omega_j + \cdots + y_{\omega-1}\,\Omega_j^{\omega-1}),$$

where $\Omega_j\,(j = 0, 1, \cdots, \omega-1)$ are the ω roots of (1).

The *generalized circulant* $N(y)$ in y of order ω degenerates to the ordinary circulant in $y_0, y_1, \cdots, y_{\omega-1}$ when in (1) we take $a_\omega = -a_0 = 1$, $a_j = 0$, $j \neq \omega, 0$.

For this special choice of the coefficients in (1), all subsequent formulas concerning generalized circulants $N(Y), N(Z), \cdots$ become the corresponding formulas for ordinary circulants.

From (20) it follows that the determinant (19) is a decomposable form of degree ω in the ω indeterminates $y_0, y_1, \cdots, y_{\omega-1}$.

The product of any number of generalized circulants of the same order ω is a generalized circulant of order ω. To obtain the explicit form of the product in terms of polynomials already defined, we note from (2), (7), (18) that

$$YZ = \sum_{k=0}^{\omega-1} B_k(y, z)\,\Omega^k.$$

Hence, by complete induction from (14), (15), we get

$$(21)\qquad YZ\cdots UV = \sum_{k=0}^{\omega-1} B_k(y, z, \cdots, u, v)\,\Omega^k;$$

and therefore, since by (20),

$$(22)\qquad N(YZ\cdots UV) = N(Y)\,N(Z)\cdots N(U)\,N(V),$$

we have

$$N(Y)\;\; B(Z)\;\cdots\; N(U)\;\; N(V)$$

$$(23)\qquad = \begin{vmatrix} B_0(0;*) & B_1(0;*) & \cdots & B_{\omega-1}(0;*) \\ B_0(1;*) & B_1(1;*) & \cdots & B_{\omega-1}(1;*) \\ \cdots & \cdots & \cdots & \cdots \\ B_0(\omega-1;*) & B_1(\omega-1;*) & \cdots & B_{\omega-1}(\omega-1;*) \end{vmatrix},$$

in which $B_k(j;*) \equiv B_k(j; B(y, z, \cdots, u, v))$, on comparing (21), (23) with Y as in (18), and (19). To include (19) in (23) we define

$$(24)\qquad B_k(y) \equiv y_k \qquad (k = 0, 1, \cdots, \omega-1),$$

so that, by (7), (15), $B_k(j; B(y)) = B_k(j; y)$.

The determinant form in (23) is the special case $n = 0$ of

$$N_n(YZ\cdots UV)$$

$$(25)\quad \equiv \begin{vmatrix} B_0(n;*) & B_1(n;*) & \cdots & B_{\omega-1}(n;*) \\ B_0(n+1;*) & B_1(n+1;*) & \cdots & B_{\omega-1}(n+1;*) \\ \cdots & \cdots & \cdots & \cdots \\ B_0(n+\omega-1;*) & B_1(n+\omega-1;*) & \cdots & B_{\omega-1}(n+\omega-1;*) \end{vmatrix}.$$

Multiply the rows of the last by $a_0, a_1, \cdots, a_{\omega-1}$ respectively; add all resulting rows, except the first, to the first and apply to the new first row thus obtained the fact that $B_k(n;*)$ is a solution of (5). Transpose the reduced first row to the last. Then it is seen that

$$a_0\, N_n(YZ\cdots UV) = (-1)^{\omega}\, a_\omega\, N_{n+1}(YZ\cdots UV),$$

where n is any integer $\geqq 0$. (If (1) be irreducible in the rational domain it is easy to extend all previous definitions and theorems so that the last holds also, in a slightly modified form, for $n<0$; but as the results are less interesting when applied to what is to follow, we have not imposed the restriction of irreducibility on (1)). In the last replace n by $0, 1, \cdots, n-1$ and multiply corresponding members of the resulting equations. Then

$$(26)\qquad N_n(YZ\cdots UV) = (-1)^{n\omega}\,(a_0/a_\omega)^n\, N_0(YZ\cdots UV),$$

which expresses $N_n(YZ\cdots UV)$ in terms of $N(YZ\cdots UV) \equiv N_0(YZ\cdots UV)$. Thus $N_n(YZ\cdots UV)$ has a kind of periodicity with respect to n. If $a_0 = a_\omega$ and ω is even, the periodicity is precise, and the period is 1.

From (26) it is seen that the necessary and sufficient condition that

$$N_n(Y)\, N_n(Z)\cdots N_n(U)\, N_n(V) = N_n(YZ\cdots UV)$$

is that

$$(-1)^{n\omega(s-1)}\,(a_0/a_\omega)^{n(s-1)} = 1,$$

where s is the number of letters $Y, Z, \cdots, U, V$. The condition is satisfied for $s = 1$ and n an arbitrary integer $\geqq 0$, or for $n = 0$ and s an arbitrary integer >0. The latter gives (22), (23); the former (20) with $B_k(j;y)$ replaced by $B_k(n+j;y)$ $(k = 0, 1, \cdots, \omega-1)$.

5. **The irregular field of extended numerical functions.** If $f(1)\neq 0$, and if $f(x)$ takes a single finite value for each integral value >0 of x, we call $f(x)$ a *numerical function* of x. If the value of $f(0)$ is not otherwise defined, we take $f(0) = 0$ by convention. Thus, on this definition, the number of divisors of 0 is 0 (not ∞, as it is sometimes useful to assert). If $g(x)$ is a numerical function of x such that $g(1) = 1$

and $g(mn) = g(m)\,g(n)$ whenever m, n are coprime integers >0, $g(x)$ is called *factorable*. Unless so stated, a numerical function will not be restricted to be factorable.

An irregular field differs from a field only in the exclusion of an infinity of elements as divisors instead of the excluded unique zero as a divisor in a field. Elements which are excluded as divisors are called *irregular*; all others *regular*. Irregular fields were fully discussed in a previous paper.* We shall now interpret the four rational operations so that, with respect to them, the set of all numerical functions, in an extended sense, is an irregular field.

If $f(x), g(x)$ are numerical functions such that $f(n) = g(n)$ for $n = 1, 2, 3, \cdots$ (all rational integers $n > 0$), we say that f, g are *equal*, and write $f = g$. Conversely, $f = g$ implies $f(n) = g(n)$ $(n = 1, 2, \cdots)$.

Consider now two resolutions of n into a product of s integral factors each >0, say $n = n_1 n_2 \cdots n_s = n_1' n_2' \cdots n_s'$. These resolutions are said to be *the same* if and only if $n_i = n_i'$ $(i = 1, \cdots, s)$. Let the summation $\sum$ refer to all distinct resolutions of n into a product of s factors as above. Then we shall write

$$\sum f_1(n_1) f_2(n_2) \cdots f_s(n_s) = f_1 f_2 \cdots f_s,$$

where $f_i(x)$ $(i = 1, \cdots, s)$ are any numerical functions, and call $f_1 f_2 \cdots f_s$ the *product* of $f_1, f_2, \cdots, f_s$. As such products will occur only in equalities (as above defined), it is unnecessary to indicate the argument n of $f_1 f_2 \cdots f_s$. If $f_1 = f_2 = \cdots = f_s$ we write $f_1 f_2 \cdots f_s \equiv f_1^s \equiv f_2^s \equiv \cdots \equiv f_s^s$. The multiplication indicated in $f_1 f_2 \cdots f_s$ is associative and commutative; we shall define presently the sense in which it is also distributive.

In applications it is frequently necessary to distinguish between the product $f_1 f_2 \cdots f_s$ and the *absolute product* $f \equiv |f_1 f_2 \cdots f_s|$, the latter being defined by

$$f(n) \equiv f_1(n) f_2(n) \cdots f_s(n) \qquad (n = 1, 2, \cdots).$$

Note that both $f_1 f_2 \cdots f_s$ and $|f_1 f_2 \cdots f_s|$ are numerical functions.

Now let α_i $(i = 1, 2, \cdots)$ be any elements of a field Φ whose zero, unity are denoted by 0, 1 as usual, and let $f_i(x)$ $(i = 1, 2, \cdots)$ be any numerical functions of x. Precisely as in the postulates for an algebra over a field† we define $\alpha_1 f_1(x) + \cdots + \alpha_s f_s(x)$. If

$$\psi_n(n) = \alpha_1 f_1(n) + \cdots + \alpha_s f_s(n) \qquad (n = 1, 2, \cdots),$$

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* *Annals of Mathematics*, vol. 27, 1926, pp. 511–536.

† Wedderburn, *Proc. Lond. Math. Soc.*, vol. 6 (1908), p. 79; Dickson, Algebras und their Arithmetics, 1st ed., p. 9.

we shall write $\psi = \alpha_1 f_1 + \cdots + \alpha_s f_s$, and say that ψ is *regular* or *irregular* according as $\psi(1) \neq 0$ or $\psi(1) = 0$.

Adjoin to the set of all numerical functions the set of all linear functions of the kind ψ just defined, where $s = 1, 2, \cdots$, the $f_i (i = 1, 2, \cdots)$ range over all numerical functions, and the α_i $(i = 1, 2, \cdots)$ over all elements of Φ. Then, f being any numerical function, ff_i (product as already defined) is a numerical function, and hence $\alpha_1 ff_1 + \cdots + \alpha_s ff_s$ is in the extended set. With ψ as above, we shall write $f\psi \equiv \alpha_1 ff_1 + \cdots + \alpha_s ff_s$, $\psi f \equiv \alpha_1 f_1 f + \cdots + \alpha_s f_s f$, and we have $f\psi = \psi f$. The extended set is therefore a ring over Φ.

If $\varphi(n)$ takes a single finite value of the type $\alpha f(n) + \beta g(n) + \cdots + \gamma h(n)$, where $\alpha, \beta, \cdots, \gamma$ are in Φ, and $f(x), g(x), \cdots, h(x)$ are numerical functions, for each integer $n > 0$, and if further $\varphi(0) = 0$ by convention, we shall call $\varphi(x)$ or φ an *extended numerical function* in Φ. Equality is defined as before. The set of all extended numerical functions is the above ring.

Let the α, β be elements of Φ, and the f, g numerical functions. The *product* $\varphi\psi$ of

$$\varphi \equiv \alpha_1 f_1 + \cdots + \alpha_s f_s, \qquad \psi \equiv \beta_1 g_1 + \cdots + \beta_t g_t$$

of the extended numerical functions φ, ψ in Φ is defined by $\chi = \varphi\psi$, where

$$\chi(n) = \sum [\alpha_1 f_1(d) + \cdots + \alpha_s f_s(d)]\,[\beta_1 g_1(\delta) + \cdots + \beta_t g_t(\delta)] \qquad (n = 1, 2, \cdots),$$

the summation extending to all pairs (d, δ) of integers $d, \delta > 0$ such that $n = d\delta$. Such multiplication is associative and commutative, as is readily seen from the corresponding sums expressing these laws in the present interpretation; it is also distributive, since evidently

$$\chi(n) = \sum [\alpha_1 f_1(d)\, \psi(\delta) + \cdots + \alpha_s f_s(d)\, \psi(\delta)].$$

The unique zero element in the ring is $\zeta(n)$, where $\zeta(n) = 0$ for all integers $n > 0$.

The unique unity is the numerical function $\varepsilon(1) = 1$, $\varepsilon(n) = 0$ for all integers $n > 1$, for evidently $\varepsilon\varphi = \varphi$, where φ is any extended numerical function in Φ. If φ be restricted to be regular, there exists a unique extended numerical function φ' in Φ such that $\varphi\varphi' = \varepsilon$. We call φ' the *reciprocal* of φ, and write $\varphi' = \varepsilon/\varphi$ or $\frac{\varepsilon}{\varphi}$ as convenient. The product $\varphi'\psi$ will be written ψ/φ or $\frac{\psi}{\varphi}$. The set of all extended numerical functions

in Φ is therefore closed under division as just defined, and ϵ is its own reciprocal; also $\chi/\chi = \epsilon$, where χ is regular.

Combining results we have the comprehensive theorem that *the set of all extended numerical functions in Φ is an irregular field.**

When Φ is the field of rational numbers, we shall call the foregoing irregular field simply the *functional field* for brevity.

6. Decomposable forms in the functional field. Let

$$y_j, z_j, \cdots, u_j, v_j \qquad (j = 0, 1, \cdots, \omega - 1)$$

be any numerical functions. Then it is clear that *the whole of* § 4 *regarding generalized circulants remains true under this interpretation of the* $y_j, \cdots, v_j$. For, in § 4 the $y_j, \cdots, v_j$ were any elements of an abstract field; the functional field defined in § 5 is abstractly identical, up to irregular elements, with an abstract field, and hence, as no divisions other than possibly by rational numbers occur in § 4, the italicized statement follows. Before specializing this extremely general theorem in an interesting direction, we give a simple example.

7. Example of § 6. When Ω in § 2 is an imaginary cube root of unity, the formulas are short enough to be written in full. A few special cases of them follow; the general result is not given.

Let $n = p^a q^b \cdots t^c$ be the resolution of the integer $n > 1$ into powers of distinct primes $p, q, \cdots, t(>1)$, and write

$$\tau(n) \equiv a + b + \cdots + c, \qquad \tau(1) = 0.$$

Then, r, s being any integers > 0, we have

$$\tau(rs) = \tau(r) + \tau(s). \tag{27}$$

Separate the rational integers > 0 into the classes $C_j (j = 0, 1, 2)$, the class C_j containing all those integers $n_j > 0$, and only those, for which $\tau(n_j) \equiv j \bmod 3$. Let $f(x)$ be any numerical function. If in a given context f takes only arguments in C_j, we indicate this by writing f_j. Otherwise stated, $f_j(n) = 0$ or $f(n)$ according as n is not, or is, in C_j.

Let $f(x), g(x)$ be any numerical functions. Define the h_j by

$$\begin{aligned} h_0 &= f_0 g_0 + f_1 g_2 + f_2 g_1, \\ h_1 &= f_0 g_1 + f_1 g_0 + f_2 g_2, \\ h_2 &= f_0 g_2 + f_1 g_1 + f_2 g_0. \end{aligned}$$

* The existence and uniqueness of the reciprocal, and hence of division, were established and amply illustrated in several previous papers, of which we may cite: *Tôhoku Math. Journ.*, vol. 17 (1920). pp. 221–231, in which an explicit form for the reciprocal of any regular Φ is given.

Then it is clear that the notation h_j is consistent with the convention regarding suffixes, for the suffixes of the f, g on the right restrict h_j to take only arguments in C_j, by (27). In full, for example,

$$h_0(n) = \sum [f(n_0)\, g(n_0') + f(n_1)\, g(n_2') + g(n_1)\, f(n_2')],$$

or all integers $n > 0$, where the summation refers to all resolutions of n into pairs of conjugate divisors n_1, n_2' and n_0, n_0', so that $n = n_0 n_0' = n_1 n_2'$, the suffixes indicating the classes.

Now write $f_i f_j f_k \equiv f_{ijk}$, and similarly for g, h, θ, where $\theta = gh$. Then, if α is an imaginary cube root of unity, we have the decompositions in the field defined by α,

$$\begin{aligned}
&\psi_{000} + \psi_{111} + \psi_{222} - 3\,\psi_{012} \\
&\quad = (\psi_0 + \psi_1 + \psi_2)(\psi_0 + \alpha\psi_1 + \alpha^2\psi_2)(\psi_0 + \alpha^2\psi_1 + \alpha\psi_2), \\
&\quad = \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 \\ \psi_2 & \psi_0 & \psi_1 \\ \psi_1 & \psi_2 & \psi_0 \end{vmatrix},
\end{aligned}$$

where ψ denotes any one of f, g, h, θ; we have also the identities

$$\begin{aligned}
&(f_{000} + f_{111} + f_{222} - 3 f_{012})(g_{000} + g_{111} + g_{222} - 3 g_{012}) \\
&= h_{000} + h_{111} + h_{222} - 3 h_{012} = \theta_{000} + \theta_{111} + \theta_{222} - 3\,\theta_{012}.
\end{aligned}$$

As a special example, let $\chi(n) = n$ for all integers $n > 0$; let $\mu(n)$ be Möbius' function, and $\varphi(n)$ the totient of n. Then $\varphi = \mu\chi$. Write

$$\begin{aligned}
\lambda_0 &= \mu_0\chi_0 + \mu_1\chi_2 + \mu_2\chi_1, \\
\lambda_1 &= \mu_0\chi_1 + \mu_1\chi_0 + \mu_2\chi_2, \\
\lambda_2 &= \mu_0\chi_2 + \mu_1\chi_1 + \mu_2\chi_0.
\end{aligned}$$

Then

$$\begin{aligned}
&(\mu_{000} + \mu_{111} + \mu_{222} - 3\,\mu_{012})(\chi_{000} + \chi_{111} + \chi_{222} - 3\,\chi_{012}) \\
&= \varphi_{000} + \varphi_{111} + \varphi_{222} - 3\,\varphi_{012} = \lambda_{000} + \lambda_{111} + \lambda_{222} - 3\,\lambda_{012}.
\end{aligned}$$

This example is one of the simplest of the general situation described next. Note that the relations $\theta = gh$, $\varphi = \mu\chi$, which are arithmetical in character rather than algebraic, in that they are consequences of the unique factorization law in rational arithmetic, yield the forms $\theta_{000} + \cdots$, $\varphi_{000} + \cdots$, for which there are no significant equivalents in § 4.

8. **Further decompositions in the functional field.** Let f, g, h be any elements (extended numerical functions as in § 5, end) of the functional field such that $fg = h$. Then if ϱ is any constant number, write for the moment

$$\varrho^{\tau(n)} f(n) \equiv f''(n) \qquad (n = 1, 2, \cdots)$$

so that f'' is defined as an element of the functional field. Similarly for g'', h''. *Each of the relations* $fg = h$, $f''g'' = h''$, *implies the other.*

For, $fg = h$ is equivalent by (27) to

$$\sum \varrho^{\tau(d)} f(d)\, \varrho^{\tau(\delta)} f(\delta) = \varrho^{\tau(n)} h(n),$$

the $\sum$ extending to all integers d, $\delta > 0$ such that $n = d\delta$. But this is $f''g'' = h''$.

We now consider a transformation of all the elements of the functional field, based upon the above, which will give the complete generalization of the example in § 7. Let k be any element of the functional field, and define k_j, also an element of the field, by

$$A_j(\tau(n))\, k(n) \equiv k_j(n) \qquad (j = 0, 1, \cdots, \omega - 1;\ n = 0, 1, \cdots), \tag{28}$$

the $A_j(n)$ $(n = 0, 1, \cdots)$ being as in § 2. If now A' be defined by

$$A_j(\tau(n)) = A'_j(n) \quad (j = 0, 1, \cdots, \omega - 1;\ n = 0, 1, \cdots), \tag{29}$$

we have, by (28),

$$k_j = |A'k| \qquad (j = 0, 1, \cdots, \omega - 1), \tag{30}$$

the significance of the absolute product $|A_j k|$ being as already defined in § 5. It is important to observe that there is no obvious relation between $|fg|^s$, the sth power of the absolute product of any two elements f, g of the field, and $|f|^s$, $|g|^s$ or f^s, g^s, as is seen from the definitions.

Now let $w, y, z, \cdots, u, v$ be any elements of the functional field between which there is the relation

$$w = yz \cdots uv, \tag{31}$$

the indicated multiplication on the right having necessarily the meaning ascribed to multiplication in the field as defined in § 5. Write

$$w' \equiv (w_0, w_1, \cdots, w_{\omega-1}), \cdots, \qquad v' \equiv (v_0, v_1, \cdots, v_{\omega-1}),$$

where the suffixes $0, 1, \cdots, \omega - 1$ have the meaning defined in (30) for the functions $w, y, z, \cdots, u, v$ respectively. *Then the form*

$$\begin{vmatrix} B_0(0; w') & B_1(0; w') & \cdots & B_{\omega-1}(0; w') \\ B_0(1; w') & B_1(1; w') & \cdots & B_{\omega-1}(1; w') \\ \cdots & \cdots & \cdots & \cdots \\ B_0(\omega-1; w') & B_1(\omega-1; w') & \cdots & B_{\omega-1}(\omega-1; w') \end{vmatrix}$$

is the product in the functional field of all the similar forms obtained by replacing in the above w' by $y', z', \cdots, u', v'$, and hence, by the theorem expressed in (23), *is equal to the generalized circulant in* (23), *with* $B_k(j; *)$ *therein replaced by* $B_k(j; B(y', z', \cdots, u', v'))$. *All algebraic operations involved in the expansions and multiplications of the determinants are, by* §§ 5, 6, *necessarily to be interpreted in the functional field.*

It is a simple exercise in the definitions to see that (31) makes this indeed obvious. The examples in § 7 are special cases of this when the equation (1) is $\Omega^3 - 1 = 0$. The theorem of course adds nothing to the comprehensive isomorphism in §§ 5, 6, but it is of independent interest.

Instead of proving such theorems, or any others in the theory, directly, it is simpler to use the method of generators explained in previous papers.* If the given functions are factorable, the variable z in the generator is replaced by Ωz, and the function generated by the ordinary algebraic product of all such transformed generators, as Ω runs through all the roots of (1), is considered as a new generator. By observing the elementary algebraic relations between several such generators, transformed and untransformed, everything follows at a glance. The indicated symbolic method also shows, as is evident from the foregoing developments, that the relations obtained in the functional field are equivalent to identities of precisely the same kind between Dirichlet series. We chose the above method here in order to avoid all matrices containing an infinity of elements, as the whole matter is essentially one of common algebra.

* *University of Washington Publications in Science*, vol. 1, No. 1 (1915), pp. 1–44; *Trans. Amer. Math. Soc.*, vol. 25 (1923), pp. 135–154.

ON CERTAIN OSCILLATION THEOREMS.*

By H. Poritsky.†

In the following we prove several theorems concerning harmonic polynomials and related functions which, we believe, are new. They deal with the distance from a point within which these functions must change sign (hence, the title) and with the extremal values at a certain distance. We then give a more general oscillation theorem that contains some of the former results as a special case.

Consider first a homogeneous harmonic polynomial p_n of degree n. Let us confine our attention to a unit sphere S with center at the origin. We shall prove

Theorem 1. *If P is any point of the unit sphere S at which the spherical harmonic p_n does not vanish and θ is the (spherical) distance from P, then p_n must change sign in the region $\theta \leqq \alpha_n$, where α_n is the smallest positive root of $P_n(\cos\theta)$, $P_n(x)$ denoting the nth Legendre polynomial of x; if the change of sign does not occur for $\theta < \alpha_n$ then $p_n = \text{const.}\, P_n(\cos\theta)$.*

Proof. Introduce spherical polar coördinates along S consisting of the above θ (the co-latitude) and a longitude φ. Resolve p_n into a Fourier series in φ

$$p_n = \sum_{i=0}^{n} (A_i \cos i\varphi + B_i \sin i\varphi) P_{n,i}(\cos\theta); \tag{1}$$

here $P_{n,i}$ represent the associated spherical harmonics. The first term of this series $A_0 P_{n,0}(\cos\theta) = A_0 P_n(\cos\theta) = (1/2\pi)\int_0^{2\pi} p_n(\theta, \varphi)\, d\varphi$ represents the "average" of the values of p_n over a small circle a distance θ from P. A_0 is the value of p_n at P and is, by hypothesis, different from zero. For definiteness, suppose it is positive. Then for $\alpha_n < \theta < \alpha_n'$, where α_n' is the next positive root of $P_n(\cos\theta)$, the average is negative. Consequently, p_n must take on negative values immediately beyond $\theta = \alpha_n$.

Now suppose that p_n is actually positive for $\theta < \alpha_n$. It then vanishes identically in φ for $\theta = \alpha_n$, as otherwise the average for $\theta = \alpha_n$ would be positive. Hence the Fourier coefficients in (1) vanish for $\theta = \alpha_n$:

$$A_i P_{n,i}(\cos\alpha_n) = B_i P_{n,i}(\cos\alpha_n) = 0.$$

* Received September 16, 1928.

† National Research Fellow in Mathematics.

But $P_{n,i}(\cos\alpha_n) \neq 0$ for $i>0$,* hence $A_i = B_i = 0$ for $i>0$. This completes the proof.

If $M(\theta)$ denotes the maximum of the values taken on by p_n along the small circle a distance θ from P, while $m(\theta)$ denotes the minimum of such values, the above formula for the average shows that $M(\theta) \geqq A_0 P_n(\cos\theta)$, $m(\theta) \leqq A_0 P_n(\cos\theta)$. The equality signs can only hold (simultaneously) for values of θ that are roots of $P_{n,i}(\cos\theta)$, $i>0$, if (1) reduces to its first term plus the several terms for whose index i the particular value of θ is a root of $P_{n,i}(\cos\theta)$; for other values of θ the equality signs above hold only if (1) reduces to its first term.

We now consider homogeneous bi-harmonic† polynomials and shall prove for them

THEOREM 2. *If P is any point on the unit sphere S with center at the origin, and q_n is a homogeneous bi-harmonic polynomial of degree $n>3$ that does not vanish at P, then q_n may be positive on S in the region $\theta \leqq \beta_n - \varepsilon$, where θ is the spherical distance from P, β_n the smallest positive root of $P_n(\cos\theta) - \cos\theta\, P_{n-1}(\cos\theta)$, and ε an arbitrary but fixed positive number, but q_n must change sign within the region $\theta<\beta_n$ (hence in any region enclosing it).*

It is interesting to point out that, while the region about P within which a change of sign of p_n must take place is a closed region, the similar region for q_n is an open one.

Proof. As is known, we may write q_n in either of the forms

$$q_n = p_n + l\,p_{n-1}, \qquad q_n = p'_n + r^2 p'_{n-2}, \tag{2}$$

where l is an arbitrary homogeneous polynomial of the first degree $\not\equiv 0$, r is the distance from the origin, and p_i, p'_i are harmonic‡ and, consequently, homogeneous polynomials of degrees indicated by their subscripts. Utilizing the former representation we choose $l = r\cos\theta$, introduce polar coördinates θ, φ as above, and resolve q_n into a Fourier series in φ along S:

* This follows readily from

$$P_{n,i}(x) = \text{const.}\,(x^2-1)^{i/2}\, d^i[P_n(x)]/dx^i$$

(see Wangerin, Theorie des Potentials und der Kugelfunktionen, II, p. 53 (2)). The roots of $P_{n,i}(x)$, except for $x = \pm 1$, are thus the same as those of $d^i P_n(x)/dx^i$. But $P_n(x)$ is a polynomial of degree n and has n roots lying -1 and 1. Hence, $P_{n,i}(x)$ for $i>0$ has $n-i$ real roots $\neq \pm 1$ that lie between those of $P_{n,i-1}(x)$. In particular, the largest root, different from ± 1, of $P_{n,i}(x)$, $i>0$, is less than the largest root of $P_n(x)$.

† That is, satisfying the equation $\nabla^4 = 0$.

‡ See Almansi, Sull' integrazione dell' Equazione differenziale $\Delta^{2n} = 0$, Annali di Mat. (3), vol. 2 (1899), §§ 1-3.

$$(3) \qquad q_n = \sum_{i=0}^{n} \cos i(\varphi - \varphi_i) \left[C_i P_{n,i}(\cos\theta) + D_i \cos\theta \, P_{n-1,i}(\cos\theta)\right].$$

The average over the circles $\theta =$ constant is now given by the first term in the summation (3)

$$C_0 P_n(\cos\theta) + D_0 \cos\theta \, P_{n-1}(\cos\theta) = (1/2\pi) \int_0^{2\pi} q_n(\theta, \varphi) \, d\varphi.$$

We shall prove that this average must change sign in the interval $0 < \theta < \beta_n$.

From the known oscillation theorems of Legendre polynomials one readily sees that the curves $y = P_n(x)$, $y = xP_{n-1}(x)$ cut each other in n distinct points whose x-coördinates are

$$(4) \qquad x = -1, \; -\cos\beta_n, \cdots, \cos\beta_n, \; 1$$

and whose y-coördinates for $x > 0$ are alternately positive and negative. To study the changes of sign of $C_0 P_n(x) + D_0 x P_{n-1}(x)$ it is sufficient to examine the case $C_0 + D_0 = 1$, since any other case is reducible to it by dividing q_n by $C_0 + D_0$, a number not equal to zero because it is the value of q_n at P. With this restriction upon the coefficients, however, the curve $y = C_0 P_n(x) + D_0 x P_{n-1}(x)$ passes through the points of intersection of $y = P_n(x)$, $y = xP_{n-1}(x)$, and, consequently, for $n > 3$, y must change sign between $x = \cos\beta_n$ and $x = 1$. q_n, therefore, must change sign in the region $0 < \theta < \beta_n$.

At both end points of the interval $\cos\beta_n \leqq x \leqq 1$ the function $xP_{n-1}(x)/P_n(x)$ is equal to 1, decreasing from 1 as x increases from the lower end point and decreasing to 1 as x approaches the upper end point; elsewhere in the interval it is different from 1. Hence values less than and close to unity are taken on by this function in the interval in question only near $x = \cos\beta_n$. If, therefore, we choose $q_n = C_0 P_n(\cos\theta) + D_0 \cos\theta \, P_{n-1}(\cos\theta)$ where $-C_0/D_0 = 1 - \varepsilon$, ε positive and sufficiently small, the change of sign of q_n will occur only as close to β_n as desired.

Had we used the other possible representation of bi-harmonic polynomials by means of harmonic ones given in (2), the average would have appeared in the form $CP_n(\cos\theta) + DP_{n-2}(\cos\theta)$ and we would have been led to investigate the intersections of the curves $y = P_n(x)$, $y = P_{n-2}(x)$. Their identity with the former intersections follows at once if the recurrence formula $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$* is written in the form $(2n-1)[P_n(x) - xP_{n-1}(x)] = (n-1)[P_n(x) - P_{n-2}(x)]$.† The

* Wangerin, loc. cit., p. 33, (31).

† For $n = 3$ q_n, being odd, must change sign in the region $\theta \leqq \pi/2$ and this is the most that may be asserted for q_3. For $n = 2$ q_n may be everywhere of one sign: we need only make the numerical value of C/D small enough in $CP_2(\cos\theta) + DP_0(\cos\theta)$.

expression for the average of q_n along the small circles $\theta = \text{const.}$ that was used above may be utilized to obtain inequalities involving $M(\theta)$, $m(\theta)$, the maximum and minimum of q_n along $\theta = \text{const.}$ To fix the average $a(\theta)$ we may assign the values 1 and a_1 to $a(0)$ aud $a(\theta_1)$ respectively. We then get

$$\begin{vmatrix} 1 & 1 & 1 \\ P_n(x_1) & P_{n-2}(x_1) & a_1 \\ P_n(x_2) & P_{n-2}(x_2) & a_2 \end{vmatrix} = 0,$$

where $\theta_2 \neq \theta_1$, $x_1 = \cos\theta_1$, $x_2 = \cos\theta_2$, $a_2 = a(\theta_2)$. If the coefficients of a_1, a_2 in the expansion of the left hand member above are both positive, we may replace them by the possibly larger quantities $M_1 = M(\theta_1)$, $M_2 = M(\theta_2)$ and, changing the equality sign to $>$, obtain a correct inequality. Such will be the case if $P_n(x_1) - P_{n-2}(x_1) < 0$, $P_n(x_2) - P_{n-2}(x_2) > 0$, that is, if x_1, x_2 lie in certain alternate intervals into which the interval $(-1, 1)$ is separated by the points of the sequence (4). In other cases similar inequalities may be asserted about (M_1, m_2), (m_1, M_2), or (m_1, m_2).

We now turn to homogeneous tri-harmonic polynomials of degree n, r_n. Their average, for $n > 3$, involves linear combinations of three functions and can be written in the form

$$C_1 P_n(\cos\theta) + C_2 P_{n-2}(\cos\theta) + C_3 P_{n-4}(\cos\theta).$$

More generally, let us consider three continuous functions of θ $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ defined in an interval $0 \leqq \theta \leqq \theta''$ and their linear combinations

$$C_1 f_1(\theta) + C_2 f_2(\theta) + C_3 f_3(\theta). \tag{5}$$

In a Euclidean space with rectangular Cartesian coördinates x_1, x_2, x_3 plot the curve $x_i = f_i(\theta)$, $i = 1, 2, 3$, and denote by $C(\theta')$ the curve segment obtained for $0 \leqq \theta \leqq \theta'$, $\theta' \leqq \theta''$, and by $B(\theta')$ the smallest convex region that contains $C(\theta')$.* We state

THEOREM 3. A. *A necessary and sufficient condition that C_i exist for which* (5) *does not vanish for $0 \leqq \theta \leqq \theta'$ is that the origin lie outside of $B(\theta')$.*

B. *A necessary and sufficient condition for the existence of C_i, not all zero, for which* (5) *does not change sign for $0 \leqq \theta \leqq \theta'$† is that the origin lie outside or on the boundary of $B(\theta')$.*

* In connection with such regions see Carathéodory, Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen, Rend. Circ. mat. Palermo, vol. 32 (1911), pp. 195 ff.

† That is, is entirely non-negative or entirely non-positive in the interval $(0, \theta')$.

Proof. All the points outside $B(\theta')$ may be characterized by the existence of planes through them that contain no points of $C(\theta')$ while all the points on the boundary of $B(\theta')$ may similarly be characterized by the existence of planes through them that do not separate the points of $C(\theta')$ but contain some of them (see Carathéodory, loc. cit., 6–8). If, therefore, the origin is outside of $B(\theta')$ and we pass a plane through it that contains no points of $C(\theta')$ and whose equation is $\bar{C}_1 x_1 + \bar{C}_2 x_2 + \bar{C}_3 x_3 = 0$, then (5) will never vanish for all θ in question if C_i are put equal to $\bar{C}_i$. Conversely, if (5) does not vanish for $0 \leqq \theta \leqq \theta'$ when $C_i = \bar{C}_i$, then $\bar{C}_i$ do not all vanish and $\bar{C}_1 x_1 + \bar{C}_2 x_2 + \bar{C}_3 x_3 = 0$ is the equation of a plane through the origin that contains none of the points of $C(\theta')$. Therefore the origin lies outside of $B(\theta')$. Part B is proved in a similar manner.

Another way of characterizing $B(\theta')$ is as the locus of centers of mass of four or less positive masses at different points of $C(\theta')$.* The coördinates of points of $B(\theta')$ may thus be expressed in the form

$$x_i = m_1 f_i(\theta_1) + m_2 f_i(\theta_2) + m_3 f_i(\theta_3) + m_4 f_i(\theta_4), \quad i = 1, 2, 3,$$
$$0 \leqq \theta_j \leqq \theta', \quad 0 \leqq m_j \text{ for } j = 1, 2, 3, 4, \quad m_1 + m_2 + m_3 + m_4 = 1.$$

Hence the distance from the origin to points of $B(\theta')$ takes on a minimum value δ somewhere. Moreover, this minimum δ varies continuously with θ' and is obviously a monotonically decreasing function of the latter. It is therefore either positive for all θ in the interval $(0, \theta'')$, or else it vanishes somewhere and then there exists a (smallest) $\bar{\theta}$ such that $\delta(\theta') > 0$ for $0 \leqq \theta' < \bar{\theta}$, $\delta(\theta') = 0$ for $\bar{\theta} \leqq \theta' \leqq \theta''$. In the former case C_i exist that render (5) positive for all values of θ in question; in the latter case (5) may be positive in any closed interval starting at 0 and of length less than $\bar{\theta}$ but must vanish in any such interval of length equal to or greater than $\bar{\theta}$.

We now suppose $\bar{\theta} > 0$ and consider the interval $0 \leqq \theta \leqq \bar{\theta}$. The origin is a point of $B(\bar{\theta})$, since $\delta(\bar{\theta}) = 0$. We shall show that it must lie on the boundary of $B(\bar{\theta})$. Let $\theta_1, \theta_2, \cdots$ be an infinite sequence of positive increasing numbers approaching $\bar{\theta}$. For each θ_i there exists a plane through the origin that does not contain any points of $C(\theta_i)$. Hence there exists a plane through the origin that does not separate the points of $C(\bar{\theta})$.† The

* Carathéodory, loc. cit., 9. Three at most are really sufficient in the present instance.

† If $C_{i,1} x_1 + C_{i,2} x_2 + C_{i,3} x_3 = 0$ are the equations of the former planes such that $C_{i,1}^2 + C_{i,2}^2 + C_{i,3}^2 = 1$, $C_{i,3} f_1(\theta) + C_{i,2} f_2(\theta) + C_{i,3} f_3(\theta) > 0$ for $0 \leqq \theta \leqq \theta_i$, there exist limiting coefficients $\bar{C}_1, \bar{C}_2, \bar{C}_3$, $\bar{C}_1^2 + \bar{C}_2^2 + \bar{C}_3^2 = 1$, such that for a proper sequence of increasing integers $i_1, i_2, \cdots$, $\lim_{n\to\infty} C_{i_n,1} = \bar{C}_1$, $\lim_{n\to\infty} C_{i_n,2} = \bar{C}_2$, $\lim_{n\to\infty} C_{i_n,3} = \bar{C}_3$. Consequently, $\bar{C}_1 f_1(\theta) + \bar{C}_2 f_2(\theta) + \bar{C}_3 f_3(\theta) \geqq 0$ for $0 \leqq \theta \leqq \bar{\theta}$ and $\bar{C}_1 x_1 + \bar{C}_2 x_2 + \bar{C}_3 x_3 = 0$ is the equation of a plane that does not separate the points of $C(\bar{\theta})$.

origin is thus a boundary point of $B(\bar{\theta})$. If, therefore, the origin be represented as a center of at most four positive point masses situated at points of $C(\bar{\theta})$, these points must be co-planar; otherwise, the origin would lie inside a tetrahedon whose vertices are on $C(\bar{\theta})$, and hence would not be on the boundary of $B(\bar{\theta})$. But the center of mass of four positive co-planar masses may always be represented as the center of mass of three or less positive masses at some of the former points. Thus the origin may be represented as the center of mass of at most three positive masses on $C(\bar{\theta})$. One of these masses must be situated precisely at the end point of the curve segment $\theta = \bar{\theta}$, for, if θ^* is the largest of the values of θ that determine the positions of the masses and $\theta^* < \bar{\theta}$, then $B(\theta^*)$ would contain the origin and we would have a contradiction since $\bar{\theta}$ is the smallest value of θ for which $B(\theta)$ contains the origin. We have thus established

THEOREM 4. *If $\bar{\theta}$ is the smallest value of θ_3 for which the equations and inequalities*

$$m_1 f_i(\theta_1) + m_2 f_i(\theta_2) + m_3 f_i(\theta_3) = 0,$$
$$i = 1, 2, 3, \quad \theta_1 < \theta_2 < \theta_3, \quad m_1 \geqq 0,\ m_2 \geqq 0,\ m_3 > 0$$

are satisfied, then there exist constants C_1, C_2, C_3 for which $C_1 f_1(\theta) + C_2 f_2(\theta) + C_3 f_3(\theta)$ is positive for $0 \leqq \theta \leqq \bar{\theta} - \varepsilon$, $\varepsilon > 0$,† but any such linear combination of $f_i(\theta)$ must vanish in $0 \leqq \theta \leqq \bar{\theta} + \varepsilon$, $\varepsilon \geqq 0$. There also exist constants C_1, C_2, C_3, not all zero, for which $C_1 f_1(\theta) + C_2 f_2(\theta) + C_3 f_3(\theta) \geqq 0$ for $0 \leqq \theta \leqq \bar{\theta}$. Conversely, if any linear combination of $f_i(\theta)$ vanishes in the interval $(0, \theta'')$ then there exists a least upper bound $\bar{\theta} < \theta''$ to the length of the intervals starting at 0 in which some linear combination will be positive; the above equations and inequalities will possess at least one solution and the smallest possible value of θ_3 will be equal to $\bar{\theta}$.‡

Returning to homogeneous tri-harmonic polynomials and their averages $C_1 P_n(\cos\theta) + C_2 P_{n-2}(\cos\theta) + C_3 P_{n-4}(\cos\theta)$, we state

† Unless otherwise stated ε is always to be taken as any *fixed* number satisfying the proper inequality.

‡ In case of three positive masses the values of C_i that render (5) non-negative for $0 \leqq \theta \leqq \bar{\theta}$ are determined except for a constant factor, being proportional to the direction cosines of the normal to the plane determined by these points. Since the distance from the points of $C(\bar{\theta})$ to that plane cannot change sign, it follows that distance must vanish to an even order at θ_2, assuming f_i differentiable to the same order. This must also happen at θ_1, unless $\theta_1 = 0$. Hence, if the interval $(0, \bar{\theta} - \varepsilon)$ throughout which (5) is positive is made to approach the interval $(0, \bar{\theta})$ while a proper one of the constants C_i is held fixed, (5) will approach a definite function that is non-negative and has in the interval $(0, \bar{\theta})$ at least several roots of the nature above specified, in addition to vanishing at $\bar{\theta}$.

If there are only two positive masses, the values of C_i that render (5) positive may be specified in terms of two linear inequalities. Finally, the case of only one positive mass is the case where the curve $C(\theta)$ passes through the origin for $\theta = \bar{\theta}$.

THEOREM 5. *If r_n is a homogeneous tri-harmonic polynomial of degree n and θ is the distance along the unit sphere S from a fixed point P on it, then r_n may be positive in the region $\theta \leqq \gamma_n - \varepsilon$, $\varepsilon > 0$, but it must vanish in the region $\theta \leqq \gamma + \varepsilon$, $\varepsilon \geqq 0$, where γ_n may be specified by means of equations and inequalities of the type given in Theorem 4 applied to the functions $P_n(\cos\theta)$, $P_{n-2}(\cos\theta)$, $P_{n-4}(\cos\theta)$. A solution of these equations and inequalities will exist for $n > 4$ with $\bar{\theta} = \gamma_n$ satisfying the inequalities $\gamma_n \leqq \alpha_n + \alpha_{n-2} + \alpha_{n-4}$, $\gamma_n < \alpha_n + \beta_{n-2}$.*

Proof. Except for the assertion of the existence of $\bar{\theta}$ and the estimate of its value the theorem is merely an application of the preceeding theorem. The existence of $\gamma_n = \bar{\theta}$ for $n > 4$ follows from the vanishing of the integral $\int_{-1}^{1} P_m(x)\,dx$ for positive integer m, in virtue of which any linear combination of the functions in question must vanish in the interval $(0, \pi)$. To prove the inequalities we proceed as follows.

Let r_n' be a new tri-harmonic polynomial whose value at all points of any circle $\theta = \text{const.}$ is equal to the average of r_n over that circle. Let P_1 be any fixed point a distance α_n away from P, θ_1 the distance from P_1. If we average r_n' over the small circles $\theta_1 = \text{const.}$ we obtain

$$r_n'' = C_2 P_{n-2}(\cos\alpha_n) P_{n-2}(\cos\theta_1) + C_3 P_{n-4}(\cos\alpha_n) P_{n-4}(\cos\theta_1).$$

By averaging this average again about a point P_2 a distance α_{n-2} away from P_1 we obtain const. $P_{n-4}(\cos\theta_2)$ where θ_2 is the distance from P_2. The last average changes sign for $\theta_2 = \alpha_{n-4}$, hence the preceeding average r_n' must change sign for $\theta_2 \leqq \alpha_{n-4}$ and therefore, certainly for $\theta_1 \leqq \alpha_{n-2} + \alpha_{n-4}$. Repeating the reasoning we arrive at the first one of the inequalities of Theorem 5. Had we stopped with the second average and applied Theorem 2 to it, we would have arrived at the second inequality.

The methods by means of which Theorem 5 was obtained could be used to establish similar results for k-harmonic polynomials. The necessary geometry of k-dimensional convex regions is available in the paper of Carathéodory cited. We could thus assert the existence of a least upper bound to the size of a spherical zone of one cap within which a k-harmonic polynomial may fail to vanish if its degree is greater than $2k$. Also, recurrence inequalities (that get, however, progressively more inaccurate) could be established on the size of these regions by the methods of successive averages. That all the upper limits to θ are less than $\pi/2$ we conclude from the vanishing of the integrals $\int_0^1 P_{2n}(x)\,dx$, $\int_0^1 x\,P_{2n+1}(x)\,dx$ for $n > 0$.

If the bi-harmonic polynomials were treated by these methods β_n would have appeared as the least value of θ' for which the smallest convex

region enclosing the curve $x_1 = P_n(\cos\theta)$, $x_2 = P_{n-2}(\cos\theta)$, $0 \leqq \theta \leqq \theta'$ contains the origin. The proof given above could be interpreted to show that this curve, starting at $x_1 = 1$, $x_2 = 1$, never crosses the line $x_1 = x_2$ till $\theta = \beta_n$.

Theorems 3 and 4 and their suggested generalizations could be applied to drawing conclusions similar to the above about the vanishing of non-homogeneous harmonic polynomials. We shall state the following result: if a harmonic polynomial $\bar{p}_n$ of degree n vanishes at the origin, then there is a least upper bound $\bar{\alpha}_n$ to the angular opening of a zone of one cap on the surface of a sphere whose center is at the origin within which $\bar{P}_n$ may be positive; $\bar{\alpha}_n$ is independent of the radius of the spherical surface; it is the least upper bound to the length of an interval starting at zero within which $C_1 P_n(\cos\theta) + \cdots + C_n P_1(\cos\theta)$ is positive for some C_i.

To give another application of the above methods consider two-dimensional solutions of the differential equations $\nabla^2 u + \lambda^2 u = 0$, $(\nabla^2 + \lambda_1^2)(\nabla^2 + \lambda_2^2) = 0$.* We may state a theorem analogous to Theorem 1 to the effect that any solution of the first differential equation must vanish within a circle of radius ϱ/λ where ϱ is the first positive root of the Bessel function of order zero $J_0(r)$,† unless it is equal to $J_0(r\lambda)$ where r is the distance from the center of the circle. Again, if the first positive root of $J_0(\lambda_2 r)$ lies between the first two positive roots of $J_0(\lambda_1 r)$, then any solution of the second differential equation above that does not vanish at a point P must change sign within a circle about P of radius σ, where σ is the first positive root of $J_0(\lambda_1 r) = J_0(\lambda_2 r)$.‡ These statements may be proved by averaging the functions over concentric circles§ and applying Theorem 3 and 4.

If one attempted to find theorems similar to the above for other than circular regions by an extension of the same methods, a difficulty is encountered not in properly generalizing Theorems 3 and 4 but in finding a proper substitute for averaging. It is by means of the latter that the

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* In the plane these differential equations are analogous to the differential equations on the sphere with center at the origin satisfied by homogeneous harmonic and bi-harmonic polynomials. The place of ∇^2 is taken by the second differential operator of Lamé-Beltrami.

† See C. Neumann, Math. Annalen, vol. 1 (1867), p. 9.

‡ For other λ_1, λ_2, $\lambda_2 < \lambda_1$ the curve $x_1 = J_0(\lambda_1 r)$, $x_2 = J_0(\lambda_2 r)$ starting at the point $x_1 = 1$, $x_2 = 1$ for $r = 0$ bends back and cuts the line $x_1 = x_2$ before the convex region B has covered the origin; the maximum value of r for which $CJ_0(\lambda_1 r) + DJ_0(\lambda_2 r) \geqq 0$, always obtainable by the specifications of Theorem 4, need not satisfy the equation $J_0(\lambda_1 r) = J_0(\lambda_2 r)$.

§ In connection with averaging harmonic and related functions over concentric circles and spheres see Chapter I of my Cornell Thesis "Topics in Potential Theory" (Cornell, 1927); the material of this chapter is contained in an article "On Certain Integrals over Spheres" recently submitted for publication to The American Journal of Mathematics.

questions considered were reduced to consideration of a linear combination of a finite number of functions. Yet a very simple generalization of the scope of Theorem 1 to arbitrary regions may be stated in the following form:

THEOREM 6. *If R is a plane region and λ_1 is the smallest value of λ for which a solution over R of $\nabla^2 u + \lambda u = 0$ exists that vanishes on the boundary of R but is not identically zero* (we shall call λ_1 the smallest "characteristic number" for the region R), *then*

(a) *if $\lambda < \lambda_1$ there exist solutions of $\nabla^2 u + \lambda u = 0$ that are positive over R.*

(b) *if $\lambda > \lambda_1$ there exists no solution of $\nabla^2 u + \lambda u = 0$ that is positive over R.*

Proof. We restrict R to regions for which at most a finite number of linearly independent solutions of $\Delta^2 u + \lambda u = 0$ vanishing at the boundary may be shown to exist for each of a discrete set of "characteristic" values of λ, say, by the methods of integral equations. The smallest characteristic value of λ, λ_1 (as well as the other characteristic values) decreases continuously if the region R is enlarged and varies continuously with a continuous deformation of the boundary.*

Suppose first that $\lambda < \lambda_1$. Enlarge the region R till a new region $\bar{R}$ is obtained enclosing R and whose first characteristic number is equal to λ. Let u be the corresponding characteristic function of the enlarged region; u does not change sign† in $\bar{R}$ and is, say, either zero or positive. The integral equation that is equivalent to the differential equation and the boundary condition satisfied by u has a kernel that is positive inside the region $\bar{R}$, namely, the Green's function for that region. u must therefore be positive inside $\bar{R}$ and, hence, over R. Part (a) is thus proved.

Now suppose that $\lambda > \lambda_1$. Let u be a function satisfying $\nabla^2 u + \lambda u = 0$ and positive over R and let u_1 be the first characteristic function of R, for definiteness, positive over R. Applying Green's Theorem to the functions u, u_1 over the region R we get

$$\iint (u \nabla^2 u_1 - u_1 \nabla^2 u)\, dR + \int \left(u \frac{\partial u_1}{\partial n} - u_1 \frac{\partial u}{\partial n} \right) ds = 0,$$

where $\partial / \partial n$ denotes differentiation along the inner normal and the rest of the notation is obvious. Hence,

$$(-\lambda_1 + \lambda) \int u u_1\, dR + \int u \frac{\partial u_1}{\partial n}\, ds = 0.$$

* Courant-Hilbert, Methoden der mathematischen Physik I, p. 331, Theorem 3 and footnote, p. 340, Theorem 10.

† Courant-Hilbert, loc. cit., p. 364.

The first term is positive and so is the factor u in the second integrand. Hence $\partial u_1/\partial n$ must be negative somewhere over the boundary of R. But this is impossible since it would make u_1 negative at some points inside R. This completes the proof of part (b).

A similar theorem may be proved for solutions of certain self-adjoint elliptic differential equations and in the same manner. Such a differential equation is the one satisfied along the unit sphere S by harmonic polynomials of Theorem 1:

$$B^2 p_n + n(n-1)p_n = 0; \tag{6}$$

here B^2 denotes the second differential operator of Lamé-Beltrami.* Now on S let us choose for the region R the spherical zone $\theta \leqq c$. For c very small the first characteristic number for R is very large; it decreases as c increases, and is equal to $n(n-1)$ when c is equal α_n. Hence for $0 < c < \alpha_n$ there will exist solutions of (6) that are positive in R, but such solution must vanish inside larger regions R corresponding to $c > \alpha_n$. Thus we arrive at results overlapping a substantial part of Theorem 1.

* Courant-Hilbert, loc. cit., pp. 266, 267.

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MULTIPLE LINES WITH FIXED COINCIDENT TANGENT PLANES.*

By Temple Rice Hollcroft.

1. **Introduction.** The number of necessary and sufficient conditions that an algebraic surface of given order contain a given line i-fold is called the postulation of the i-fold line on the surface.

Noether† determined the postulation of an i-fold line on an algebraic surface of order n. His postulation formula applies only to an i-fold line such that a plane section of the surface has at its intersection with the line an ordinary i-fold point with distinct tangents.

Various kinds of multiple points of order i occur on an algebraic plane curve when the singularity contains cusps or consecutive multiple points or both. These singularities may be considered plane sections of i-fold lines of algebraic surfaces. Thus an algebraic surface may have as many varieties of i-fold lines as a plane algebraic curve has varieties of singular points of multiplicity i.

In general the i tangent planes to an algebraic surface along an i-fold line are torsal, that is, they vary from point to point along the line. Any or all of the tangent planes may be fixed in position. This gives rise to many types of i-fold lines on a surface, each with a different postulation, but all having the same type of i-fold point on the curve of the plane section.

In Noether's postulation formula, the i simple tangent planes are all considered torsal. In the present paper, unless otherwise specified, the simple tangent planes to the surface through an i-fold line will be considered torsal and those consisting of two or more coincident tangent planes through the i-fold line will be considered fixed planes.

The purpose of this paper is to determine the posulation of the types of i-fold lines on an algebraic surface with coincident tangent planes when these tangent planes are fixed.

For brevity, we shall use the statement, "the plane section of an i-fold line is an i-fold point", meaning thereby "the plane section of the surface containing an i-fold line at the point where the plane cuts the i-fold line is an i-fold point".

* Received September 19, 1928. Presented to the American Mathematical Society Sept. 7, 1928.

† Max Noether, Sulle curve multiple di superficie algebriche, Annali di Mat., (2), vol. 5 (1872), pp. 163–176.

2. Multiple lines whose plane sections contain cusps. Given an algebraic surface f of order n whose equation is $f(x, y, z) = 0$. In order that it contain the line $x = y = 0$ i-fold all the partial derivatives of f with respect to x and y up to and including the $(i-1)$st must vanish for all values of z. This causes the following number of coefficients of $f(x, y, z)$ to disappear:

$$n+1+2n+3(n-1)+4(n-2)+\cdots+i(n-i+2)$$
$$=\frac{1}{6}i(i+1)(3n-2i+5).$$

This number is the postulation of an i-fold line on f. It was determined by Noether in the paper cited.

In any plane $z = c$, the i tangents to the i-fold point at the origin Q on the curve $f(x, y, c) = 0$ are defined by the equation

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^i f(x, y, c) = 0.$$

In the case of the ordinary i-fold line whose postulation is given above, the tangents at 0 are all distinct, that is, there exist no relations among the ith partial derivatives of $f(x, y, c)$.

The necessary and sufficient condition that two of the tangent planes to f through the line $x = y = 0$ coincide with the plane $y = 0$ is that

$$\frac{\partial^i f}{\partial x^i} = \frac{\partial^i f}{\partial x^{i-1}\partial y} = 0$$

for all values of c. Each of the partial derivatives of order i in which $x = y = 0$ contains $n-i+1$ independent coefficients of the equation $f(x, y, z) = 0$. Therefore, in order that two of the tangent planes to f through the i-fold line be fixed coincident planes, $2(n-i+1)$ additional conditions on f are necessary. The surface f has a cuspidal edge to which the fixed plane is tangent and the plane section of the i-fold line is an i-fold point containing one cusp.

For $i = 2$, the postulation of a cuspidal line with fixed tangent plane on a surface of order n is $5n-1$.

Similarly, if an i-fold line has t coincident tangent planes such that all coincide with a fixed plane, the postulation of the line is

$$\frac{1}{6}i(i+1)(3n-2i+5)+t(n-i+1).$$

The plane section of the i-fold line is an i-fold point with t coincident tangents.

The postulation given above of an i-fold line for which t tangent planes coincide with the same fixed plane is the same as the postulation of an i-fold line for which t tangent planes coincide with any number t_1 of fixed planes, s_j tangent planes coinciding with each fixed plane, $s_j \geqq 0$ and $\Sigma s_j = t$. This holds also for $t_1 = t$ in which case each of the t distinct planes is fixed.

3. Consecutive lines on a surface. We shall now determine the postulation of an i-fold line on f such that a plane section of the line consists of two consecutive i-fold points.

In order that the curve $f(x, y, c) = 0$ have two consecutive i-fold points at the origin with the tangent $y = 0$, two conditions must be satisfied:

(1) The function f and all of its partial derivatives up to and including the $(i-1)$st vanish when $x = y = 0$.

(2) In the i sets of partial derivatives each composed of $i+a$ partial derivatives of order $i+a-1$, $1 \leqq a \leqq i$, there shall vanish $i-a+1$ of each set, that is, all those of order r with respect to x such that $i+a-1 \geqq r \geqq 2a-1$.

The first condition determines the multiplicity i of the origin; the second condition is derived from the limiting form of the general equation of a curve of order $n \geqq 2i$ with an i-fold point at the origin and a second i-fold point at $(c, 0)$ when $c = 0$.

The line $x = y = 0$ will be i-fold on $f(x, y, z) = 0$ with the tangent plane $y = 0$ and such that the plane section of the line consists of two consecutive i-fold points when the above two sets of conditions are satisfied by $f(x, y, c) = 0$ for all values of c. Since each jth partial derivative in which $x = y = 0$ contains $n-j+1$ independent coefficients of f, when a jth derivative vanishes, $n-j+1$ coefficients of f are determined.

The sum of the following series, therefore, defines the total number of coefficients of the surface f that must be determined in order that f have an i-fold line of this kind:

$$n+1+2n+3(n-1)+4(n-2)+\cdots+i(n-i+2)$$
$$+i(n-i+1)+(i-1)(n-i)+\cdots+2(n-2i+3)+n-2i+2.$$

This series contains $2i$ terms. If we add the first i terms, term by term, to the series formed by the last i terms taken in reverse order, the sum is readily found to be

$$\frac{1}{2}i(i+1)(2n-2i+3).$$

This is, therefore, the postulation of an i-fold line with a fixed tangent plane on f such that a plane section of the line consists of two consecutive i-fold points.

Two intersecting i-fold lines on a surface may rotate into coincidence about their common point (which need not remain fixed) and thus become two consecutive i-fold lines of the surface. The plane determined by the two lines approaches a definite limiting position and becomes the tangent plane to the i sheets of the surface which have contact with each other and with this plane along the limiting position of the line.

The postulation of two intersecting i-fold lines on a surface f of order n is

$$\frac{1}{3}i(i+1)(3n-2i+5)-\frac{1}{6}i(i+1)(2i+1)=\frac{1}{2}i(i+1)(2n-2i+3).$$

The postulation of two intersecting i-fold lines is therefore equal to the postulation of an i-fold line with a fixed tangent plane whose plane section is two consecutive i-fold points. This fact is interpreted geometrically as follows: Two intersecting i-fold lines on an algebraic surface have the same postulation whether they are distinct or consecutive.

The postulation of a rational space curve of order s and multiplicity i on an algebraic surface is the same as the postulation of a system of s lines each of multiplicity i containing $s-1$ intersections.* This system may be represented schematically by an open, skew polygon, that is, each line intersects the next line in order, but the first and last lines have no point in common. The s lines may all become consecutive, each rotating successively into coincidence with its neighbor. By extending the above process it results that the postulation of the system of s distinct i-fold lines containing $s-1$ intersections is the same as the postulation of s consecutive i-fold lines on the surface. This result gives rise to the theorem:

The postulation of a rational space curve of order s and multiplicity i on an algebraic surface of order n is the same as the postulation of s consecutive i-fold lines on that surface. This postulation is

$$\frac{1}{6}i(i+1)[s(3n-2i+5)-(s-1)(2i+1)].$$

4. Consecutive lines of different orders. Assume that the surface f contains a system of s lines involving $s-1$ intersections and forming an open, skew polygon as described in the preceding section. Let the lines be of the multiplicities $i_1, i_2, i_3, \cdots, i_s$ such that $i_1 \geqq i_2 \geqq i_3 \geqq \cdots \geqq i_s$.

Noether† has proved that the postulation of two lines of orders i_1 and i_2 on a surface of order n ($i_1 \geqq i_2$ and $n \geqq i_1+i_2-2$) is

$$\frac{1}{6}[i_1(i_1+1)(3n-2i_1+5)+i_2(i_2+1)(3n-2i_2+5)-i_2(i_2+1)(3i_1-i_2+1)].$$

* Hilda Hudson, Cremona Transformations, Cambridge (1927), pp. 220–221.

† M. Noether, loc. cit., p. 173.

Since in the system of lines described above each i_j-fold line intersects an i_{j+1}-fold line of equal or lower multiplicity, $j = 1, \cdots, s-1$, the postulation of this system of s lines of multiplicities i_j, $j = 1, 2, \cdots, s$ is

$$\frac{1}{6}\sum_{j=1}^{s} i_j(i_j+1)(3n-2i_j+5) - \frac{1}{6}\sum_{j=1}^{s-1} i_{j+1}(i_{j+1}+1)(3i_j-i_{j+1}+1).$$

A further condition that the surface contain this set of lines is

$$\sum_{j=1}^{s} i_j \leqq n+2.$$

Now let the i_s-fold line approach coincidence with the i_{s-1}-fold line, these with the i_{s-2}-fold, etc. until all of the s lines have become consecutive. The resulting singular line is of multiplicity i_1 and is such that its plane section consists of s consecutive i_j-fold points, $j = 1, 2, \cdots, s$. The postulation of this singular line on f is the same as the postulation of the system of s distinct multiple lines given above.

The preceding result is the space analogue of the plane theorem proved by Enriques* that s multiple points of given order of a plane curve have the same postulation whether distinct or consecutive.

5. **Special singular lines.** To find the postulation of a ramphoid cuspidal line $x = y = 0$ on the surface $f(x, y, z) = 0$, the conditions must be found that the plane curve $f(x, y, c) = 0$ have a ramphoid cusp at the origin for all values of c.

The curve $f(x, y, c) = 0$ has a ramphoid cusp at $(0, 0)$ with the tangent $y = 0$ provided that

$$f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^3 f}{\partial x^3} = 0$$

and

$$R \equiv \left(\frac{\partial^3 f}{\partial x^2\,\partial y}\right)^2 - \frac{1}{3}\left(\frac{\partial^4 f}{\partial x^4}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = 0 \quad \text{when} \quad x = y = 0.$$

Each ith derivative is of order $n-i$ in c. In order that the six linear conditions be satisfied for all values of c, there must be $6n-3$ conditions on the surface. Since these six conditions determine a tacnode on $f(x, y, c) = 0$, the postulation of a tacnodal line with fixed tangent plane on a surface of order n is $6n-3$.

The cuspidal condition R is of order $2n-6$ in c. Therefore in order that R be an identity in c, $2n-5$ coefficients of R must vanish. This gives rise to $2n-5$ additional independent relations among the coefficients

* F. Enriques, Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, Bologna (1918), vol. II, p. 338.

of $f(x, y, z) = 0$. Then the total number of conditions on f that it contain the line $x = y = 0$ as a ramphoid cuspidal line with a fixed tangent plane is

$$8(n-1).$$

This is the postulation of a double line on a surface of order n such that a plane section of the line is a ramphoid cusp and such that the tangent plane through the line is fixed in position.

If the curve $f(x, y, c) = 0$ has an inflection at the origin with the x-axis the inflectional tangent, the conditions are:

$$f = \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = 0.$$

When these conditions hold for all values of c, the line $x = y = 0$ is determined as a line of the surface f such that the plane section of f at every point of the line is an inflection. As shown in section 2, when these three conditions are satisfied for all values of c, $3n$ coefficients of f are determined. Then the postulation of an inflectional line of a surface of order n is $3n$ when the tangent plane to the surface along the line is given.

If $f(x, y, c) = 0$ has a flecnode at the origin with $y = 0$ as flecnodal tangent, in addition to the three conditions for the inflection given above, the first partial derivative with respect to y and the third partial derivative with respect to x must vanish. When the above conditions are made to hold for an arbitrary value of c so that the line $x = y = 0$ becomes a flecnodal line of f whose flecnodal tangent plane only is fixed, the number of independent coefficients of f is reduced by $5n-2$. This, then, is the postulation of a flecnodal line with a fixed flecnodal tangent plane.

If $f(x, y, c) = 0$ has a biflecnode at the origin with $x = 0$, $y = 0$ the tangents, in addition to the conditions given above for the flecnode, the second and third derivatives with respect to y must vanish. When these conditions are satisfied for all values of c, the line $x = y = 0$ is a biflecnodal line on f with both tangent planes defined. The postulation of such a line is $7n-5$.

In general, if any number t of the i tangent planes of an i-fold line are fixed inflectional tangent planes to the surface along that line, the postulation of such an i-fold line is

$$\frac{1}{6} i(i+1)(3n-2i+5)+t(2n-2i+1).$$

WELLS COLLEGE.

ON THE FUNDAMENTAL TRANSFORMATION OF SURFACES.*

By Ernest P. Lane.

1. **Introduction.** Two surfaces are in the relation of a *fundamental transformation,* or transformation F, in case their points are in a one-to-one correspondence such that the lines joining corresponding points form a congruence whose developables intersect both surfaces in conjugate nets, neither surface being a focal surface of the congruence. The congruence is called *the conjugate congruence* of the transformation, because it is conjugate to both nets. If the congruence is a bundle of lines, the two surfaces are *radially related.*

The tangent planes at corresponding points of the two surfaces intersect in the lines of *the harmonic congruence* of the transformation, which is harmonic to both nets. If this congruence is a ruled plane the two surfaces are called *perspective.*

The researches of Eisenhart,† Jonas,‡ and others have shown how extensive and significant a generalization this transformation is of several well kown transformations of surfaces. It would seem to be of interest therefore to study transformations F in ordinary space by the method developed by Wilczynski, Green, and others, which has recently been used by Grove§ to extend the notation of a transformation F to nets not conjugate. We employ a completely integrable system of linear homogeneous partial differential equations and a group of transformations that leaves our configuration invariant, neglecting however the tedium of computing a complete system of invariants.

After reducing the defining system of differential equations to a canonical form, we calculate certain invariants of a net, and then compute the change in them due to the transformation F under consideration. Then it is easy to write down conditions on the coefficients of the differential equations necessary and sufficient that the transformation shall preserve certain properties of a net. Introducing the axis and ray of a point with respect to a net, and employing a local coördinate system, we obtain new inter-

* Received October 17, 1928.

† Eisenhart, *Transformations of surfaces,* Princeton, 1923, p. 34 et seq.

‡ Jonas, Sitzungsber. Berl. Math. Ges., vol. 14 (1915), p. 103.

§ Grove, *Transformations of nets,* Trans. Amer. Math. Soc., vol. 30 (1928), p. 483.

pretations for the conjugate and harmonic invariants used by Graustein* and Slotnick as tools for the study of a transformation F.

By means of certain power series expansions for the coördinates of a point on a surface, we introduce two quadric surfaces which it seems natural to associate with a fundamental transformation. And we investigate to some extent their relations and properties.

Finally, transformations F whose conjugate and harmonic congruences are of special types receive brief attention.

2. **The differential equations.** Green,† using a somewhat different notation from that which we shall employ, called attention to the fact that in ordinary space two surfaces S_x, S_y with their points in one-to-one correspondence so that corresponding points P_x, P_y have the same curvilinear coördinates, and such that P_y is not in the tangent plane of S_x at P_x, are a pair of integral surfaces of a system of differential equations of the form

$$
\begin{aligned}
x_{uu} &= Cx + Ax_u + Gx_v + Ly,\\
x_{uv} &= cx + ax_u + bx_v + My,\\
x_{vv} &= Dx + Ex_u + Bx_v + Ny,\\
y_u &= px + \alpha x_u + \beta x_v + ly,\\
y_v &= qx + \gamma x_u + \delta x_v + ny,
\end{aligned}
\tag{1}
$$

and be calculated the integrability conditions therefore. Of these conditions the last four, obtained by equating the two expressions for y_{uv} that are consequences of the last two of equations (1), are

$$
\begin{aligned}
\gamma_u - \alpha_v + \gamma A - \beta E + q + a(\delta - \alpha) &= l\gamma - n\alpha,\\
\beta_v - \delta_u + \beta B - \gamma G + p + b(\alpha - \delta) &= n\beta - l\delta,\\
p_v + \beta D + lq + c(\alpha - \delta) &= q_u + \gamma C + np,\\
l_v + \beta N + (\alpha - \delta)M &= n_u + \gamma L.
\end{aligned}
\tag{2}
$$

We wish first of all to modify system (1) so that every pair of integral surfaces will be in relation F. For this purpose we shall determine the developables of the congruence Γ_{xy} of lines xy. The point η defined by

$$\eta = y + kx$$

* Graustein, *An invariant of a general transformation of surfaces*, Bull. Amer. Math. Soc., vol. 32 (1926), p. 357.

† Green, *Memoir on the general theory of surfaces*, etc., Trans. Amer. Math. Soc., vol. 20 (1919), § 20.

is on l_{xy}. As l_{xy} varies over a ruled surface of the family $dv - \lambda du = 0$ in Γ_{xy}, P_η generates a curve whose tangent at P_η is determined by P_η and the point η' given by

$$\eta' = y_u + y_v\lambda + k(x_u + x_v\lambda) + k'x.$$

Expressing η' as a linear combination of x, x_u, x_v, y and equating to zero the coefficients of x_u, x_v we obtain conditions on k, λ necessary and sufficient that l_{xy} may generate a developable and have P_η for focal point, namely,

$$\alpha + k + \gamma\lambda = 0, \qquad \beta + (\delta + k)\lambda = 0.$$

Eliminating k and replacing λ by dv/du we obtain *the differential equation of the developables of* Γ_{xy},

$$\beta\, du^2 + (\delta - \alpha)\, du\, dv - \gamma\, dv^2 = 0. \tag{3}$$

Incidentally, eliminating λ we obtain

$$k^2 + (\alpha + \delta)k + \alpha\delta - \beta\gamma = 0. \tag{4}$$

The two points η corresponding to the two roots of this equation are the focal points of l_{xy}.

If now $\beta = \gamma = 0$, $\delta - \alpha \neq 0$, the developables of Γ_{xy} intersect S_x and S_y in the parametric curves thereon, the foci η, ζ of l_{xy} being given by

$$\eta = y - \alpha x, \qquad \zeta = y - \delta x. \tag{5}$$

We shall suppose from now on that $\beta = \gamma = 0$, and in order that S_y may not be a focal surface of Γ_{xy} we shall suppose that $\alpha\delta \neq 0$. If $\delta - \alpha = 0$, the developables of Γ_{xy} are indeterminate and Γ_{xy} is a bundle of lines with its center at the fixed point $y - \alpha x$, the surfaces S_x, S_y being radially related.

The parametric curves on S_x form a conjugate net N_x in case $M = 0$. We shall suppose from now on that this condition is satisfied, and in order that S_x may not be developable we shall suppose that $LN \neq 0$. Under the suppositions $\beta = \gamma = M = 0$, the integrability conditions (2) become

$$\begin{aligned} \alpha_v + a(\alpha - \delta) &= q + n\alpha, \\ \delta_u + b(\delta - \alpha) &= p + l\delta, \\ p_v + lq + c(\alpha - \delta) &= q_u + np, \\ l_v &= n_u. \end{aligned} \tag{6}$$

It is not difficult to calculate the system of equations corresponding to (1) when the roles of x and y are interchanged. We shall compute

all the coefficients of such a system later, but at the moment the only coefficient that is needed is the one corresponding to M, which vanishes when the parametric curves on S_y form a conjugate net N_y. This condition, which we shall suppose from now on to be satisfied, is found to be

$$(7) \qquad p\delta(\alpha_v + a\alpha) + q\alpha(p + b\alpha + l\delta) - \alpha\delta(p_v + c\alpha + lq) = 0.$$

The conditions that we have imposed insure that every pair of integral surfaces of system (1) will be in relation F.

It is possible to simplify system (1) still more by means of a transformation of the form

$$(8) \qquad x = \lambda\bar{x}, \qquad y = \mu\bar{y}.$$

The effect of this transformation on the coefficients p, q, l, n is found to be given by the formulas

$$(9) \qquad \begin{aligned} \mu\bar{p} &= \lambda(p + \alpha\lambda_u/\lambda), & \mu\bar{q} &= \lambda(q + \delta\lambda_v/\lambda), \\ \bar{l} &= l - \mu_u/\mu, & \bar{n} &= n - \mu_v/\mu. \end{aligned}$$

The last of (6) shows that μ can be chosen so that $\bar{l} = \bar{n} = 0$. We shall suppose from now on that this choice has been made. A condition necessary and sufficient that λ can be chosen so that $\bar{p} = \bar{q} = 0$ is

$$(p/\alpha)_v = (q/\delta)_u.$$

By means of (6) this condition can be shown to be equivalent to (7) if $\delta - \alpha \neq 0$, and to be a consequence of (6) if $\delta - \alpha = 0$. And so we shall suppose from now on that this choice of λ has been made. In the presence of the conditions $p = q = 0$, $\alpha\delta \neq 0$, equation (7) reduces to $c = 0$. We reach thus the following conclusion. *Any system* (1) *such that every pair of integral surfaces is in the relation of a transformation* F, *the surfaces* S_x *being non-developable, can be reduced to the form*

$$(10) \qquad \begin{aligned} x_{uu} &= Cx + Ax_u + Gx_v + Ly, \\ x_{uv} &= \quad\; ax_u + bx_v, \\ x_{vv} &= Dx + Ex_u + Bx_v + Ny, \\ y_u &= \alpha x_u, \qquad y_v = \delta x_v \qquad (\alpha\delta LN \neq 0). \end{aligned}$$

The most general transformation (8) which leaves the form of the canonical system (10) invariant has λ and μ constant. The only coefficients of (10) not absolutely invariant under such a transformation are α, δ, L, N, for which we find

$$(11) \qquad \bar{\alpha} = \alpha\lambda/\mu, \quad \bar{\delta} = \delta\lambda/\mu, \quad \bar{L} = \mu L/\lambda, \quad \bar{N} = \mu N/\lambda.$$

The effect of the transformation of parameters

$$\bar{u} = U(u), \qquad \bar{v} = V(v) \tag{12}$$

on the coefficients of (10) is given by

$$\begin{gathered} \bar{C} = C/U'^2, \qquad U'\bar{A} = A - U''/U', \qquad \bar{G} = V'G/U'^2, \\ \bar{L} = L/U'^2, \qquad \bar{a} = a/V', \qquad \bar{\alpha} = \alpha \end{gathered} \tag{13}$$

and the formulas obtainable from these by the substitution

$$\begin{pmatrix} u & a & \alpha & A & C & E & L \\ v & b & \delta & B & D & G & N \end{pmatrix}. \tag{14}$$

If *the coefficients of the equations corresponding to* (10) *when the roles of* x *and* y *are interchanged* are indicated by dashes, we find

$$\begin{aligned} &\bar{C} = \alpha L, && \bar{A} = A + \alpha_u/\alpha, && \bar{G} = \alpha G/\delta, && \bar{L} = \alpha C, \\ &\bar{a} = a\delta/\alpha, && \bar{b} = b\alpha/\delta, && \bar{\alpha} = 1/\alpha, && \bar{\delta} = 1/\delta, \\ &\bar{D} = \delta N, && \bar{E} = \delta E/\alpha, && \bar{B} = B + \delta_v/\delta, && \bar{N} = \delta D. \end{aligned} \tag{15}$$

In order that S_y may be non-developable we shall suppose that $CD \neq 0$.

All of *the integrability conditions for system* (10) are found to be

$$\begin{aligned} &a_u + ab = A_v + EG, && b_v + ab = B_u + EG, \\ &b_u + b^2 + aG = G_v + bA + \delta L + BG + C, \\ &a_v + a^2 + bE = E_u + aB + \alpha N + AE + D, \\ &C_v = aC - DG, && D_u = bD - CE, \\ &L_v = aL - GN, && N_u = bN - EL, \\ &\alpha_v = a(\delta - \alpha), && \delta_u = b(\alpha - \delta). \end{aligned} \tag{16}$$

These equations are of course invariant under the transformation (15). And it follows from the first two of them that there exists a function θ determined, except for an additive constant, by

$$\theta_u = b + A, \qquad \theta_v = a + B. \tag{17}$$

We shall now find the differential equations corresponding to (10) in plane coördinates, so that it will be possible to study by means of them the two surfaces S_x, S_y after the dualistic transformation that converts

each point of a surface into the tangent plane at the point. If the coördinates ξ of the tangent plane of S_x at P_x are defined by

$$\xi = e^{-\theta}(x, x_u, x_v) \tag{18}$$

while the coördinates η of the tangent plane of S_y at P_y are defined by

$$\eta = e^{-\varphi}(y, y_u, y_v), \tag{19}$$

where φ is defined by

$$\varphi = \theta + \log \alpha\delta, \tag{20}$$

then ξ, η are found to satisfy the system of equations

$$\begin{aligned} \xi_{uu} &= \alpha L\xi - (A - L_u/L)\,\xi_u - aL\xi_v/N - L\eta, \\ \xi_{uv} &= -GN\xi_u/L - EL\xi_v/N, \\ \xi_{vv} &= \delta N\xi - bN\xi_u/L - (B - N_v/N)\,\xi_v - N\eta, \\ \eta_u &= -C\xi_u/L, \qquad \eta_v = -D\xi_v/N. \end{aligned} \tag{21}$$

If the roles of ξ, η in the formulation of those equations are interchanged, ξ and η are found by actual calculation, or by (15), to satisfy

$$\begin{aligned} \eta_{uu} &= C\eta - (A - C_u/C)\,\eta_u - aC\eta_v/D - \alpha C\xi, \\ \eta_{uv} &= -DG\eta_u/C - CE\eta_v/D, \\ \eta_{vv} &= D\eta - bD\eta_u/C - (B - D_v/D)\,\eta_v - \delta D\xi, \\ \xi_u &= -L\eta_u/C, \qquad \xi_v = -N\eta_v/D. \end{aligned} \tag{22}$$

3. **Invariants and geometric theorems.** There are two types of problems that arise in the investigation of transformations F. In a problem of the first type we consider a conjugate net having a special property, being perhaps isothermally conjugate, and inquire under what conditions the F transformed net may have the same property. And in a problem of the second type we consider as a whole the configuration composed of two nets in relation F and the conjugate (or harmonic) congruence, and study properties thereof whose definition depends on the relation F. We shall consider each type in turn.

In ordinary space a surface referred to a conjugate net was studied* by Green who used a certain system of two partial differential equations of the second order. For N_x one of these equations, in a notation different from Green's, is the Laplace equation appearing as the second of (10).

* Green, *Projective differential geometry of one-parameter families of space curves*, etc., Amer. Journ. Math., vol. 37 (1915), p. 215, and vol. 38 (1916), p. 287.

The other is found by eliminating y from the first and third of (10) and solving the result for x_{uu}. It can be written in the form

$$x_{uu} = a'x_{vv} + b'x_u + c'x_v + d'x \tag{23}$$

where

$$a' = L/N, \quad b' = A - a'E, \quad c' = G - a'B, \quad d' = C - a'D. \tag{24}$$

Some of the invariants of N_x are found to have in our notation the following expressions:

$$\begin{aligned}
H &= ab - a_u, \qquad K = ab - b_v, \\
W^{(u)} &= 2b_v + a_u - B_u - (\log L)_{uv}, \\
W^{(v)} &= 2a_u + b_v - A_v - (\log N)_{uv}, \\
\mathcal{H} &= K + W^{(u)} = a_u + EG - (\log L)_{uv} = N(G_u + bG - GL_u/L)/L, \\
\mathfrak{H} &= H + W^{(v)} = b_v + EG - (\log N)_{uv} = L(E_v + aE - EN_v/N)/N, \\
\mathfrak{D} &= L(E_u - 2bE + AE + \alpha N)/N - (G_v - 2aG + BG + \delta L), \\
8\mathfrak{B}' &= 6a - 2B - 3L_v/L + N_v/N, \\
8\mathfrak{C}' &= 6b - 2A - 3N_u/N + L_u/L.
\end{aligned} \tag{25}$$

By use of (15), the corresponding invariants for N_y, indicated by dashes, are found to have the following expressions:

$$\begin{aligned}
\bar{H} &= H - (\log \alpha)_{uv}, & \bar{K} &= K - (\log \delta)_{uv}, \\
\bar{W}^{(u)} &= W^{(u)} + (\log \delta L/C)_{uv}, & \bar{W}^{(v)} &= W^{(v)} + (\log \alpha N/D)_{uv}, \\
\bar{\mathcal{H}} &= \mathcal{H} + (\log L/C)_{uv} = D(G_u + bG - GC_u/C)/C, \\
\bar{\mathfrak{H}} &= \mathfrak{H} + (\log N/D)_{uv} = C(E_v + aE - ED_v/D)/D, \\
\bar{\mathfrak{D}} &= C[E_u - (1 + \alpha/\delta)bE + AE + D]/D \\
&\qquad - \alpha[G_v - (1 + \delta/\alpha)aG + BG + C]/\delta, \\
8\bar{\mathfrak{B}}' &= 8\mathfrak{B}' + (\log \alpha^3 DL^3/\delta C^3 N)_v, \\
8\bar{\mathfrak{C}}' &= 8\mathfrak{C}' + (\log \delta^3 CN^3/\alpha D^3 L)_u.
\end{aligned} \tag{26}$$

From (25) it is easy to read off conditions on the coefficients of (10) necessary and sufficient that N_x may have special projective properties. For instance, N_x has equal point invariants in case $H = K$, so that $a_u = b_v$. Similarly we may treat a net N_x if it has equal tangential invariants, $\mathcal{H} = \mathfrak{H}$; if it is isothermally conjugate, so that $W^{(u)} = W^{(v)}$, this condition being equivalent to $(\log N/L)_{uv} = 0$; if it is harmonic,* $\mathfrak{D} = 0$; if it is quadratic, $\mathfrak{B}' = \mathfrak{C}' = 0$; and if it is an R net, $W^{(u)} = W^{(v)} = 0$.

* Wilczynski, *Geometrical significance of isothermal conjugacy of a net of curves*, Amer. Journ. Math., vol. 42 (1920), p. 211.

And from (26) it is then easy to read off conditions on the coefficients of (10) necessary and sufficient that a net N_x having one or more of these properties may correspond by a transformation F to a net N_y having the same property or properties. We shall refrain from entering upon a detailed discussion of these conditions, remarking that conditions similar to some of them have been obtained by Slotnick who has also stated some theorems as consequences thereof.*

We shall now discuss certain cross ratios used by Slotnick and Graustein, and obtain new interpretations for them, thus exemplifying problems of the second type previously mentioned. The cross ratio R of the points x, y and the focal points η, ζ of the generator l_{xy} of the conjugate congruence Γ_{xy} of the transformation F under discussion is given by

$$R = (x, y, \eta, \zeta) = (\infty, 0, \alpha, \delta) = \delta/\alpha. \tag{27}$$

This is the reciprocal of Slotnick's invariant C which is equal to 1 for a radial transformation. There are other cross ratios which are equal to R. For instance, if local coördinates $x_1, \cdots, x_4$ based on the tetrahedron x, x_u, x_v, y with suitably chosen unit point are introduced, the equations of *the ray* of P_x with respect to N_x, which joins the Laplace transformed points σ, ϱ defined by

$$\sigma = x_v - ax, \qquad \varrho = x_u - bx, \tag{28}$$

is found to have the local equations

$$x_4 = x_1 + bx_2 + ax_3 = 0. \tag{29}$$

This line crosses the generator $x_1 = x_4 = 0$ of the harmonic congruence of the transformation F in the point $(0, a, -b, 0)$. Similarly, the ray of P_y with respect to N_y has the local equations

$$x_1 = b\alpha x_2 + a\delta x_3 + \alpha\delta x_4 = 0 \tag{30}$$

and crosses the line $x_1 = x_4 = 0$ in the point $(0, a\delta, -b\alpha, 0)$. *The cross ratio of the second and first† focal points of the line* $x_1 = x_4 = 0$, *namely the points* $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, *and the two points where the rays of* P_x *and* P_y *meet this line is equal to* R. Still another way to

* Slotnick, *A contribution to the theory of fundamental transformations of surfaces*, Trans. Amer. Math. Soc., vol. 30 (1928), p. 190.

† In this paper the convention is observed that the first focal point of a generator of a congruence is the point which generates a u-curve with the generator of the congruence for tangent. The first focal plane is tangent to the first focal surface at the first focal point.

define a cross ratio which turns out to be equal to R is to consider *the first and second focal planes* $x_2 = 0$, $x_3 = 0$ *of* l_{xy}, *the plane* $x_2 dv - x_3 du = 0$ *determined by* l_{xy} *and the tangent to any curve through* P_x *on* S_x *in the direction* dv/du, *and the plane* $x_2 \delta dv - x_3 \alpha du = 0$ *determined by* l_{xy} *and the corresponding curve through* P_y *on* S_y. *The cross ratio of these planes in the order named is equal to* R.

Dual considerations lead to the discussion of another cross ratio. The first focal plane of the line $x_1 = x_4 = 0$ of the harmonic congruence, which is the tangent plane of the surface generated by the point x_v, intersects l_{xy} in the point $(D, 0, 0, N)$, and the second focal plane of $x_1 = x_4 = 0$ intersects l_{xy} in the point $(C, 0, 0, L)$. Therefore the cross ratio S of the tangent planes of S_x and S_y and the first and second focal planes of the generator $x_1 = x_4 = 0$ of the harmonic congruence of the transformation F under discussion is given by

$$S = (\infty, 0, D/N, C/L) = CN/DL. \tag{31}$$

This is Slotnick's invariant H which is equal to 1 for a perspective transformation. Therefore Slotnick's Theorem XI becomes that the asymptotic curves on S_x and S_y correspond in case $R = S$. There is another way to define a cross ratio which is equal to S. *The axis* of P_x with respect to N_x is the line of intersection of the osculating planes of C_u and C_v at P_x on S_x. Equations (23), (24), (10) show that the axis of P_x joins P_x to the point $(C, EL/N, G, L)$. Therefore the local equation of the plane containing l_{xy} and the axis of P_x is

$$GNx_2 - ELx_3 = 0. \tag{32}$$

Similarly, the equation of the plane containing l_{xy} and the axis of P_y is

$$DGx_2 - CEx_3 = 0. \tag{33}$$

Therefore *the cross ratio of the second and first focal planes* $x_3 = 0$ *and* $x_2 = 0$ *of* l_{xy}, *the plane of* l_{xy} *and the axis of* P_x, *and the plane of* l_{xy} *and the axis of* P_y *is equal to* S.

4. **The associated quadric surfaces.** The coördinates of a point X near P_x on S_x can be represented by power series of the form

$$X = x + x_u \Delta u + x_v \Delta v + (x_{uu} \Delta u^2 + 2x_{uv} \Delta u \Delta v + x_{vv} \Delta v^2)/2 + \cdots$$

in which Δu, Δv denote the increments of u, v that correspond to displacement from P_x to the point X. When the derivatives of x of the second and higher orders are replaced by the expressions given for them by (10) and the equations obtained therefrom by differentiation, the result is

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 y,$$

where

$$\begin{aligned}
x_1 &= 1 + \cdots, \\
x_2 &= \Delta u + (A\Delta u^2 + 2a\Delta u\Delta v + E\Delta v^2)/2 + \cdots, \\
x_3 &= \Delta v + (G\Delta u^2 + 2b\Delta u\Delta v + B\Delta v^2)/2 + \cdots, \\
x_4 &= (L\Delta u^2 + N\Delta v^2)/2 \\
&\quad + [(L_u + AL)\Delta u^3 + 3aL\Delta u^2\Delta v + 3bN\Delta u\Delta v^2 + (N_v + BN)\Delta v^3]/6 + \cdots.
\end{aligned} \tag{34}$$

These are *the local parametric equations of* S_x. In a similar way we obtain *the local parametric equations of* S_y:

$$\begin{aligned}
y_1 &= (\alpha C\Delta u^2 + \delta D\Delta v^2)/2 + [(2\alpha_u C + \alpha C_u + \alpha AC)\Delta u^3 + 3a\delta C\Delta u^2\Delta v \\
&\quad + 3b\alpha D\Delta u\Delta v^2 + (2\delta_v D + \delta D_v + \delta BD)\Delta v^3]/6 + \cdots, \\
y_2 &= \alpha\Delta u + [(\alpha_u + \alpha A)\Delta u^2 + 2a\delta\Delta u\Delta v + \delta E\Delta v^2]/2 + \cdots, \\
y_3 &= \delta\Delta v + [\alpha G\Delta u^2 + 2b\alpha\Delta u\Delta v + (\delta_v + \delta B)\Delta v^2]/2 + \cdots, \\
y_4 &= 1 + \cdots,
\end{aligned} \tag{35}$$

Since it is three conditions for a quadric surface to have contact of the first order with an analytic surface at a point of the latter, and since it is six conditions for a quadric to have contact of the second order, it is not surprising that *there is a unique quadric* Q_x *which has contact of the second order with* S_x *at* P_x *and contact of the first order with* S_y *at* P_y. On writing the equation of a general quadric, and demanding that this equation be satisfied by the series (34) identically in Δu, Δv as far as the terms of the second degree, and by (35) as far as the terms of the first degree, the equation of Q_x is found to be

$$Lx_2^2 + Nx_3^2 - 2x_1 x_4 = 0. \tag{36}$$

There is also *a unique quadric* Q_y *with second order contact with* S_y *at* P_y *and first order contact with* S_x *at* P_x, the equation of this quadric being found similarly to be

$$\delta Cx_2^2 + \alpha Dx_3^2 - 2\alpha\delta x_1 x_4 = 0. \tag{37}$$

The quadrics Q_x, Q_y will be called *the associated quadric surfaces of the transformation* F.

The associated quadrics Q_x, Q_y determine a pencil of quadrics. *Two of the cones in this pencil are the planes*

$$\delta(\alpha L - C)x_2^2 + \alpha(\delta N - D)x_3^2 = 0, \tag{38}$$

which intersect in the line xy of the conjugate congruence and separate harmonically the focal planes $x_2 = 0$ *and* $x_3 = 0$ *of this line.* Another cone in the pencil has the equation

$$(\delta CN - \alpha DL)\, x_3^2 + 2\delta(\alpha L - C)\, x_1 x_4 = 0, \tag{39}$$

its vertex being at the point (0, 1, 0, 0), and the fourth cone has the equation

$$(\alpha DL - \delta CN)\, x_2^2 + 2\alpha(\delta N - D)\, x_1 x_4 = 0, \tag{40}$$

its vertex being at the point (0, 0, 1, 0).

The quadric Q_x intersects S_x in a curve with a triple point at P_x the directions of the triple point tangents being given by

$$\begin{aligned}(L_u - 2AL)\, du^3 - 3(aL + GN)\, du^2\, dv \\ - 3(bN + EL)\, du\, dv^2 + (N_v - 2BN)\, dv^3 = 0,\end{aligned} \tag{41}$$

and intersects S_y in a curve with a double point, the directions of the double point tangents being given by

$$\alpha(\alpha L - C)\, du^2 + \delta(\delta N - D)\, dv^2 = 0. \tag{42}$$

The corresponding equations for Q_y are

$$\begin{aligned}[(\alpha_u + 2\alpha A)\, C - \alpha C_u]\, du^3 + 3(a\delta C + \gamma DG)\, du^2\, dv \\ + 3(b\alpha D + \delta CE)\, du\, dv^2 + [(\delta_v + 3\delta B)\, D - \delta D_v]\, dv^3 = 0,\end{aligned} \tag{43}$$

and

$$(\delta C - L)\, du^2 + (\alpha D - N)\, dv^2 = 0. \tag{44}$$

The planes containing the intersection of Q_x and Q_y intersect S_x in curves having the directions

$$\delta(\alpha L - C)\, du^2 + \alpha(\delta N - D)\, dv^2 = 0, \tag{45}$$

and intersect S_y in curves having the directions

$$\delta\alpha^2(\alpha L - C)\, du^2 + \alpha\delta^2(\delta N - D)\, dv^2 = 0. \tag{46}$$

The curves thus defined on S_x for instance, and the curves on S_x corresponding to the curves thus defined on S_y, would seem to be worthy of more detailed study than space permits here.

5. **Special conjugate and harmonic congruences.** We have already seen that the conjugate congruence is a bundle of lines in case

$R = 1$, a condition equivalent to $\alpha = \delta$. In this case we have $\alpha_u = \alpha_v = 0$ by (16) and by (11) we can make $\alpha = \delta = 1$. Dually, the harmonic congruence is a ruled plane in case $S = 1$, a condition equivalent to $CN = DL$. In this case we have $(C/L)_u = (C/L)_v = 0$ by (16) and by (11) we can make $C = L$, $D = N$. *The surfaces S_x and S_y are projectively equivalent in case $R = S = 1$,* since the differential equations of the kind used by Green for S_x are identical with the corresponding equations for S_y if, and only if, these conditions are satisfied.

Besides the extremely special conjugate and harmonic congruences just considered, which will be excluded from our remaining considerations, other possibilities suggest themselves. For instance the focal surfaces of the conjugate congruence may reduce to curves. We find from (5) by actual calculation

$$\begin{aligned} \eta_u &= -\alpha_u x, & \eta_v &= (\delta - \alpha)\sigma, \\ \zeta_u &= (\alpha - \delta)\varrho, & \zeta_v &= -\delta_v x, \end{aligned} \tag{47}$$

where ϱ, σ are defined by (28). Imposing the condition that η shall satisfy a linear homogeneous partial differential equation of the first order, we find that *the locus of P_η is a curve* (which is not a fixed point) *in case* $\alpha_u = 0$. Therefore *if the locus of P_η is a curve at all it is a v-curve.* Similarly, *the locus of P_ζ is a curve in case $\delta_v = 0$, so that if it is a curve at all it is a u-curve.*

Imposing the condition that η shall satisfy a linear homogeneous partial differential equation of the second order containing only v-derivatives we find that *a condition necessary and sufficient that the surface S_η may be ruled with the v-curves for generators is $E = 0$.* Incidentally it can be shown that *S_η is developable if, and only if, $E = 0$,* so that (the transformation F being non-radial) *S_η is a ruled surface with its v-curves for generators in case it is developable.* Similarly, *S_ζ is ruled for its u-curves for generators in case this surface is developable,* a necessary and sufficient condition therefore being $G = 0$. Combining these results with those of the preceding paragraph we see that *the conjugate congruence of a transformation F is linear in case*

$$\alpha_u = \delta_v = E = G = 0. \tag{48}$$

The first focal surface S' of the harmonic congruence, which is generated by the point x_v, reduces to a curve in case $a = 0$, and is then a v-curve. The second focal surface S'' reduces to a curve in case $b = 0$, and is then a u-curve. The first focal surface S' is ruled with its v-curves for generators in case $(D/N)_v = 0$. In fact, S' is developable if, and only if, $(D/N)_v = 0$ so that (the transformation F being non-perspective) S'

is a ruled surface with its v-curves for generators in case it is developable. Similarly, *S'' is ruled with its u-curves for generators in case this surface is developable,* a necessary and sufffcient condition therefore being $(C/L)_u = 0$. Combining these results we see that *the harmonic congruence of a transformation F is linear in case*

$$a = b = (D/N)_v = (C/L)_u = 0. \tag{49}$$

The v-curves on S_x are plane curves in case $\mathfrak{K} = 0$, a condition that can be seen by (25) to be implied by $E = 0$, and which can be shown by (25) and (16) to be implied by $(D/N)_v = 0$ when F is non-perspective. Therefore *if S_η is developable, the v-curves on S_x are plane curves; and these curves are plane if S' is developable.* Similarly, *if either S_ζ or S'' is developable, the u-curves on S_x are plane curves.* Dually, the u-curves on S_x are cone curves in case $H = 0$, a condition that can be seen by (25) to be implied by $a = 0$ and which can be shown by (25) and (16) to be implied by $\alpha_u = 0$ when F is non-radial. Therefore *if either S' or S_η reduces to a curve, the u-curves on S_x are cone curves.* Similarly, *if either of S'' or S_ζ reduces to a curve, the v-curves on S_x are cone curves.*

The osculating plane of C_v on S_x is shown by (10) to contain l_{xy} in case $E = 0$ and in this case (15) shows that the osculating plane of C_v on S_y also contains l_{xy}. Moreover it is geometrically obvious, besides being easy to demonstrate analytically that these planes coincide in this case, the v-curves on S_x and S_y being plane curves in the same plane. Similar remarks can be made with u and v interchanged. Therefore the axis of P_x with respect to N_x, and the axis of P_y with respect to N_y, coincide with l_{xy} in case $E = G = 0$. *A transformation F has for its conjugate congruence the axis congruence of one net if, and only if, its conjugate congruence is also the axis congruence of the other net, and then both nets consist of plane curves, two corresponding curves lying in the same plane.* Dually, *the harmonic congruence is the ray congruence of one net if, and only if, it is also the ray congruence of the other net, and then both nets consist of cone curves, two corresponding curves lying on cones with the same vertex.*

Another kind of specially restricted congruence is a W congruence. If $(\alpha - \delta)\alpha_u \delta_v \neq 0$ so that the focal surfaces of the conjugate congruence do not reduce to a fixed point nor to curves, the asymptotic curves on S_η are found to have the differential equation

$$\alpha_u\, du^2 + (\alpha - \delta) E\, dv^2 = 0, \tag{50}$$

and the asymptotic curves on S_ζ,

$$(\delta - \alpha) G\, du^2 + \delta_v\, dv^2 = 0. \tag{51}$$

Imposing the condition that these equations be equivalent, we see that *the conjugate congruence is a W congruence in case*

$$\alpha_u \delta_v + (\alpha - \delta)^2 EG = 0. \tag{52}$$

Dually, if $ab(CN - DL) \neq 0$ so that the focal surfaces of the harmonic congruence do not reduce to a fixed plane nor to curves, the asymptotic curves on S' are found to have the differential equation

$$(ND_v - DN_v)\,du^2 + a(CN - DL)\,dv^2 = 0, \tag{53}$$

and the asymptotic curves on S'',

$$b(DL - CN)\,du^2 + (LC_u - CL_u)\,dv^2 = 0. \tag{54}$$

Therefore *the harmonic congruence is a W congruence in case*

$$(ND_v - DN_v)(LC_u - CL_u) + ab(CN - DL)^2 = 0. \tag{55}$$

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NOTE ADDED TO PROOF, July 9, 1929: Grove has pointed out to the author the close connection of the results at the bottom of p. 464 and at the end of section 3 with his theorems at the end of section 5 on p. 493 of his paper already cited. Mr. Clifford Mendel has remarked that the quadric $ax_1x_3 - b\alpha x_2x_4 = 0$ contains the lines xx_u, yx_v, $\eta\varrho$, $\zeta\sigma$ and also $\varrho\sigma$, xy, while the quadric $bx_1x_2 - a\delta x_3x_4 = 0$ contains the lines xx_v, yx_u, $\eta\sigma$, $\zeta\varrho$ and also $\varrho\sigma$, xy.

THE FREQUENCY FUNCTION OF y/x.*

By C. C. Craig.

1. **Introduction.** The problem of obtaining the frequency distribution of the quotient y/x when the distributions of both x and y are known had until recently so far as I know been studied only by Karl Pearson.† He gave general formulas for the first four moments of y/x but remarked that they are "practically unworkable if x and y are correlated as we should have to find the third and fourth order product moments." He gave the formulas in case x and y are uncorrelated which he attributed to Dr. M. Greenwood, Jr. These however were rejected because Dr. Greenwood was unable to make them work sufficiently well in practice and Pearson finally adopted the method of finding the constants of the product xz, with z set equal to $1/y$, in the case in which x and y are uncorrelated, obtaining very satisfactory results in the examples cited.

In a paper on spurious correlation, S. D. Wicksell has, in regard to expressing the moments of $1/x$ in terms of those of x, made some remarks pertinent to the problem in hand‡. In my doctor's thesis (soon to appear in Metron) I have made use of a method of finding approximations to the semi-invariants of $\alpha_3 = \nu_3/\nu_2^{3/2}$ and $\alpha_4 = \nu_4/\nu_2^2$ due to sampling which is applicable to this more simple and similar problem under restrictions similar to those imposed there. In this paper I propose to go into the question of the availability of this method in this problem with some care. Formulas in terms of semi-invariants equivalent to those of Pearson in terms of moments are readily obtained and a sufficient and "almost" necessary condition for their validity is given. It is especially noteworthy that the use of semi-invariants enables me to readily include the important case in which x and y *are correlated,* particularly if x and y are normally distributed. Two numerical illustrations of the method are given, one being a careful attempt to check the theory by applying it to a case in which the correlation surface for x and y is as nearly perfectly normal with $r_{xy} = .60$ as it was possible to make it by actual graduation, calculating cell frequencies to the nearest tenth with a total frequency of 1000.

2. **The semi-invariants of y/x.** The definition of Thiele's semi-invariants, $L_1, L_2, L_3, \cdots$ for y/x is

* Received November 12, 1928.

† Pearson, Karl, On the constants of index distributions, etc., Biometrika, vol. 7 (1910), pp. 531–541.

‡ Wicksell, S. D., An exact formula for spurious correlation, Metron, vol. I, No. 4 (1921), pp. 33–40.

$$(1) \qquad e^{L_1 t+\frac{1}{2!}L_2 t^2+\frac{1}{3!}L_3 t^3+\cdots} = \int_{-\infty}^{\infty} f(y/x)\, e^{\frac{y}{x}t}\, d(y/x),$$

in which $f(y/x)$ is the probability or frequency function for y/x. Now I write,

$$x = m_x+\varepsilon_1 = m_1+\varepsilon_1 = m_1\left(1+\frac{\varepsilon_1}{m_1}\right),$$

and

$$y = m_y+\varepsilon_2 = m_2+\varepsilon_2 = m_2\left(1+\frac{\varepsilon_2}{m_2}\right),$$

which define ε_1 and ε_2 as the deviations of x and y respectively from their means. Then by a well-known theorem in semi-invariants, I can rewrite the above,

$$(2) \qquad e^{L_1 t+\frac{1}{2!}L_2 t^2+\frac{1}{3!}L_3 t^3+\cdots} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y)\, e^{\frac{m_2}{m_1}\left(1+\frac{\varepsilon_2}{m_2}\right)\left(1+\frac{\varepsilon_1}{m_1}\right)^{-1}t}\, dx\, dy,$$

in which now $F(x, y)$ is the probability or correlation function of x and y. On expanding the right member of (2) and equating the coefficients of like powers of t in the usual manner, I obtain,

$$\begin{aligned}
L_1 &= \frac{m_2}{m_1}\left[1+\frac{\mu_{20}}{m_1^2}-\frac{\mu_{30}}{m_1^3}+\frac{\mu_{40}}{m_1^4}-\cdots\right] \\
&\quad +\frac{1}{m_1}\left[-\frac{\mu_{11}}{m_1}+\frac{\mu_{21}}{m_1^2}-\frac{\mu_{31}}{m_1^3}+\frac{\mu_{41}}{m_1^4}-\cdots\right], \\
L_2 &= \frac{m_2^2}{m_1^2}\left[1+\frac{3\mu_{20}}{m_1^2}-\frac{4\mu_{30}}{m_1^3}+\frac{5\mu_{40}}{m_1^4}-\cdots\right] \\
&\quad +\frac{2m_2}{m_1^2}\left[-\frac{2\mu_{11}}{m_1}+\frac{3\mu_{21}}{m_1^2}-\frac{4\mu_{31}}{m_1^3}+\frac{5\mu_{41}}{m_1^4}-\cdots\right] \\
&\quad +\frac{1}{m_1^2}\left[\mu_{02}-\frac{2\mu_{12}}{m_1}+\frac{3\mu_{22}}{m_1^2}-\frac{4\mu_{32}}{m_1^3}+\frac{5\mu_{42}}{m_1^4}-\cdots\right]-L_1^2, \\
L_3 &= \frac{m_2^3}{m_1^3}\left[1+\frac{6\mu_{20}}{m_1^2}-\frac{10\mu_{30}}{m_1^3}+\frac{15\mu_{40}}{m_1^4}-\cdots\right] \\
&\quad +\frac{3m_2^2}{m_1^3}\left[-\frac{3\mu_{11}}{m_1}+\frac{6\mu_{21}}{m_1^2}-\frac{10\mu_{31}}{m_1^3}+\frac{15\mu_{41}}{m_1^4}-\cdots\right] \\
&\quad +\frac{3m_2}{m_1^3}\left[\mu_{02}-\frac{3\mu_{12}}{m_1}+\frac{6\mu_{22}}{m_1^2}-\frac{10\mu_{32}}{m_1^3}+\frac{15\mu_{42}}{m_1^4}-\cdots\right] \\
&\quad +\frac{1}{m_1^3}\left[\mu_{03}-\frac{3\mu_{13}}{m_1}+\frac{6\mu_{23}}{m_1^2}-\frac{10\mu_{33}}{m_1^3}+\frac{15\mu_{43}}{m_1^4}-\cdots\right] \\
&\quad -3L_2L_1-L_1^3,
\end{aligned} \qquad (3)$$

$$
\begin{aligned}
L_4 = {} & \frac{m_2^4}{m_1^4}\left[1+\frac{10\mu_{20}}{m_1^2}-\frac{20\mu_{30}}{m_1^3}+\frac{35\mu_{40}}{m_1^4}-\cdots\right] \\
& +4\frac{m_2^3}{m_1^4}\left[-\frac{4\mu_{11}}{m_1}+\frac{10\mu_{21}}{m_1^2}-\frac{20\mu_{31}}{m_1^3}+\frac{35\mu_{41}}{m_1^4}-\cdots\right] \\
& +\frac{6m_2^2}{m_1^4}\left[\mu_{02}-\frac{4\mu_{12}}{m_1}+\frac{10\mu_{22}}{m_1^2}-\frac{20\mu_{32}}{m_1^3}+\cdots\right] \\
& +\frac{4m_2}{m_1^4}\left[\mu_{03}-\frac{4\mu_{13}}{m_1}+\frac{10\mu_{23}}{m_1^2}-\frac{20\mu_{33}}{m_1^3}+\cdots\right] \\
& +\frac{1}{m_1^4}\left[\mu_{04}-\frac{4\mu_{14}}{m_1}+\frac{10\mu_{24}}{m_1^2}-\frac{20\mu_{34}}{m_1^3}+\cdots\right] \\
& -4L_3L_1-3L_2^2-6L_2L_1^2-L_1^4.
\end{aligned}
$$

In these μ_{rs} is obviously the rs-th moment of $F(x, y)$ about the means of x and y as origin. These formulas are, of course, equivalent to those given by Pearson, as mentioned above. But when the moments in these are replaced by their values in terms of the semi-invariants of $F(x, y)$ I will have some results that will under suitable conditions be practically workable if x and y are correlated.

It is obvious, as Wicksell remarks in his article previously referred to, that the series, the sums of which make up the L's, will under the most general and theoretical conditions diverge. In particular, if the probability of the occurrence of a zero value of x together with a value of y not zero is more than an infinitesimal the problem is not at all susceptible to mathematical treatment. It is therefore desirable to restrict the frequency distribution for x in such a way that the probability of a zero value of x is an infinitesimal of a sufficiently high order that the integral in (1) will have meaning. The obvious thing is to require that the range of x be finite, or else, if the range be infinite in either direction, that the order of contact of the frequency curve for x be sufficiently high there, and that accordingly the zero for x fall either entirely outside the admissible values of x or else sufficiently far into the tails.

But in practice only distributions of x occur with finite ranges of values for x. If the zero for x falls outside the range of the observed values of x difficulty of the kind under consideration will not arise. I then shall impose the following conditions on the distribution of values of x:

1. $m_x = k\sigma_x \qquad k>1,$
2. $m_x - k\sigma_x < x < m_x + k\sigma_x.$

This means in practice that m_x shall be large compared with σ_x, so large that in the actual distributions of x to be considered no zero values of x will occur.

Then if $\bar{x} = x - m_x$, and if $|\bar{x}| \leqq a < k\sigma_x$, I have, denoting the probability function of x by $\varphi(x)$, since $0 \leqq \varphi(x) < 1$,

$$\left|\frac{\mu_r}{m_1^r}\right| \leqq \frac{1}{m_1^r}\left|\int_{-a}^{a} \bar{x}^r \varphi(x)\,dx\right| \leqq \frac{2}{m_1^r}\int_{-a}^{a} \bar{x}^r\,dx = \frac{2a}{r+1}\left(\frac{a}{m_1}\right)^r.$$

The ratio a/m_1 is less than unity and it is now apparent that the first series in the value of L_1 is convergent. But since the y-subscript in the second series of L_1 is constant, it is readily seen that the same argument can be applied here also and that this series under the given conditions is also convergent. But in any of the series that are found in the L's the y-subscript is constant. The additional fact is that of increasing coefficients but since the ratio of any two successive ones rapidly approaches unity, convergence in all these series is secured.

Now in the inequality used just above, $\varphi(x)$ is not only less than unity but in practice it is always small for $\bar{x}$ larger than two or three times σ_x. Also in practice k will have to be at least three and had better be taken as five or more. These two circumstances cause convergence to be rapid in the actual applications of the formulas given.

The next step is to express the L's in terms of the semi-invariants of $F(x, y)$. If λ_{rs} is the rs-th semi-invariant of $F(x, y)$, then $\lambda_{10} = \lambda_{01} = 0$. I have carried out the reductions far enough to give all the terms up to and including those of the 6th order in the expressions for L_1, L_2, L_3, and L_4 under the assumption that $F(x, y)$ is completely characterized by its semi-invariants of the first four orders, i. e., that

$$\lambda_{rs} = 0, \qquad r + s \geqq 5.$$

The necessary formulas for moments in terms of semi-invariants needed in this case follow.

$$\begin{aligned}
&\mu_{10} = \lambda_{10} = 0, && \mu_{30} = \lambda_{30},\\
&\mu_{20} = \lambda_{20}, && \mu_{21} = \lambda_{21},\\
&\mu_{11} = \lambda_{11},\\
&\mu_{40} = \lambda_{40} + 3\lambda_{20}^2,\\
&\mu_{31} = \lambda_{31} + 3\lambda_{20}\lambda_{11},\\
&\mu_{22} = \lambda_{22} + \lambda_{20}\lambda_{02} + 2\lambda_{11}^2,\\
&\mu_{50} = \cdots + 10\lambda_{30}\lambda_{20},\\
&\mu_{41} = \cdots + 4\lambda_{30}\lambda_{11} + 6\lambda_{21}\lambda_{20},\\
&\mu_{32} = \cdots + \lambda_{30}\lambda_{02} + 6\lambda_{21}\lambda_{11} + 3\lambda_{12}\lambda_{20},\\
&\mu_{60} = \cdots + 15\lambda_{40}\lambda_{20} + 10\lambda_{30}^2 + 15\lambda_{20}^3,
\end{aligned} \tag{4a}$$

$$\mu_{51} = \cdots + 5\lambda_{40}\lambda_{11} + 10\lambda_{31}\lambda_{20} + 10\lambda_{30}\lambda_{21} + 15\lambda_{20}^2\lambda_{11},$$

$$\mu_{42} = \cdots + \lambda_{40}\lambda_{02} + 8\lambda_{31}\lambda_{11} + 6\lambda_{22}\lambda_{20} + 4\lambda_{30}\lambda_{12} + 6\lambda_{21}^2 + 3\lambda_{20}^2\lambda_{02} + 12\lambda_{20}\lambda_{11}^2, \tag{4b}$$

$$\mu_{33} = \cdots + 3\lambda_{31}\lambda_{02} + 9\lambda_{22}\lambda_{11} + 3\lambda_{13}\lambda_{20} + \lambda_{30}\lambda_{03} + 9\lambda_{21}\lambda_{12} + 9\lambda_{20}\lambda_{11}\lambda_{02} + 6\lambda_{11}^3.$$

On permutation of subscripts μ_{12} is obtained from μ_{21}, etc.

Substituting these into (3), I get, after setting $l_{rs} = \lambda_{rs}/m_1^{r+s}$, and $\omega = m_2/m_1$,

$$\begin{aligned}
L_1 &= \omega[1 + l_{20} - l_{30} + l_{40} + 3l_{20}^2 - 10l_{30}l_{20} + 15l_{40}l_{20} + 10l_{30}^2 + 15l_{20}^3] \\
&\quad + [-l_{11} + l_{21} - l_{31} - 3l_{20}l_{11} + 4l_{30}l_{11} + 6l_{21}l_{20} - 5l_{40}l_{11} - 10l_{31}l_{20} \\
&\quad - 10l_{30}l_{21} - 15l_{20}^2l_{11}], \\
L_2 &= \omega^2[l_{20} - 2l_{30} + 3l_{40} + 8l_{20}^2 - 38l_{30}l_{20} + 73l_{40}l_{20} + 49l_{30}^2 + 69l_{20}^3] \\
&\quad + 2\omega[-l_{11} - 2l_{21} - 3l_{31} - 8l_{20}l_{11} + 15l_{30}l_{11} + 23l_{21}l_{20} - 24l_{40}l_{11} \\
&\quad - 49l_{31}l_{20} - 49l_{30}l_{21} - 69l_{20}^2l_{11}] + [l_{02} - 2l_{12} + 3l_{22} + 3l_{20}l_{02} + 5l_{11}^2 \\
&\quad - 4l_{30}l_{02} - 22l_{21}l_{11} - 12l_{12}l_{20} + 5l_{40}l_{02} + 38l_{31}l_{11} + 30l_{22}l_{20} \\
&\quad + 20l_{30}l_{12} + 29l_{21}^2 + 15l_{20}^2l_{02} + 54l_{20}l_{11}^2], \\
L_3 &= \omega^3[-l_{30} + 3l_{40} + 6l_{20}^2 - 51l_{30}l_{20} + 138l_{40}l_{20} + 94l_{30}^2 + 116l_{20}^3] \\
&\quad + 3\omega^2[l_{21} - 3l_{31} - 6l_{20}l_{11} + 20l_{30}l_{11} + 31l_{21}l_{20} - 45l_{40}l_{11} - 93l_{31}l_{20} \\
&\quad - 94l_{30}l_{21} - 116l_{20}^2l_{11}] + 3\omega[-l_{21} + 3l_{22} + 2l_{20}l_{02} + 4l_{11}^2 - 5l_{30}l_{02} \\
&\quad - 30l_{21}l_{11} - 16l_{12}l_{20} + 9l_{40}l_{02} + 72l_{31}l_{11} + 57l_{22}l_{20} + 38l_{30}l_{12} \\
&\quad + 56l_{21}^2 + 24l_{20}^2l_{02} + 92l_{20}l_{11}^2] + [l_{03} - 3l_{13} - 6l_{02}l_{11} + 6l_{03}l_{20} \\
&\quad + 30l_{12}l_{11} + 15l_{21}l_{02} - 27l_{31}l_{02} - 81l_{22}l_{11} - 30l_{13}l_{20} - 10l_{30}l_{03} \\
&\quad - 84l_{21}l_{12} - 72l_{20}l_{11}l_{02} - 44l_{11}^3], \\
L_4 &= \omega^4[l_{40} - 24l_{30}l_{20} + 120l_{40}l_{20} + 84l_{30}^2 + 72l_{20}^3] + 4\omega^3[-l_{31} + 9l_{30}l_{11} \\
&\quad + 15l_{21}l_{20} - 38l_{40}l_{11} - 82l_{31}l_{20} - 84l_{30}l_{21} - 72l_{20}^2l_{11}] + 6\omega^2[l_{22} \\
&\quad - 2l_{30}l_{02} - 14l_{21}l_{11} - 8l_{12}l_{20} + 7l_{40}l_{02} + 62l_{31}l_{11} + 51l_{22}l_{20} + 34l_{30}l_{12} \\
&\quad + 50l_{21}^2 + 14l_{20}^2l_{02} + 58l_{20}l_{11}^2] + 4\omega[-l_{31} + 3l_{03}l_{20} + 15l_{12}l_{11} \\
&\quad + 6l_{21}l_{02} - 21l_{31}l_{02} - 72l_{22}l_{11} - 27l_{13}l_{20} - 9l_{30}l_{03} - 75l_{21}l_{12} \\
&\quad - 42l_{20}l_{11}l_{02} - 30l_{11}^3] + [l_{04} - 12l_{03}l_{11} - 12l_{12}l_{02} + 10l_{04}l_{20} \\
&\quad + 68l_{13}l_{11} + 42l_{22}l_{02} + 36l_{03}l_{21} + 48l_{12}^2 + 12l_{20}l_{02}^2 + 60l_{11}^2l_{02}].
\end{aligned} \tag{5}$$

In case the correlation surface $z = F(x, y)$ is normal, $\lambda_{rs} = 0$, $r + s \geqq 3$, and the L's reduce to:

$$L_1 = \omega[1 + l_{20} + 3l_{20}^2 + 15l_{20}^3] + [-l_{11} - 3l_{20}l_{11} - 15l_{20}^2l_{11}],$$

$$L_2 = \omega^2[l_{20} + 8l_{20}^2 + 69l_{20}^3] + 2\omega[-l_{11} - 8l_{20}l_{11} - 69l_{20}^2l_{11}] + [l_{02} + 3l_{20}l_{02} + 5l_{11}^2 + 15l_{20}^2l_{02} + 54l_{20}l_{11}^2], \tag{6a}$$

$$
\begin{aligned}
L_3 &= \omega^3[6\,l_{20}^2 + 116\,l_{20}^3] + 3\,\omega^2[-6\,l_{20}\,l_{11} - 116\,l_{20}^2\,l_{11}] + 3\,\omega[2\,l_{20}\,l_{02} \\
&\quad + 4\,l_{11}^2 + 24\,l_{20}^2\,l_{02} + 92\,l_{20}\,l_{11}^2] + [-6\,l_{02}\,l_{11} - 72\,l_{20}\,l_{11}\,l_{02} - 44\,l_{11}^3], \\
L_4 &= \omega^4[72\,l_{20}^3] + 4\omega^3[-72\,l_{20}^2 l_{11}] + 6\omega^2[14\,l_{20}^2 l_{02} + 58\,l_{20}\,l_{11}^2] + 4\omega[-42\,l_{20}\,l_{11}\,l_{02} \\
&\quad - 30\,l_{11}^3] + [12\,l_{20}\,l_{02}^2 + 60\,l_{11}^2\,l_{02}].
\end{aligned} \tag{6 b}
$$

And under the further condition that x and y are uncorrelated, i. e., that $\lambda_{11} = 0$, these still further reduce to:

$$
\begin{aligned}
L_1 &= \omega\,[1 + l_{20} + 3\,l_{20}^2 + 15\,l_{20}^3], \\
L_2 &= \omega^2\,[l_{20} + 8\,l_{20}^2 + 69\,l_{20}^3] + [l_{02} + 3\,l_{20}\,l_{02} + 15\,l_{20}^2\,l_{02}], \\
L_3 &= \omega^3\,[6\,l_{20}^2 + 116\,l_{20}^3] + 3\,\omega^2\,[2\,l_{20}\,l_{02} + 24\,l_{20}^2\,l_{02}], \\
L_4 &= 72\,\omega^4\,l_{20}^3 + 84\,\omega^2\,l_{20}^2\,l_{02} + 12\,l_{20}\,l_{02}^2.
\end{aligned} \tag{7}
$$

It is striking that in each of the L's the sum of the coefficients of each set of terms homogeneous with regard to the subscripts of the l's, that is, of the same weight, is the same from bracket to bracket, except for alternation of signs. For example, in L_4, in the first bracket there is $72\,l_{20}^3$, in the second, $-72\,l_{20}^2\,l_{11}$, in the third, $14\,l_{20}^2\,l_{02} + 58\,l_{20}\,l_{11}^2$, in the fourth, $-42\,l_{20}\,l_{11}\,l_{02} - 30\,l_{11}^3$, and in the fifth $12\,l_{20}\,l_{02}^2 + 60\,l_{11}^2\,l_{02}$. A proof that this holds in general seems to be not at all immediate, but it is not required in the present paper. Verifying that this held good in all cases so far as I went helped in checking my computations and the fact that it does hold so far as my computations are carried is important both in the immediate sequel and in contributing to the rapid convergence found in the numerical application of the formulas for the L's.

In the case that $F(x, y)$ is normal several conclusions can be drawn at once from the expressions (6) for the L's.

1. It is to be noted that $r_{xy} = l_{11}/\sqrt{l_{20}\,l_{02}}$, whence $l_{11} = r_{xy}\sqrt{l_{20}\,l_{02}}$. If I denote r_{xy} simply by r and if further I set $\sqrt{l_{02}/l_{20}} = \sigma_y/\sigma_x = \nu$, it is seen readily that,

$$L_1 = \omega\,[1 + l_{20} + 3\,l_{20}^2 + 15\,l_{20}^3] - \nu r[l_{20} + 3l_{20}^2 + 15\,l_{20}^3] = \omega + a^2(\omega - \nu r),$$

with

$$a^2 = l_{20} + 3\,l_{20}^2 + 15\,l_{20}^3.$$

Since a^2 is essentially positive, L_1 is an inverse linear function of r. For given values of m_1, σ_x, m_2, and σ_y, the maximum value of L_1 is obtained for $r = -1$ and the minimum for $r = 1$. If $r = 0$, L_1 is always greater numerically than ω. In particular, $L_1 = \omega$ only when

$$r = \frac{m_2}{\sigma_y} \Big/ \frac{m_1}{\sigma_x}.$$

If ω is positive, then L_1 is greater than or less than ω accordingly as r is less than or greater than $\frac{m_2}{\sigma_y}\Big/\frac{m_1}{\sigma_x}$. As a slight generalization, it is to be noted that in case ω is positive, any negative value of r insures that L_1 is greater than ω. And if the ratio $\frac{m_2}{\sigma_y}\Big/\frac{m_1}{\sigma_x}$ is at all near unity, then in the case that ω is positive, it will require a high positive value of r for L_1 to be less than ω even in this case.

2. Using the same notation, it is easy to obtain,

$$L_2 = b^2(\omega - r\nu)^2 + c^2\nu^2(1 - r^2)$$

with

$$b^2 = l_{20} + 8l_{20}^2 + 69l_{20}^3,$$

and

$$c^2 = l_{20} + 3l_{20}^2 + 15l_{20}^3 = a^2.$$

Since b^2 and c^2 are both essentially positive, L_2 can only vanish for $r = \pm 1$, and then only if also,

$$\frac{m_2}{m_1} = \pm\frac{\sigma_y}{\sigma_x}$$

which it is readily seen is the condition that the two series in x's and y's be identical except for a proportionality factor, or that y/x be constant, which conclusion was, of course, anticipated. By inspection of the expression for L_2 in (6) it is apparent that the maximum value for m_1, m_2, σ_x, and σ_y given is attained for $r = \mp 1$ accordingly as ω is positive or negative.

3. By a similar reduction, I get,

$$L_3 = (\omega - \nu r)\,[d^2(\omega - \nu r)^2 + 6e^2\nu^2(1 - r^2)],$$

with

$$d^2 = 6l_{20}^2 + 116l_{20}^3,$$

and

$$e^2 = l_{20}^2 + 12l_{20}^3.$$

The expression inside the bracket is positive or zero and, as in the case immediately preceding, it can assume the latter value only if y/x be constant. And the factor outside the bracket is positive, zero, or negative accordingly as r is less than, equal to, or greater than $\frac{m_2}{\sigma_y}\Big/\frac{m_1}{\sigma_x}$. This last is also the condition that the skewness be positive, zero, or negative. In particular, if ω be positive, the skewness will be positive for any negative value of r, for $r = 0$, and for any positive value of r up to $\frac{m_2}{\sigma_y}\Big/\frac{m_1}{\sigma_x}$.

To gain an idea of the magnitude of the skewness,

$$\alpha_3 = \frac{L_3}{L_2^{3/2}} = \frac{d^2\left[(\omega - \nu r)^3 + \frac{6e^2}{d^2}\nu^2(1-r^2)(\omega - \nu r)\right]}{b^3\left[(\omega - \nu r)^2 + \frac{c^2}{b^2}(1-r^2)\nu^2\right]^{3/2}}$$

it is only necessary to estimate it from the expression itself. It will help to note that d^2/b^3 is rather closely approximated by $6\sqrt{l_{20}}$ and that neither $6e^2/d^2$ nor c^2/b^2 differs greatly from unity. The quotient of the bracketed terms is rather a variable quantity but in ordinary cases it will be of the order of magnitude of unity. In the example appended in which $\omega = \nu = 1$ and $r = .60$, $d^2/b^3 = .464$, and $6\sqrt{l_{20}} = .449$. The quotient of the bracketed terms, on taking $6e^2/d^2$ and c^2/b^2 each as unity, gives .447 and this times $6\sqrt{l_{20}}$ gives .201 for the skewness whereas the value of $L_3/L_2^{3/2}$ obtained by using all the terms of (6) gave .210.

4. After somewhat more algebraic manipulation, I obtain,

$$L_4 = 12 l_{20}^3 [6(\omega - \nu r)^2 + \nu^2(1-r^2)]\,[(\omega - \nu r)^2 + \nu^2(1-r^2)].$$

As it is to be expected this can vanish only in the case that y/x is constant. That is, except for this trivial case, the distribution of y/x's always has positive excess. Again it is apparent from the expression in (6) that the maximum value for L_4 is attained for $r = \pm 1$, accordingly as ω is negative or positive.

To gain an idea of the order of magnitude of

$$\alpha_4 - 3 = \frac{L_4}{L_2^2},$$

I estimate it, in the same manner as for $L_3/L_2^{3/2}$, to be of the order of $72 l_{20}$. In the example referred to $72 l_{20} = .403$ and $L_4/L_2^2 = .130$.

In the case that x and y are uncorrelated, $r = 0$ and the remarks just made apply when r is set equal to zero throughout. But in this case it is more simple to look directly at the expressions (7). I read off from them that in the case that $z = F(x, y)$ is normal with zero correlation,

1. The mean of y/x is always greater than ω.

2. L_2 is rather closely approximated by $\omega^2 l_{20} + l_{02}$.

3. The skewness, $L_3/L_2^{3/2}$, is always positive, except for $\omega = 0$ (in which case it is zero), and can be estimated from $6\sqrt{l_{20}} = 6\sigma_x/m_1$.

4. The excess L_4/L_2^2 is always positive and can be estimated from $72 l_{20}$ unless ω is small, in which case the value will be between $72 l_{20}$ and $12 l_{20}$.

3. **Numerical tests of results.** By way of illustration, application, and check of the theory developed, I have carried out two numerical examples in all detail. The second and more elaborate of these has, I think, considerable interest in its own right.

Example 1. I tried here to get as simple a check as possible on the numerical usefulness and validity of the method. For this purpose I employed some data I had obtained in connection with my thesis. From a normal parent distribution I took 400 samples of two hundred values of the variate each and computed the characteristics of each sample. In this way I obtained distributions of 400 sample means, of 400 sample second moments about the mean, etc. (See my thesis for more details.) The distributions of ν_2's and ν_3's I took as my distributions of x's and y's, each y/x being the quotient of the ν_3 by the ν_2 for the same sample. (For a study of the distribution of the quotients, $\nu_3/\nu_2^{3/2}$, more interesting in themselves, also see my thesis.) Here the distributions of both x and y are both very close to normal and r_{xy} is very close to zero.

The accompanying correlation table shows the actual data used. (See p. 480.) For the frequency distribution of ν_2's I computed the characteristics as follows: (ν_2's and ν_3's were each tabulated to the nearest .001)

$$M_x = 4.077, \quad \lambda_2 = 0.1309, \quad \alpha_3 = 0.0502 = \frac{\lambda_3}{\lambda_2^{3/2}},$$

$$\sigma_x = 0.3619, \quad \lambda_3 = 0.0024, \quad \alpha_4 - 3 = -0.1834 = \frac{\lambda_4}{\lambda_2^2},$$

$$\lambda_4 = 0.0031.$$

And for the marginal frequency distribution of sample ν_3's, I found the following characteristics:

$$M_x = 0.0735, \quad \lambda_2 = 1.9076, \quad \alpha_3 = -0.0629,$$

$$\sigma_x = 1.3811, \quad \lambda_3 = -0.1656, \quad \alpha_4 - 3 = 0.1205,$$

$$\lambda_4 = 0.4383.$$

Then I computed r_{xy} from the table itself obtaining the value, -0.0605—, with a probable error of .032. This appeared, then, to be as near to a normal frequency distribution of two independent variables as one could hope to encounter barring artificial examples and in what follows the formulas (7) were used.

Next I computed the value of $y/x = \nu_3/\nu_2$ for each of the samples and tabulated the results as follows: (Values were recorded to the nearest .001.)

CORRELATION TABLE FOR SAMPLE ν_3'S AND ν_8'S.

Class Intervals $\nu_3 = x$ / $\nu_8 = y$	4.90–5.039	4.76–	4.62–	4.48–	4.34–	4.20–	4.06–	3.92–	3.78–	3.64–	3.50–	3.36–	3.22–	3.08–	2.94–	2.80–2.939	f_y	y'	$x'y'f_{xy}$
—5.20—4.401	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	1	—6	—30
—4.40–	..	..	..	..	..	..	..	1	1	..	..	..	..	..	..	..	2	—5	5
—3.60–	..	..	..	..	1	..	..	1	1	..	1	..	..	..	..	..	4	—4	4
—2.80–	..	1	..	2	2	4	3	4	3	..	..	..	..	..	..	..	19	—3	—84
—2.00–	..	3	3	4	4	2	7	5	3	5	3	2	..	..	..	..	41	—2	—84
—1.20–	..	4	3	5	8	14	13	13	8	5	4	2	1	..	..	..	80	—1	—81
—0.40–	2	4	..	4	11	8	16	11	16	9	7	2	1	1	..	..	92	0	0
0.40–	..	1	3	2	4	11	14	14	8	13	6	3	4	..	..	1	84	1	—15
1.20–	1	1	..	1	6	4	4	5	5	4	5	..	1	..	..	..	37	2	28
2.00–	..	..	3	5	3	4	2	8	1	4	1	..	..	..	..	..	31	3	126
2.80–	..	..	..	1	1	1	1	1	2	1	..	..	..	..	..	..	8	4	24
3.60–4.399	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	1	5	20
f_x	3	14	13	25	40	48	60	63	48	41	27	9	7	1	0	1	400	..	—87
x'	7	6	5	4	3	2	1	0	—1	—2	—3	—4	—5	—6	—7	—8	..		
$x'y'f_{xy}$	14	—60	—15	36	9	10	—4	0	3	—44	—15	12	—25	0	0	—8	—87		

Class intervals		$f_{y/x}$
−1.05−	−0.951	1
−0.95−		3
−0.85−		1
−0.75−		3
−0.65−		9
−0.55−		15
−0.45−		21
−0.35−		25
−0.25−		38
−0.15−		48
−0.05−		49
0.05−		52
0.15−		39
0.25−		33
0.35−		18
0.45−		20
0.55−		12
0.65−		7
0.75−		3
0.85−	0.949	3
		400

Computation gave for the characteristics

$$m_{y/x} = 0.0232,$$
$$\sigma_{y/x} = 0.3340,$$
$$\lambda_{2:y/x} = 0.1115+,$$
$$\lambda_{3:y/x} = -0.0013,$$
$$\lambda_{4:y/x} = 0.0006,$$
$$\alpha_{3:y/x} = -0.0337,$$
$$\alpha_{4:y/x}-3 = 0.0467.$$

Then setting $\omega = 0$ in the formulas (7) and using

$$l_{20} = \frac{\lambda_{20}}{m_1^2} = 0.007\,876 \quad \text{and} \quad l_{02} = \frac{\lambda_{02}}{m_1^2} = 0.1148$$

these gave the following predicted values to be compared with the above observed values:

$$m_{y/x} = 0.0,$$
$$\sigma_{y/x} = 0.3429,$$
$$\lambda_{2:y/x} = 0.1176,$$
$$\lambda_{3:y/x} = 0.0,$$
$$\lambda_{4:y/x} = 0.001\,246,$$
$$\alpha_{3:y/x} = 0.0,$$
$$\alpha_{4:y/x}-3 = 0.0901.$$

In each case it will be found that the difference between the predicted and the observed value is less than the standard deviation of the quantity predicted. (Of course, here the error of prediction is not a function of the size of the sample alone.)

NORMAL CORRELATION SURFACE CONSTRUCTED USING
$r_{xy} = .60$, $\sigma_x = \sigma_y = 2.0$, AND $n = 1000$.

Classes x / y	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	f_y
20	..	.1	.1	.1	.1	.1	..	..	..	..	..	..	..	..	..	0.5
21	.1	.2	.4	.6	.6	.4	.2	.1	..	..	..	..	..	..	..	2.6
22	.1	.4	1.0	1.8	2.3	1.9	1.1	.4	.1	..	..	..	..	..	..	9.1
23	.1	.6	1.8	4.1	6.3	6.7	4.8	2.4	.8	.2	..	..	..	..	..	27.8
24	.1	.6	2.3	6.3	12.1	15.9	14.4	8.9	3.8	1.1	.2	..	..	..	..	65.7
25	.1	.4	1.9	6.7	15.9	26.1	29.4	22.8	12.1	4.4	1.1	.2	..	..	..	121.1
26	..	.2	1.1	4.8	14.4	29.4	41.4	40.0	26.6	12.1	3.8	.8	.1	..	..	174.7
27	..	.1	.4	2.4	8.9	22.8	40.0	48.2	40.0	22.8	8.9	2.4	.4	.1	..	197.4
28	..	..	.1	.8	3.8	12.1	26.6	40.0	41.4	29.4	14.4	4.8	1.1	.2	.1	174.7
29	..	..	..	.2	1.1	4.4	12.1	22.8	29.4	26.1	15.9	6.7	1.9	.4	.1	121.1
30	..	..	..	..	.2	1.1	3.8	8.9	14.4	15.9	12.1	6.3	2.3	.6	.1	65.7
31	..	..	..	..	..	.2	.8	2.4	4.8	6.7	6.3	4.1	1.8	.6	.1	27.8
32	..	..	..	..	..	..	.1	.4	1.1	1.9	2.3	1.8	1.0	.4	.1	9.1
33	..	..	..	..	..	..	..	.1	.2	.4	.6	.6	.4	.2	..	2.6
34	..	..	..	..	..	..	..	..	..	.1	.1	.1	.1	.1	..	0.5
f_x	0.5	2.6	9.1	27.8	65.7	121.1	174.7	197.4	174.7	121.1	65.7	27.8	9.1	2.6	0.5	1000.4

Example 2. The case in which both ω and r_{xy} are both zero is certainly very special and naturally the second example chosen has neither of these features. Moreover it occured to me to attempt to test the theory free from the perturbing effects of sampling in $F(x, y)$. These latter could be studied separately by means of considering the variation in the terms that compose the L's due to sampling and hence the variation in the L's themselves. Given a $z = F(x, y)$ which is an ideally normal correlation surface, prediction of the constants for the distribution of y/x's by means of any theoretical results will be a check on the theory by which they were obtained alone.

With this in mind I constructed the accompanying normal correlation table. (See p. 482). In order to obtain the symmetry that should be exhibited by such a table when cell frequencies are saved to the nearest tenth I found it necessary to perform the graduation, using a table of areas under the normal curve, with class widths half as wide as shown, thus obtaining first a correlation table with four times as many cells, with each cell-frequency saved to the nearest hundredth. Then the cells were combined in fours and total frequencies were saved to the nearest tenth. Even then I had two cells which read 14.4 and two others in symmetric positions which read 14.3. But since I had previously performed the graduation with fifteen classes each way, I was able to tell the effect of using finer divisions and thus to choose 14.4 as the frequency in all four cells with no further calculation.

Graduation with fifteen classes each way served very well to get a symmetric table if cell-frequencies were saved to the nearest unit. But for this table I found that $\lambda_{40} = \lambda_{04} = -2.30$ which is a quantity of the same order of magnitude as $\lambda_{20} = 3.87$, making l_{40} a quantity that could not be neglected in comparison with l_{20}^3.

Values of x and y were assigned as shown, making $M_x = M_y = 27$ and thus $\omega = 1$. This may seem to be making this case too special also but this is a matter of appearance rather than reality. Inspection of the formulas (6) shows that if all the y's were doubled, making $\omega = 2$, the decreasing powers of ω in successive terms is precisely offset in effect by the increasing orders of y-subscripts. If ω were increased without increasing σ_y proportionately, the effect would be to increase the importance of the first groups of terms of the L's at the expense of the latter ones but the nature of the convergence in each group would not be affected at all. The relative contributions of groups of terms of the same weight would still be unchanged in any L.

After the graduation was completed as shown, the following characteristics for the correlation table were obtained:

$$n = 1000.4, \qquad m_x = m_y = 27.0.$$
$$\lambda_{20} = \lambda_{02} = 4.0800, \quad \lambda_{11} = 2.4140, \quad \lambda_{30} = \lambda_{03} = 0.0, \quad \lambda_{40} = \lambda_{04} = -0.3422,$$
$$\sigma_x = \sigma_y = 2.0199, \quad \alpha_{3:x} = \alpha_{3:y} = 0.0, \quad \alpha_{4:x} - 3 = \alpha_{4:y} - 3 = -0.02056,$$
$$r_{xy} = 0.5917.$$

(Sheppard's corrections were not used in the above computations.) These are, of course, the characteristics to be used for $F(x, y)$.

Next the l's to be used in calculating the predicted characteristics from the formulas (6) were found to be,

$$l_{20} = l_{02} = 0.0055967, \qquad l_{11} = 0.0033114.$$

With these set in (6), I obtained for the predicted L's:

$$L_1 = 1.002325, \qquad L_2 = 0.004678,$$
$$L_3 = 0.0000672, \qquad L_4 = 0.00000284,$$

with accordingly,

$$\sigma_{y|x} = 0.06840, \qquad \alpha_{3:y|x} = 0.210.$$
$$\alpha_{4:y|x} - 3 = 0.130, \qquad \mu_{4:y|x} = 0.00006849 = L_4 + 3L_2^2.$$

Then from the correlation table the actual distribution of y/x's was computed. Simply each cell-frequency was assigned to the quotient y/x for that cell. One tabulation of the resulting y/x's into a frequency distribution was as follows: (Values of y/x's were recorded to the nearest .001.)

Class intervals	$f_{y\|x}$
0.777–0.806	0.9
0.807–	3.9
0.837–	13.6
0.867–	36.2
0.897–	75.3
0.927–	74.0
0.957–	187.3
0.987–	218.0
1.017–	187.3
1.047–	47.4
1.077–	76.1
1.107–	53.6
1.137–	12.8
1.167–	9.2
1.197–	3.2
1.227–	1.2
1.257–1.286	0.4
	1000.4

The characteristics of this distribution were found to be

$$m_{y|x} = 1.0034,$$
$$\sigma_{y|x} = 0.0672,$$
$$\lambda_{2:y|x} = 0.00451 = \mu_{2:y|x},$$
$$\lambda_{3:y|x} = 0.0000720 = \mu_{3:y|x},$$
$$\mu_{4:y|x} = 0.0000689,$$
$$\lambda_{4:y|x} = 0.00000784,$$
$$\alpha_{3:y|x} = 0.238,$$
$$\alpha_{4:y|x} - 3 = 0.385+.$$

(All the values in the 0.987–1.016 class are precisely 1.000 and were so taken in the calculation.)

There is some agreement between predictions and observations but particularly for the means and fourth semi-invariants it is not as good as it ought to be. An inspection of the serial tabulation of y/x's suggested that the values were not evenly enough distributed in each of the frequency classes to justify the use of classes so wide. To test this the data were retabulated using 0.010 as the class width. This new frequency distribution gave the following results:

$$\begin{aligned} m_{y/x} &= 1.0022, & \mu_{4:y/x} &= 0.0000676, \\ \sigma_{y/x} &= 0.0685—, & \lambda_{4:y/x} &= 0.00000157, \\ \lambda_{2:y/x} &= 0.00469, & \alpha_{3:y/x} &= 0.213, \\ \lambda_{3:y/x} &= 0.0000684, & \alpha_{4:y/x} - 3 &= 0.071. \end{aligned}$$

Matters were so much improved that I set myself the laborious task of computing the characteristics serially, that is, from the values of y/x obtained from the correlation table with no loss of detail from grouping. The new results were,

$$\begin{aligned} m_{y/x} &= 1.0023255, & \mu_{4:y/x} &= 0.00006738, \\ \sigma_{y/x} &= 0.0683, & \lambda_{4:y/x} &= 0.00000204, \\ \lambda_{2:y/x} &= 0.00467, & \alpha_{3:y/x} &= 0.207, \\ \lambda_{3:y/x} &= 0.0000660, & \alpha_{4:y/x} - 3 &= 0.094. \end{aligned}$$

Now I think it will be admitted that the agreement is very excellent except in the case of L_4 and $\alpha_4 - 3$. The first consideration with respect to this discrepancy is naturally concerning the effect of the terms of higher order omitted. I found that l_{40} l_{20} and l_{60} are approximately -0.000000004 and 0.00000004 respectively and I think it can be assumed without computation that the other similar sixth order terms are of the same order of magnitude. It looks then as if terms of the second kind might affect the value of L_4 in the seventh decimal place. I also found the correction obtained by including the eighth order terms under the assumption that $z = F(x, y)$ is perfectly normal. This *adds* 0.00000024 to L_4 which only makes things a little worse. Of course the correlation table is not a normal correlation table and this leads to what appears to me a more important consideration. I have had to use x and y as discrete variates whereas the theory is developed and the results are obtained for x and y continuous. That is, there is an inherent grouping in the table itself which cannot be corrected for and which introduces a systematic error in defect in the even

moments of the distribution of y/x's compared with what would be found if the x and y were really continuous. After comparing the three different sets of values computed with more and less grouping in the distributions it seems to me quite reasonable to suppose that this is a sufficient explanation for the difference. If I had a normal $z = F(x, y)$ with the data listed serially instead of in frequency-cells this supposition could be tested!

In concluding this section it ought to be pointed out that the formulas (6) will still give very good results in case $z = F(x, y)$ is not normal but does not depart greatly from normality. To judge of the practicability of this procedure it is only necessary to compute the magnitude of the terms in (5) which are omitted in (6). Particularly for the mean and the standard deviation will (6) be found to give good results in a wide class of such cases without the use of the more extended expressions (5).

3. **The extension of the method to y/x^2, etc.** Obviously the method employed in developing the characteristics of y/x from those of $F(x, y)$ can be extended to find the characteristics also of y/x^2, y/x^3, y^2/x, and so on. In my thesis, as I have previously mentioned, I had occasion to study the distributions of $y/x^{3/2}$ and y/x^2 and the method of this paper was developed therein. By a mere transcription of one set of results obtained there, I have for y/x^2; setting now $\omega = m_2/m_1^2$ and $l_{rs} = \lambda_{rs}/m_1^{r+2s}$, in case $z = F(x, y)$ is normal, up to and including terms of the sixth order,

$$
\begin{aligned}
L_1 &= \omega(1 + 3l_{20} + 15l_{20}^2 + 105l_{20}^3) + (-2l_{11} - 12l_{20}l_{11} - 90l_{20}^2 l_{11}),\\
L_2 &= \omega^2(4l_{20} + 66l_{20}^2 + 960l_{20}^3) + 2\omega(-2l_{11} - 42l_{20}l_{11} - 648l_{20}^2 l_{11})\\
&\quad + (l_{02} + 10l_{20}l_{02} + 16l_{11}^2 + 105l_{20}^2 l_{02} + 372l_{20}l_{11}^2),\\
L_3 &= \omega^3(72l_{20}^2 + 2664l_{20}^3) + 3\omega^2(-40l_{20}l_{11} - 1680l_{20}^2 l_{11}) + 3\omega(8l_{20}l_{02} + 14l_{11}^2\\
&\quad + 228l_{20}^2 l_{02} + 816l_{20}l_{11}^2) + (-12l_{11}l_{02} - 408l_{20}l_{11}l_{02} - 232l_{11}^3),\\
L_4 &= \omega^4(2496l_{20}^3) + 4\omega^3(-1368l_{20}^2 l_{11}) + 6\omega^2(176l_{20}^2 l_{02} + 648l_{20}l_{11}^2)\\
&\quad + 4\omega(-288l_{20}l_{11}l_{02} - 180l_{11}^3) + (48l_{20}l_{02}^2 + 216l_{11}^2 l_{02}).
\end{aligned}
$$

A study of my thesis will enable the reader to get similar results for other indices.

(NOTE.) Since the completion of the above paper there has appeared in Biometrika; vol. 20A, Parts 1 and 2, July, 1928; a paper entitled, "Frequency distribution of an index when both the components follow the normal law", by A. S. Merrill. He has carried out an investigation along "Pearsonian" lines in the case that both x and y are normally distributed.

ON A THEOREM OF KUMMER'S CONCERNING POWER CHARACTERS OF UNITS IN A CYCLOTOMIC FIELD.*

By H. S. Vandiver.

Consider the cyclotomic field $k(\alpha)$, where $\alpha = e^{2i\pi/l}$ and l is an odd prime. If ω is an integer in this field and $\mathfrak{q}$ is an ideal prime, then $\omega^{N(\mathfrak{q})-1} \equiv 1 \pmod{\mathfrak{q}}$ where $N(\mathfrak{q})$ is the norm of $\mathfrak{q}$ in $k(\alpha)$ and (ω) is prime to $(\mathfrak{q})$. Also there is one and but one integer a in the set $(0, 1, 2, \cdots, l-1)$ such that

$$\omega^{\frac{N(\mathfrak{q})-1}{l}} \equiv \alpha^a \pmod{\mathfrak{q}}$$

provided $\mathfrak{q}$ is prime to (l). We write

$$\left\{\frac{\omega}{\mathfrak{q}}\right\} = \alpha^a.$$

The present paper is concerned with these power characters for the case where ω is a unit in $k(\alpha)$.

Kummer† proved that, if

$$E_n(\alpha) = e(\alpha)^{1+sr^{-2n}+s^2r^{-4n}+\cdots+s^{l'-1}r^{-2n(l'-1)}}$$

where $l' = (l-1)/2$, r is a primitive root of l; the exponent in this relation is written in the notation of the Kronecker-Hilbert‡ symbolic powers and

$$e(\alpha) = \left[\frac{(1-\alpha^r)(1-\alpha^{-r})}{(1-\alpha)(1-\alpha^{-1})}\right]^{\frac{1}{2}} = \pm\frac{\alpha^{\frac{1}{2}(1-r)}(1-\alpha^r)}{1-\alpha}$$

then

$$(1) \quad \text{ind. } E_n(\alpha) \equiv \frac{r^{2n}-1}{2(1+a^{l-2n}-(a+1)^{l-2n})} \cdot \frac{d_0^{l-2n}\log \psi_a(e^v)}{dv^{l-2n}}, \pmod{l};$$

where

$$\left\{\frac{E_n(\alpha)}{\mathfrak{q}}\right\} = \alpha^{\text{ind. } E_n(\alpha)},$$

a is any integer not congruent to 0 or -1 modulo l,

$$\psi_a(\alpha) = \sum \alpha^{-(a+1)h+\text{ind.}(g^h+1)},$$

$\mathfrak{q}$ is an ideal prime in $k(\alpha)$ whose norm is q^t, q odd, g is a primitive root of $\mathfrak{q}$, h ranges over the integers $0, 1, 2, \cdots, q^t-2$ excepting $(q^t-1)/2$, and g is selected so that $g^{(q^t-1)/l} \equiv \alpha \pmod{\mathfrak{q}}$; whence $g^{\text{ind } x} \equiv x \pmod{\mathfrak{q}}$. The symbol

* Received November 10, 1928.

† Journ. für Math., 44 (1852), 121–130.

‡ Hilbert, Die Theorie der algebraischen Zahlkörper, Ber. Deut. Math. Ver., 1894, 271.

$$\frac{d_0^{l-2n} \log \psi_a(e^v)}{dv^{l-2n}}$$

means that the $(l-2n)$th derivative of $\log|\psi_a(e^v)|$ is taken with respect to v and $v=0$ substituted in the result, where e is the Napierian base.

In an article* by the writer it was noted that Kummer's Theorem was subject to the restriction that q is odd as the function $\psi_a(\alpha)$ has no meaning for $q=2$. In the latter case we may take the function, however, as

$$\psi_a(\alpha) = \alpha^{-(a+1)h+\text{ind.}(g^h+1)}$$

with h ranging over the integers $1, 2, \cdots, 2^t-2$, and Kummer's argument will also hold with slight modification. As his work in connection with this theorem is all given in the language of his theory of ideals and this particular result is not included apparently in any of the theorems of succeeding writers on the power characters of units in a cyclotomic field, I shall give Kummer's argument here in modern form as in later papers I intend to make extensive use of these results.

In another paper† I use similar methods in deriving analogous results concerning power characters in the cyclotomic field defined by a primitive pnth root of unity where n is prime to p. These theorems, however, are given in a form such that the special case of power characters corresponding to pth roots of unity is not included. Set

$$\psi_a(e^v) = \sum e^{v(-(a+1)h+\text{ind.}(g^h+1))}$$

where the summation extends over the values $h=0, 1, 2, \cdots, q^t-2$ with the exception of $h=(q^t-1)/2$ if q is odd and 0 if q is even, with e the Napierian base. If $\log \psi_a(e^v)$ stands for the Napierian logarithm of ψ_a, then

$$\frac{d \log \psi_a(e^v)}{dv} = \frac{d\psi_a(e^v)}{dv} \cdot \frac{1}{\psi_a(e^v)}.$$

It then follows from Leibnitz's theorem that

$$(2) \qquad \frac{d^{l-2n} \log \psi_a(e^v)}{dv^{l-2n}} = \frac{d^{m+1} \psi_a(e^v)}{dv^{m+1}} U + \frac{m}{1} \frac{d^m \psi_a(e^v)}{dv^m} \frac{dU}{dv} + \cdots$$

where $l-2n-1 = m$ and $1/\psi_a(e^v) = U$.

We have‡

$$\psi_a(\alpha)\, \psi_a(\alpha^{-1}) = q^t,$$

* Bull. Amer. Math. Soc., 28 (1922), 400–407.

† Amer. Jour. Math., 47 (1925), 140–147.

‡ Kummer, l. c., 111.

hence the expression $\Psi_a(u)\,\Psi_a(u^{-1}) - q^t$ vanishes for $u = \alpha$, hence, since the function $(u^p - 1)/(u - 1)$ is irreducible in the rational field, the former function is divisible by the latter and, if we put $u = e^v$, we may write

$$\Psi_a(e^v)\,\Psi_a(e^{-v}) = q^t + VW,$$

where $V = 1 + e^v + e^{2v} + \cdots + e^{(l-1)v}$ and W is a rational function in e^v with rational integral coefficients. Then

$$\Psi_a(e^{-v}) = \frac{q^t + VW}{\Psi_a(e^v)} = (q^t + VW)\,U,$$

and therefore

$$\frac{d^i\,\Psi_a(e^{-v})}{dv^i} = \frac{d^i\,U}{dv^i}(q^t + VW) + \frac{i}{1}\,\frac{d^{i-1}\,U}{dv^{i-1}}\,\frac{d(VW)}{dv} + \cdots.$$

We have

$$\frac{d_0^i(V)}{dv^i} \equiv 0 \pmod{l}, \qquad i = 0, 1, \cdots, l-2,$$

where the symbol at the left indicates that the ith derivative of V is taken with respect to v and $v = 0$ substituted in the result. This congruence applied to the last equation given yields

$$\frac{d_0^i\,\Psi_a(e^{-v})}{dv^i} \equiv \frac{d_0^i\,U}{dv^i}, \mod l,$$

where i ranges as before.

Using this in connection with (2) we have

$$\frac{d_0^{l-2n}\log\Psi_a(e^v)}{dv^{l-2n}} \equiv D_{m+1}\,D_0 - \frac{m}{1}\,D_m\,D_1 + \frac{m(m-1)}{1\cdot 2}\,D_{m-1}\,D_2 - \cdots, \mod l,$$

where $D_i = d_0^i\,\Psi_a(e^v)/dv^i$. Carrying out the differentiation of the D's and setting

$$-(a+1)h + \text{ind.}(g^h+1) = C_h, \qquad -(a+1)k + \text{ind.}(g^k+1) = C_k,$$

we obtain the congruence

$$\frac{d_0^{l-2n}\log\Psi_a(e^v)}{dv^{l-2n}} \equiv \sum\sum C^{m+1}\,C_k^0 - \frac{m}{1}\,C_h^m\,C_k + \frac{m(m-1)}{1\cdot 2}\,C_h^{m-1}\,C_k^2 - \cdots$$

modulo l where the double summation sign indicates that h and k range independently over the values 0, 1, $\cdots$, $q^t - 2$ with the exception of $h = (q^t-1)/2$ and $k = (q^t-1)/2$ if q is odd and $h = 0$, $k = 0$ if q is even. This relation may be written

$$\frac{d_0^{l-2n}\log\Psi_a(e^v)}{dv^{l-2n}} \equiv \sum\sum C_k\,(C_h - C_k)^m,$$

or, what is the same thing,

$$\frac{d_0^{l-2n} \log \Psi_a(e^v)}{dv^{l-2n}} \equiv \sum\sum(-(a+1)h + \text{ind.}(g^h+1))(-(a+1)(h-k) + \text{ind.}(g^h+1) - \text{ind.}(g^k+1))^{l-2n-1}.$$

Consider the relation

$$g^{k-h} \equiv g^k, \qquad \frac{g^k+1}{g^h+1} \equiv g^h, \mod q;$$

if we except the values for which $h = k$, in which case it is the congruence (3), the terms in the right hand member of congruence (3) vanish. We may observe that corresponding to any set (h', k') there is a unique set (h, k) unless $k' = h'$. Conversely, for any set (h, k) with h not equal to k there is a unique set (h', k') where both are included in the set $1, 2, \cdots, q^t-2$ and h' not equal to k'. The right hand member of (3) then becomes

$$\sum\sum(-(a+1)\text{ind.}(g^{h'}-1)+\text{ind.}(g^{k'}-1)+a\,\text{ind.}(g^{k'}-g^{h'}))((a+1)k'-h')^{l-2n-1}$$

where h' and k' range as stated. For simplicity write $h = h'$ and $k = k'$. This sum may be written in three parts as follows:

$$-(a+1)\sum\sum \text{ind.}(g^h-1)((a+1)k-h)^{l-2n-1},$$
$$\sum\sum \text{ind.}(g^k-1)((a+1)k-h)^{l-2n-1},$$
$$a\sum\sum \text{ind.}(g^k-g^h)((a+1)k-h)^{l-2n-1},$$

and the summation of each part we consider separately.

The first part may be written in the form

$$a^{l-2n-1}(a+1)\sum h^{l-2n-1}\text{ind.}(g^h-1)-(a+1)\sum\sum\text{ind.}(g^h-1)((a+1)k-h)^{l-2n-1}$$

where now h and k range independently over all values from 1 to q^t-2 inclusive without exception. Carrying out the summation with respect to powers of k we have, since the sum of $k^i \equiv 0$ for $i = 1, 2, 3, \cdots, l-2$ and q^t-1 is congruent to 0 modulo l,

$$\sum((a+1)k-h)^{l-2n-1} \equiv (q^t-2)h^{l-2n-1} \equiv -h^{l-2n-1},$$

so that the first part of our sum becomes

$$(a+1)(1+a^{l-2n-1})\sum h^{l-2n-1}\,\text{ind.}(g^h-1).$$

In a similar way we find for the second part

$$-(a^{l-2n-1}+(a+1)^{l-2n-1})\sum h^{l-2n-1}\,\text{ind.}(g^h-1).$$

The third part requires somewhat different treatment. Introduce in it the terms corresponding to $h = 0$ and $k = 0$ with h not equal to k. In order to do this we subtract from the original expression the quantity

$$a \sum h^{l-2n-1} \text{ ind. } (g^h - 1) + a\,(a+1)^{l-2n-1} \sum k^{l-2n-1} \text{ ind. } (g^k - 1).$$

With the values zero of h and k inserted we take the third part and we can write $h + k$ instead of h so that it becomes

$$a \sum \sum (\text{ind. } (g^h - 1) + k)\,(ak - h)^{l-2n-1},$$

where h ranges over the values 1, 2, 3, $\cdots$, $q^t - 2$. Keeping h fixed we carry out the summation by expanding and allowing k to vary and this gives the result that the expression is congruent to zero modulo l. The third part then becomes

$$-a\,(1 + (a+1)^{l-2n-1}) \sum h^{l-2n-1} \text{ ind. } (g^h - 1),$$

hence our entire original sum reduces to

$$\frac{d_0^{l-2n} \log \Psi_a(e^v)}{dv^{l-2n}} = (1 + a^{l-2n} - (a+1)^{l-2n}) \sum h^{l-2n-1} \text{ ind. } (g^h - 1)$$

for $h = 1, 2, 3, \cdots, q^t - 2$.

To transform the right hand member of the above relation we note that

$$\sum h^{l-2n-1} \text{ ind. } (g^h - 1) \equiv \sum \sum i^{l-2n-1} \text{ ind. } (g^{i+kl} - 1)$$

for $i = 1, 2, 3, \cdots, l-1$, $k = 0, 1, 2, \cdots, c-1$, where $c = (q^t - 1)/l$. But we also have

$$(g^i - 1)\,(g^{i+l} - 1)\,(g^{i+2l} - 1) \cdots (g^{i+(c-1)l} - 1) \equiv 1 - g^{ci}, \mod q,$$

and

$$\sum \text{ind. } (g^{i+kl} - 1) \equiv \text{ind. } (1 - g^{ci}) \equiv \text{ind. } (1 - \alpha^i), \mod l,$$

since $g^c \equiv \alpha \pmod q$. Carrying out the summation with respect to k we have

$$\sum h^{l-2n-1} \text{ ind. } (g^h - 1) \equiv \sum i^{l-2n-1} \text{ ind. } (1 - \alpha^i)$$

for $i = 1, 2, 3, \cdots, l-1$. Setting in this relation ir instead of i, multiplying by r^{2n} and subtracting the second congruence from the original we have

$$(r^{2n} - 1) \sum h^{l-2n-1} \text{ ind. } (g^h - 1) \equiv \sum i^{l-2n-1} \text{ ind. } \left(\frac{1 - \alpha^{ri}}{1 - \alpha^i}\right),$$

expressing i as a power of l and then it is easy to see that the expression on the right hand side of the above congruence is twice the index of the unit $E_n(\alpha)$ which yields the relation (1).

ON THE GEOMETRY OF THE WEDDLE SURFACE.

By Walter L. Moore.*

1. Introduction. The general Weddle surface, W, was first defined as the locus of the vertices of quadric cones which pass through six generic points $A_1, A_2, A_3, A_4, A_5, A_6$ in space.[1] It appears thus as the Jacobian[2] of a Webb of quadrics having the six points as base points. These points are nodes on the surface. It contains the 15 lines, l_{ik}, joining the nodes $A_i A_k$, the 10 lines g_{ijk}, of intersection of the pairs of planes on the nodes $A_i A_j A_k$ and $A_m A_n A_o$ and the twisted cubic determined by the six points.[3] This curve, which is contained in the quadric cones tangent to W at the nodes, is asymptotic to the surface. A model of W designed to show these elementary properties has been constructed.[4]

An historical account of the application of hyperelliptic functions to the problem of uniformizing the surface,[5] and of its relation to Kummer's surface[6] is omitted.

2. The specialized Weddle surface having its nodes pairs of a skew involution.[7] Two points (x) and (y) are pairs of a skew involution, I, if they form with the intersections of their join with two skew lines, g and l, an harmonic quadruple. If g and l are imaginary I is said to be elliptic, if real, hyperbolic. Unless otherwise specified the symbol I will stand for an hyperbolic skew involution.

If g and l are the joins of the points $P(0, 0, 1, 1)$ $Q(1, 1, 0, 0)$ and $P'(0, 0, 1, -1)$ $Q'(1, -1, 0, 0)$ respectively, I will have the form

$$\varrho x_1' = x_2, \qquad \varrho x_2' = x_1, \qquad \varrho x_3' = x_4, \qquad \varrho x_4' = x_3,$$

which we designate by the symbol $I_{12\cdot 34}$.

Theorem 1. *The Weddle surface whose nodes are pairs of a skew involution contains, besides the* 25 *lines* l_{ik} *and* g_{ijk}, *the two axes* g *and* l *of the involution and hence contains* 27 *lines.*

Let P be any point on l, one of the axes of I. From P project the configuration of the six nodes A_i and the axes g and l upon a generic plane. The six nodes A_i and the axes g and l will project into six points A_i', the line g' and the point L' respectively. The three joins of $A_i A_k$ which intersect g and l will project into the joins of $A_i' A_k'$ which pass thru L' and intersect g' in a point G' such that A_i', A_k', L' and G' form an harmonic

* Received July 12, 1928.

quadruple, since a cross ratio is invariant under a projection. The points A_i' then lie on a conic and P, any point of l, is the vertex of a quadric cone thru A_i. The line l then must lie on W, likewise g, and the theorem follows.

The Weddle surface of Theorem 1 is obviously invariant under $I_{12\cdot34}$.

3. An involutorial correspondence between pairs of points in a skew involution which are base points of an invariant net of quadrics.

THEOREM 2. *Each quadric of a pencil of quadrics on* 8 *points which are pairs of a skew involution is invariant under the involution.*

A space quartic C_4 is uniquely determined by 8 generic points in space and is the base curve of a pencil of quadrics thru the 8 points. If the 8 points are pairs of I then clearly C_4 is invariant under I though not pointwise.

Any quadric of the pencil cuts the axes, g and l, in two points each of which are invariant under I, hence 12 points on the quadric are transformed into the same set of 12 by I. Since a quadric is uniquely determined by 9 points the theorem follows.

Theorem 2 enables us to find the relations existing between the four pairs of I which are the base points of an invariant net of quadrics. This relation we state as

THEOREM 3. *Three pairs of points* A_1A_2, A_3A_4, A_5A_6 *of a skew involution, of which no two pairs are collinear, determine a fourth pair,* A_7A_8, *uniquely, in such a manner that the four pairs determine the eight base points of a net of quadrics invariant under the involution. If two of the pairs are fixed the remaining pairs are connected by an involutorial cubic transformation whose F points are the fixed points of the pairs.*

There is no loss of generality in choosing I to be the particular skew involution $I_{12\cdot34}$ and the two pairs of fixed points to be the vertices of the reference tetrahedron. The quadrics, on these points, which are invariant under $I_{12\cdot34}$ form a webb:

$$a\,x_1x_2+b\,x_3x_4+c\,(x_1x_3+x_2x_4)+d\,(x_1x_4+x_2x_3)\;=\;0.$$

The condition that the points $A_5(x_1, x_2, x_3, x_4)$, $A_6(x_2, x_1, x_4, x_3)$, $A_7(y_1, y_2, y_3, y_4)$, $A_8(y_2, y_1, y_4, y_3)$ lie on a net of these quadrics is that the matrix

$$\begin{pmatrix} x_1x_2 & x_3x_4 & x_1x_3+x_2x_4 & x_1x_4+x_2x_3 \\ y_1y_2 & y_3y_4 & y_1y_3+y_2y_4 & y_1y_4+y_2y_3 \end{pmatrix}$$

be of rank 1. This condition yields three independent quadratic equations in the y's giving as solutions the four vertices of the reference tetrahedron, the

two points A_5, A_6 and the points $A_7\left(\frac{1}{x_4}, \frac{1}{x_3}, \frac{1}{x_2}, \frac{1}{x_1}\right)$ and $A_8\left(\frac{1}{x_3}, \frac{1}{x_4}, \frac{1}{x_1}, \frac{1}{x_2}\right)$ from which we observe the truth of the theorem.

The cubic involution sending the pair $A_5 A_6$ into the pair $A_7 A_8$ we designate by the symbol $S_{14\cdot 23}$ and that sending $A_5 A_6$ into $A_8 A_7$ by the symbol $S_{13\cdot 24}$. It is evident that the transformations $I_{12\cdot 34}$, $S_{14\cdot 23}$ and $S_{13\cdot 24}$ form a group of order 4.

Concerning the eight associated points we may state

THEOREM 4. *The eight associated points of a net of quadrics which are pairs of the skew involution $I_{12\cdot 34}$ are vertices of two in- and circumscribed tetrahedra $A_1 A_3 A_6 A_8$ and $A_2 A_4 A_5 A_7$.*

The proof of the theorem follows from eight identities similar to the identity

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_1 & x_2 & x_3 & x_4 \\ \frac{1}{x_3} & \frac{1}{x_4} & \frac{1}{x_1} & \frac{1}{x_2} \end{vmatrix} \equiv 0$$

proving that the points $A_1 A_3 A_5 A_8$ are coplanar.

Among the 28 Weddle surfaces whose nodes are points of this associated set, 24 are degenerate. The four non-degenerate ones whose nodes are $A_1 A_2 A_3 A_1 A_5 A_6$, $A_1 A_2 A_3 A_4 A_7 A_8$, $A_1 A_2 A_5 A_6 A_7 A_8$, $A_3 A_4 A_5 A_6 A_7 A_8$ we will designate by I, II, III, IV respectively.

By direct substitution in the equations of I and II we could establish the theorem.

THEOREM 5. *The set of two Weddle surfaces, whose nodes are among the eight associated points which are pairs of a skew involution, and having the four F points of the cubic involution, determined by the associated points, in common, is invariant under the group of transformations whose elements constists of the identity and the involutorial transformations $I_{12\cdot 34}$, $S_{14\cdot 23}$, and $S_{13\cdot 24}$.*

By the same method used to prove Theorem 1 the following theorem may now be established:

THEOREM 6. *The Schur[8] sextic associated with the eight base points of a net of quadrics degenerates into six lines if the eight base points are pairs of a skew involution.*

The six lines are the four intersections of the pairs of planes on the eight points, which do not contain any of these points, and the two axes, g and l of the involution.

4. A collineation group, C_8, leaving invariant eight associated points of a net of quadrics which are pairs of the skew involution $I_{12\cdot34}$.

THEOREM 7. *There exists an Abelian group, C_8, of collineations leaving invariant the four non-degenerate Weddle surfaces having six of the eight associated points $A_1, \cdots, A_8$ as nodes and hence having the degenerate Schur sextic, associated with these points, as an invariant curve.*

The collineations $T_{13\cdot24\cdot57\cdot68}$, $T_{17\cdot35\cdot28\cdot46}$, $T_{15\cdot37\cdot26\cdot48}$, $T_{12\cdot34\cdot56\cdot78}$, $T_{14\cdot23\cdot58\cdot67}$, $T_{18\cdot27\cdot36\cdot45}$, $T_{16\cdot25\cdot38\cdot47}$ may be easily determined by asking that the general collineation transform the points $A_1, \cdots, A_8$ of section 3 in the manner indicated by the subscripts. That these form a group and are commutative is easily seen. The nodes of the Weddle surfaces are permuted as shown and a collineation transforms a Weddle surface into a Weddle surface hence the theorem follows.

5. Groups of cyclic collineations of order six and their invariant hexahedra.[9] A collineation group may always be found which leaves a set of five generic points invariant. We wish to prove the following theorem:

THEOREM 8. *If five generic points $B_1, \cdots, B_5$ of a general hexahedron are fixed, then there exists only one point B_6 such that the hexahedron on this and the five generic points is transformed into itself cyclically in a definite order by a proper collineation. The collineations leaving invariant a given hexahedron form a cyclic group C_6.*

The collineations are determined as in section 4. If four of the points $B_1 B_2 B_3 B_4$ be taken as the vertices of the reference tetrahedron and $B_5(y_1, y_2, y_3, y_4)$, $B_6(z_1, z_2, z_3, z_4)$ the remaining two points, the collineation which transforms these in the order $(B_1 B_2 B_3 B_4 B_5 B_6 \cdots)$ is

$$\begin{aligned}
\varrho x_1' &= y_1^2 y_2 y_3 z_1 x_4, \\
\varrho x_2' &= y_2 y_3 y_4 (y_1 z_2 - y_2 z_1) x_1 + y_1 y_2^2 y_3 z_1 x_4, \\
\varrho x_3' &= y_1 y_3 y_4 (y_1 z_3 - y_3 z_1) x_2 + y_1 y_2 y_3^2 z_1 x_4, \\
\varrho x_4' &= y_1 y_2 y_4 (y_1 z_4 - y_4 z_1) x_3 + y_1 y_2 y_3 y_4 z_1 x_4.
\end{aligned}$$

To put on the further condition that B_6 be transformed into B_1 leads to equations involving the y's and z's whose solutions give the points B_1, B_2, B_3, B_4, B_5 and $B_6(6y_1, 4y_2, 3y_3, 2y_4)$. The remaining solutions give singular transformations, hence the theorem follows. The group C_6 may be written 1, T_{123456} $T_{135\cdot246}$, $T_{14\cdot25\cdot36}$, $T_{153\cdot264}$, T_{165432}.

We may add the corrollary.

COROLLARY 1. *The Weddle Surface whose nodes are vertices of an invariant hexahedron, is invariant under the group C_6.*

It may be readily verified that if the order of permutation of the five points be changed, except for the inverse, the sixth point and the corresponding transformation will be changed. There are then 60 such groups associated with the five given points. The products of these groups do not give a group leaving a hexahedron invariant.

Among these groups are some whose generating transformations raised to the third power give the axial involution already discussed. One such is the group generated by T_{135246} which has the invariant hexahedron B_1, B_2, B_3, B_4, B_5, and $B_6(-3y_1, y_2, -y_3, 2y_4)$.

This hexahedron is also invariant under the collineation,

$$T_{14\cdot23\cdot5\cdot6}: \quad \begin{aligned} \varrho x_1' &= y_1^2 y_2 y_3 x_4, & \varrho x_2' &= y_1 y_2^2 y_4 x_3, \\ \varrho x_3' &= y_1 y_3^2 y_4 x_2, & \varrho x_4' &= y_2 y_3 y_4^2 x_1, \end{aligned}$$

T_{135246} and $T_{14\cdot23\cdot5\cdot6}$ generate a group, C_{12}, of collineations:

$$1,\ T_{135246},\ T_{154\cdot326},\ T_{12\cdot34\cdot56},\ T_{145\cdot362},\ T_{164253},\ T_{14\cdot23\cdot5\cdot6},$$
$$T_{12\cdot35\cdot46},\ T_{15\cdot26\cdot3\cdot4},\ T_{13\cdot24\cdot56},\ T_{1\cdot2\cdot36\cdot45},\ T_{16\cdot25\cdot34}.$$

There results the theorem:

THEOREM 9. *The Weddle Surface, whose nodes are the vertices of an hexahedron invariant under a cyclic group of collineations containing the axial involution, $I_{12\cdot34}$, is invariant under a group, C_{12}, of collineations.*

If the vertices be real then I is elliptic, if I be hyperbolic then a pair of the vertices must be imaginary.

6. **The group of collineations leaving invariant eight associated points of which six are the vertices of an hexahedron invariant under the six-ary cyclic collineation group containing the skew involution $I_{12\cdot34}$.** By Theorem 4 we find the eight associated points to be $A_1, A_2, A_3, A_4, A_5(y_1, y_2, y_3, y_4)$, $A_6(-3y_1, y_2, -y_3, 3y_4)$, $A_7(3y_1, y_2, y_3, 3y_4)$, $A_8(y_1, -y_2, y_3, -y_4)$. A_7 and A_8 are permuted by T_{135246} and invariant under $T_{14\cdot23\cdot5\cdot6}$ hence the associated set is invariant under the group, C_{12}.

The group, C_{12}, of this section and the C_8 of section 4 have the transformation $T_{12\cdot34\cdot56\cdot78}$ in common. Their product is a group of order 48.[10] The 4 non-degenerate Weddle surfaces I, II, III, IV are clearly invariant under the transformations of this C_{48}. This establishes the theorem:

THEOREM 10. *The system of eight associated points of an elliptic skew involution determined by three pairs of a six-ary cyclic collineation is invariant under a group, C_{48}, of collineations. The 4 non-degenerate Weddle surfaces whose nodes are among the eight points form an invariant system under the C_{48}. This has the degenerate Schur sextic, associated with the eight points, as an invariant curve.*

The table given below shows the G_{48}. One observes that each Weddle surface is invariant under one of the four subgroups of order 12 contained in the G_{48}.

Collineations	Collineations	Permutations of W
$T_{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}$	$T_{12\cdot34\cdot56\cdot78}$	I · II · III · IV
$T_{1\cdot2\cdot376\cdot485}$	$T_{12\cdot386475}$	I II III · IV
$T_{1\cdot2\cdot367\cdot458}$	$T_{12\cdot357468}$	I III II · IV
$T_{18\cdot27\cdot3\cdot4\cdot5\cdot6}$	$T_{17\cdot28\cdot34\cdot56}$	I IV · II · III
$T_{1548\cdot2637}$	$T_{1647\cdot2538}$	I IV II III
$T_{1458\cdot2367}$	$T_{1357\cdot2468}$	I IV III II
$T_{154\cdot263\cdot7\cdot8}$	$T_{164253\cdot78}$	I · II III IV
$T_{14\cdot23\cdot58\cdot67}$	$T_{13\cdot24\cdot57\cdot68}$	I II · III IV
$T_{184\cdot273\cdot5\cdot6}$	$T_{174283\cdot56}$	I III IV · II
$T_{1485\cdot2376}$	$T_{1386\cdot2475}$	I II IV III
$T_{1845\cdot2736}$	$T_{1746\cdot2835}$	I III II IV
$T_{15\cdot26\cdot3\cdot4\cdot7\cdot8}$	$T_{16\cdot25\cdot34\cdot78}$	I · II IV · III
$T_{145\cdot286\cdot7\cdot8}$	$T_{135246\cdot78}$	I · II IV III
$T_{185\cdot276\cdot3\cdot4}$	$T_{175268\cdot34}$	I II IV · III
$T_{15\cdot26\cdot37\cdot48}$	$T_{16\cdot25\cdot38\cdot47}$	I III · II IV
$T_{1\cdot2\cdot3\cdot4\cdot58\cdot67}$	$T_{12\cdot34\cdot57\cdot68}$	I II · III · IV
$T_{1\cdot2\cdot37\cdot48\cdot5\cdot6}$	$T_{12\cdot38\cdot47\cdot56}$	I III · II · IV
$T_{1\cdot2\cdot36\cdot45\cdot7\cdot8}$	$T_{12\cdot35\cdot46\cdot78}$	I · II III · IV
$T_{1584\cdot2673}$	$T_{1683\cdot2574}$	I III IV II
$T_{14\cdot23\cdot5\cdot6\cdot7\cdot8}$	$T_{18\cdot24\cdot56\cdot78}$	I · II · III IV
$T_{1854\cdot2763}$	$T_{1753\cdot2864}$	I II III IV
$T_{18\cdot27\cdot36\cdot45}$	$T_{17\cdot28\cdot35\cdot46}$	I IV · II III
$T_{158\cdot267\cdot3\cdot4}$	$T_{168257\cdot34}$	I IV II · III
$T_{148\cdot237\cdot5\cdot6}$	$T_{138247\cdot56}$	I IV III · II

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3 Chasles, C. R., vol. 52 (1861), pp. 1157–1162; Cayley, C. R., vol. 52 (1861), pp. 1216–1218; Coll. Wks., vol. 5, pp. 4–7; Poudra, N. A. Math., vol. 2 (1863), p. 307. See also Hierholzer, Math. Ann., vol. 4 (1872), pp. 172–180; Caspary, Bull. Sc. Math., vol. 1 (1887), pp. 222–236, vol. 13 (1889), pp. 202–240; Jessop, "Treatise on Quartic Surfaces"; and Hudson, "Kummers Quartic Surface".

4 For a description of the model see: University of Ill. Bull., Math. models. series 4 (1928), p. 37.

5 Schottky, Jour. für Math., vol. 105 (1891), pp. 398–317; Baker, "Multiply Periodic Functions".

6 Darboux, ibid. p. 354; Reye, Crelle's Jour., vol. 86 (1878), p. 86; Schottky and Humbert, ibid.; Baker, Proc. Lond. Math. Soc., vol. 1 (1903), p. 247.

7 The theorem of this section was first proved by Hutchinson, Ann. of Math., vol. 11 (1896–1897), pp. 158–160, by means of hyperelliptic functions and later by Snyder, Am. Math. Soc. Trans., vol. 12 (1910), pp. 354–366, using properties of transformations. In this paper is an elementary proof based on the definition of the Weddle Surface.

8 Schur, Math. Ann., vol. 18 (1881), pp. 1–32. Cf. Emch, Am. Jour. of Math., vol. 48 (1926), p. 42.

9 R. Krause, Inaug.-Diss. (1903).

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DER ZUSAMMENHANG ZWISCHEN DER SYMMETRISCHEN UND DER LINEAREN GRUPPE.*

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Einleitung.

Die Komponenten eines willkürlichen Tensors F von f^{ter} Stufe in einem n-dimensionalen Vektorraum bezeichne ich mit $F(i_1\, i_2 \cdots i_f)$, die lineare Mannigfaltigkeit aller dieser Tensoren mit $\mathfrak{R}$. Unter dem Einfluß einer willkürlichen linearen Abbildung oder Transformation des Vektorraums

$$(1) \qquad B\colon\ x_i' = \sum_{k=1}^{n} b(i\,k)\, x_k$$

erfahren sie die zugehörige Transformation

$$(2) \qquad B_f\colon\ F'(i_1 \cdots i_f) = \sum_{(k)} b(i_1\, k_1) \cdots b(i_f\, k_f) \cdot F(k_1 \cdots k_f).$$

Durch die Korrespondenz $B \to B_f$ wird eine bestimmte *Darstellung* $(\lambda)^f$ *der Gruppe* λ *aller linearen Transformationen* B definiert. Ihre Zerlegung in irreduzible Bestandteile hängt aufs engste zusammen mit den Darstellungen der endlichen symmetrischen Gruppe $\pi = \pi_f$ der Permutationen von f Dingen, wie auf verschiedene Weise von I. Schur und dem Verfasser gezeigt wurde.† Da diese Beziehung in der Quantenmechanik eine große Rolle spielt, habe ich ihr insbesondere das letzte Kapitel meines Buches „Gruppentheorie und Quantenmechanik" (Leipzig 1928) gewidmet. Die Quantentheorie ist aber nicht eigentlich an der Gruppe der B_f interessiert, sondern an einer umfassenderen, der Gruppe σ der *symmetrischen Transformationen*. Ich nenne die lineare Transformation $F \to F'$ in $\mathfrak{R}$:

$$(C) \qquad F'(i_1 \cdots i_f) = \sum_{(k)} c(i_1 \cdots i_f;\ k_1 \cdots k_f)\ F(k_1 \cdots k_f)$$

symmetrisch, wenn der Koeffizient $c(i_1 \cdots i_f;\ k_1 \cdots k_f)$ seinen Wert nicht ändert, falls man die Subindizes $1, 2, \cdots, f$ einer beliebigen Permutation s unterwirft — oder, was auf dasselbe hinauskommt, falls beide Reihen von Indizes $i_1 \cdots i_f;\ k_1 \cdots k_f$ simultan dieselbe Permutation s erleiden. Die Quantentheorie stellt durch einen Tensor F den Zustand eines physikalischen Gebildes dar, das aus f gleichartigen Korpuskeln, etwa f Elektronen besteht.

* Received February 6, 1929.

† I. Schur, Inauguraldissertation, Berlin 1901. H. Weyl, Mathem. Zeitschr. **23** (1925), p. 271. I. Schur, Sitzungsber. Berl. Akad. 1927, p. 58; 1928, p. 100.

Ein linearer Teilraum $\mathfrak{P}$ von $\mathfrak{R}$, der invariant ist gegenüber allen symmetrischen Transformationen, hat nach dem dynamischen Grundgesetz die Eigenschaft, daß der Zustand des physikalischen Gebildes, wenn einmal in $\mathfrak{P}$ befindlich, durch keinerlei Einwirkung daraus vertrieben werden kann. Die Transformation B_f ist offenbar symmetrisch, ich nenne sie eine *spezielle* symmetrische Transformation. Wenn B ganz λ durchläuft, so durchläuft B_f eine mit λ isomorphe Untergruppe σ_0 von σ. Ich habe nachträglich bemerkt, daß die Gruppe σ der symmetrischen Transformationen das natürliche Bindeglied zwischen den Gruppen π und λ abgibt; in der Tat weist sie schon durch ihren Begriff auf die Permutationsgruppe π hin und führt auf der andern Seite durch die „Spezialisierung" sogleich auf die Gruppe λ. Dies Mittelglied gestattet, die Theorie wesentlich direkter, elementarer und vollständiger als bisher aufzubauen, so daß an Durchsichtigkeit nichts mehr zu wünschen übrig bleibt. Ich möchte dies hier ab ovo und in rein mathematischer Formulierung entwickeln.

§ 1 enthält einige vorbereitende, übrigens längst geläufige Definitionen und Betrachtungen. § 2 gibt die benötigten Hauptsätze über π, §§ 3 und 4 sind dem Zusammenhang zwischen π und σ gewidmet, § 5 demjenigen zwischen σ und λ. § 6 betrifft Verallgemeinerungen, in denen die symmetrische durch eine beliebige Permutationsgruppe ersetzt wird. Nur in ein paar Fußnoten nehme ich in diesem Hauptteil Bezug auf das oben zitierte Buch. Der Anhang § 7 jedoch enthält einen kurzen Hinweis auf daran anschließende Entwicklungen, die in jenem Buch gegeben und keiner Abänderung bedürftig sind, nebst einer ergänzenden Bemerkung über dasjenige gruppentheoretische Problem, das in der Quantentheorie durch das Pauli-Verbot zusammen mit der Existenz des Elektronenspins hervorgerufen wird.

§ 1. Symmetrieoperatoren.

1. Ich erinnere rasch an die Bedeutung der Worte *invariant, irreduzibel, äquivalent.* Gegeben sei eine Gruppe γ von linearen homogenen Transformationen im linearen Vektorraum $\mathfrak{G}$. Ein linearer Teilraum $\mathfrak{G}'$ von $\mathfrak{G}$ heißt invariant gegenüber γ, wenn die Vektoren von $\mathfrak{G}'$ durch die Transformationen von γ in Vektoren von $\mathfrak{G}'$ übergehen. Ein solcher invarianter Teilraum $\mathfrak{G}'$ ist irreduzibel, wenn er keine andern invarianten Teilräume enthält als $\mathfrak{G}'$ selbst und den nur aus dem Vektor 0 bestehenden. Zwei invariante Teilräume $\mathfrak{G}_1$, $\mathfrak{G}_2$ sind äquivalent, wenn nach geeigneter Wahl des affinen Koordinatensystems in dem einen und andern, die beiden Transformationen S_1 und S_2, welche die willkürliche Transformation S der Gruppe γ in $\mathfrak{G}_1$ und $\mathfrak{G}_2$ hervorruft, dieselbe Koeffizientenmatrix besitzen (identisch in S). Ohne von Koordinatensystemen zu reden, kann man diese Forderung so fassen: es soll sich eine eineindeutige lineare Zuordnung $x_1 \to x_2$ zwischen

den Vektoren x_1 von $\mathfrak{G}_1$ und x_2 von $\mathfrak{G}_2$ so stiften lassen, daß, wenn $x_1' = Sx_1$ in $\mathfrak{G}_1$ ist, für die zugeordneten Vektoren x_2 und x_2' in $\mathfrak{G}_2$ die gleiche Beziehung $x_2' = Sx_2$ gilt, unter S eine beliebige Transformation der Gruppe γ verstanden. Die Zuordnung $x_1 \to x_2$ heißt eine ähnliche Abbildung von $\mathfrak{G}_1$ auf $\mathfrak{G}_2$. — Die Begriffe invariant, irreduzibel, äquivalent sind relativ auf eine Gruppe γ. In unserm Raum $\mathfrak{R}$ der Tensoren f^{ter} Stufe sind sie stets so zu verstehen, daß als Gruppe γ *die Gruppe σ der symmetrischen Transformationen zugrunde gelegt wird.*

2. Ein Tensor F kann einer Permutation

$$s\colon\ 1 \to 1',\quad 2 \to 2',\quad \cdots,\quad f \to f'$$

unterworfen werden; dadurch entsteht der Tensor $\mathbf{s}F$, dessen Komponenten durch die Gleichung gegeben sind,

$$\mathbf{s}F(i_1 \cdots i_f) = F(i_{1'} \cdots i_{f'}).$$

Bei solcher Festsetzung geht der Tensor $\mathbf{s}(\mathbf{t}F)$ aus F durch die zusammengesetzte Permutation st hervor, die zustande kommt, wenn erst t, dann s ausgeführt wird.* Sind $a(s)$ irgendwelche den $f!$ Permutationen s zugeordnete Zahlen, so können wir allgemeiner den Symmetrieoperator

$$\mathbf{a} = \sum_s a(s) \cdot \mathbf{s}$$

betrachten, der aus F den Tensor

$$\mathbf{a}F = \sum_s a(s) \cdot \mathbf{s}F$$

erzeugt. Solche Symmetrieoperatoren lassen sich addieren und mit Zahlen multiplizieren: sie bilden einen linearen Raum $\mathfrak{r}$ von $f!$ Dimensionen. Aber sie lassen sich auch multiplizieren: übt man erst den Operator $\mathbf{a}$, dann $\mathbf{b}$ aus, so geht das Resultat $\mathbf{b}(\mathbf{a}F)$ aus F durch einen einzigen Operator $\mathbf{c}$ hervor, den wir das Produkt $\mathbf{ba}$ nennen. Es wird geliefert durch die Gleichung

$$c(s) = \sum b(t')\, a(t), \tag{3}$$

in welcher sich die Summe rechts über alle Paare von Permutationen t', t erstreckt, deren Produkt $t't = s$ ist. Die Symmetrieoperatoren bilden daher, was man hierzulande eine *Algebra* nennt; in dieser Rolle — als Elemente, die sich addieren und multiplizieren lassen, aber unter Abstraktion davon, daß sie Operatoren sind, welche auf Tensoren wirken — bezeichnen wir sie als Symmetriegrößen oder kurz als Größen. In der gleichen Weise

* Unglücklicherweise enthält mein Buch hinsichtlich dieser Definition einen Lapsus (trotz richtiger Fassung in meinen älteren Arbeiten, namentlich Mathem. Zeitschr. **23** (1925), p. 271).

gehört übrigens zu jeder endlichen abstrakten Gruppe der Ordnung h eine h-dimensionale Algebra, die *Gruppenalgebra* oder der *Gruppenring*.

3. Dem Tensor F eine Symmetriebedingung auferlegen, heißt ihn einer Gleichung

$$G \equiv \mathfrak{a}F \equiv \sum_t a(t) \cdot \mathfrak{t}F = 0 \tag{4}$$

unterwerfen. Da mit G selbstverständlich auch $\mathfrak{s}G$ verschwindet, erfüllt dann F die weitere Gleichung

$$\sum_t a(t) \cdot \mathfrak{st}F = 0.$$

Führen wir die Symmetriegröße $\mathfrak{F}$ mit den tensoriellen Komponenten $F(s) = \mathfrak{s}F$ ein und definieren außerdem $\hat{\mathfrak{a}}$ durch

$$\hat{\mathfrak{a}}(s) = a(s^{-1}),$$

so nimmt diese Beziehung die knappe Form an: $\mathfrak{F}\hat{\mathfrak{a}} = 0$. Sie ist mit der Symmetriebedingung (4) gleichbedeutend. Die Lösungen $\mathfrak{x}$ der Gleichung

$$\mathfrak{x}\hat{\mathfrak{a}} = 0 \tag{5}$$

bilden eine links-invariante Subalgebra; d. h. eine lineare Mannigfaltigkeit, welche mit $\mathfrak{x}$ zugleich $\mathfrak{cx}$ enthält, welches auch der von links her herantretende Faktor $\mathfrak{c}$ ist. Dies bleibt in Geltung, wenn statt (5) mehrere derartige Gleichungen simultan postuliert werden. Das Resultat unserer vorläufigen Betrachtung fixieren wir in der Definition:

Eine linksinvariante Subalgebra $\mathfrak{p}$ der zur Permutationsgruppe π gehörigen Gruppenalgebra $\mathfrak{r}$ bestimmt eine Symmetrieklasse $\mathfrak{P}$ von Tensoren: der Tensor F gehört dieser Symmetrieklasse an, wenn $\mathfrak{F}$ in $\mathfrak{p}$ liegt.

Es ist evident, daß $\mathfrak{P}$, die Mannigfaltigkeit der Tensoren von der Symmetrie $\mathfrak{p}$, ein gegenüber allen symmetrischen Transformationen invarianter Teilraum von $\mathfrak{R}$ ist. Die Hauptfrage hinsichtlich des Zusammenhangs von π und σ, welche in §§ 3 und 4 bejahend beantwortet wird, ist die, ob umgekehrt jeder invariante Teilraum $\mathfrak{P}$ aus $\mathfrak{R}$ durch Symmetriebedingungen ausgeschieden werden kann.

Eine linksinvariante Subalgebra ist ein Teilraum $\mathfrak{p}$, der durch alle Transformationen

$$(c): \qquad \mathfrak{x} \to \mathfrak{x}' = \mathfrak{cx}$$

in sich übergeführt wird. Für deren Zusammensetzung gilt offenbar $(c'c) = (c')(c)$. Ordnen wir jeder Permutation s die Transformation (s): $\mathfrak{x}' = \mathfrak{sx}$ des Gruppenraumes $\mathfrak{r}$ zu, so entsteht die sog. *reguläre Darstellung* der Permutationsgruppe π vom Grade $f!$. Die Transformationen (s) bilden eine mit π isomorphe Gruppe (π) von linearen Abbildungen im Gruppenraum $\mathfrak{r}$.

Und der Begriff linksinvariante Subalgebra kann jetzt so gefaßt werden: linearer Teilraum von $\mathfrak{r}$, welcher gegenüber der Gruppe (π) invariant ist. Im folgenden sind *die Termini invariant, irreduzibel, äquivalent im Gruppenraum* $\mathfrak{r}$ *stets auf die Gruppe* (π) *zu beziehen.* — Wir fassen ergänzend zusammen:

Satz 1. *Ein invarianter Teilraum* $\mathfrak{p}$ *von* $\mathfrak{r}$ *bestimmt einen invarianten Teilraum* $\mathfrak{P}$ *von* $\mathfrak{R}$: *der Tensor* F *gehört zu* $\mathfrak{P}$, *wenn die Größe* $\mathbf{F}$ *in* $\mathfrak{p}$ *liegt. Ist* $\mathfrak{p}' \prec \mathfrak{p}$, *so auch* $\mathfrak{P}' \prec \mathfrak{P}$. *Ist* $\mathfrak{p}$ *zerlegt in zwei linear unabhängige invariante Teilräume* $\mathfrak{p}_1 + \mathfrak{p}_2$, *so gilt für die zugehörigen Teilräume von* $\mathfrak{R}$ *im gleichen Sinne* $\mathfrak{P} = \mathfrak{P}_1 + \mathfrak{P}_2$.

Der letzte Zusatz wird erst zu Beginn von § 3 vollständig bewiesen.

§ 2. Invariante Teilräume im Gruppenraum $\mathfrak{r}$.

1. Satz 2. *Der invariante Teilraum* $\mathfrak{p}$ *von* $\mathfrak{r}$ *besitzt eine erzeugende Einheit* $\mathbf{e}$; *das heißt*: $\mathbf{e}$ *ist eine* $\mathfrak{p}$ *angehörige idempotente Größe,* $\mathbf{ee} = \mathbf{e}$, *von der Art, daß alle Größen von* $\mathfrak{p}$ *in der Form* $\mathbf{xe}$ *darstellbar sind. Jede Größe* $\mathbf{x}$ *von* $\mathfrak{p}$ *genügt insbesondere der Gleichung* $\mathbf{xe} = \mathbf{x}$.

Beweis. Der h-dimensionale Teilraum $\mathfrak{p}$ werde durch die Vektoren $\mathbf{e}_1, \cdots, \mathbf{e}_h$ aufgespannt. Eine Transformation $\mathbf{x} \to \mathbf{x}'$:

$$x'(s) = \sum_t d(s, t)\, x(t)$$

verwandelt 1. jede Größe $\mathbf{x}$ in eine in $\mathfrak{p}$ liegende $\mathbf{x}'$ und führt 2. jede in $\mathfrak{p}$ liegende Größe $\mathbf{x}$ in sich über, wenn sie von der folgenden Form ist:

$$d(s, t) = \sum_{i=1}^{h} e_i(s)\, \check{e}_i(t) \tag{6}$$

und die $\check{e}_i(s)$ den Gleichungen genügen

$$\sum_s e_i(s)\, \check{e}_k(s) = \delta_{ik}.$$

Ihnen gemäß mögen die Zahlen $\check{e}_i(s)$ bestimmt sein. Wenn r eine gegebene Permutation ist, muß mit $\mathbf{x}$ auch die durch $x_r(s) = x(rs)$ erklärte Größe $\mathbf{x}_r$ zu $\mathfrak{p}$ gehören, weil $\mathfrak{p}$ invariant ist. Infolgedessen hat die Transformation $d_r(s, t) = d(rs, rt)$ dieselben beiden Eigenschaften 1. und 2. wie $d(s, t)$ und darum auch die Summe

$$e(s, t) = \frac{1}{f!} \sum_r d(rs, rt). \tag{7}$$

Diese Funktion von s und t ändert sich aber offenbar nicht, wenn s und t durch rs und rt ersetzt werden; d. h. sie ist eine Funktion $e(t^{-1}s)$ von $t^{-1}s$ allein. Daß die Transformation $\mathbf{x}' = \mathbf{xe}$ besagte beide Eigenschaften besitzt, ist aber alles, was wir zu beweisen wünschen. — Die Konstruktion (7)

führt eine Normierung der nicht eindeutig bestimmten $\check{e}_i(s)$ in solcher Weise herbei, daß (6) eine Funktion von $t^{-1}s$ allein wird.

Derselbe Gedankengang liefert den allgemeineren

Satz 3. *Sind* $\mathfrak{p}$, $\mathfrak{p}'$ *äquivalente invariante Teilräume, so wird die ähnliche Abbildung von* $\mathfrak{p}$ *auf* $\mathfrak{p}'$ *durch eine Formel von der Gestalt* $\mathbf{x} \to \mathbf{x}' = \mathbf{x}\mathfrak{b}$ *vermittelt;* $\mathfrak{b}$ *ist eine feste in* $\mathfrak{p}'$ *gelegene Größe.*

Beweis. Die ähnliche Abbildung $\mathfrak{B}$ möge die $\mathfrak{p}$ aufspannenden Vektoren $\mathbf{e}_1, \cdots, \mathbf{e}_h$ überführen in $\mathbf{e}'_1, \cdots, \mathbf{e}'_h$. Die Transformation $\mathbf{x} \to \mathbf{x}'$:

$$x'(s) = \sum_t a(s,t)\, x(t) \quad \text{mit} \quad a(s,t) = \sum_{i=1}^{h} e'_i(s)\, \check{e}_i(t)$$

hat dann die beiden Eigenschaften: 1. sie führt jede Größe $\mathbf{x}$ in eine in $\mathfrak{p}'$ gelegene $\mathbf{x}'$ über; 2. für die in $\mathfrak{p}$ liegenden Größen $\mathbf{x}$ fällt sie mit $\mathfrak{B}$ zusammen. Geht $\mathbf{x}$ durch $\mathfrak{B}$ in $\mathbf{x}'$ über, so verwandelt sie auch $\mathbf{x}_r$ mit den Komponenten $x_r(s) = x(rs)$ in $\mathbf{x}'_r$ mit den Komponenten $x'_r(s) = x'(rs)$: das ist die Voraussetzung, daß $\mathfrak{B}$ eine *ähnliche* Abbildung der *invarianten* Teilräume $\mathfrak{p}$ und $\mathfrak{p}'$ aufeinander ist; und dies hat zur Folge, daß $a(rs, rt)$ die beiden Eigenschaften 1. und 2. mit $a(s,t)$ teilt und daher auch

$$\frac{1}{f!} \sum_r a(rs, rt) = b(t^{-1}s).$$

Die zu $\mathfrak{B}$ inverse Abbildung von $\mathfrak{p}'$ auf $\mathfrak{p}$ wird durch eine analoge Formel $\mathbf{x} = \mathbf{x}'\mathfrak{b}'$ vermittelt.

2. *Hilfsbetrachtung.* Der invariante Teilraum $\mathfrak{p}$ mit der erzeugenden Einheit $\mathbf{e}$ sei in zwei unabhängige invariante Teilräume $\mathfrak{p}_1$, $\mathfrak{p}_2$ zerlegt. Dies besagt, daß jede Größe $\mathbf{x}$ in $\mathfrak{p}$ sich auf eine und nur eine Weise als eine Summe $\mathbf{x}_1 + \mathbf{x}_2$ darstellen läßt, deren Glieder in $\mathfrak{p}_1$ und $\mathfrak{p}_2$ liegen. *Insbesondere nehmen wir diese Zerlegung mit* $\mathbf{e}$ *vor:* $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$. Für jede Größe $\mathbf{x}$ aus $\mathfrak{p}$ gilt

$$\mathbf{x} = \mathbf{x}\mathbf{e} = \mathbf{x}\mathbf{e}_1 + \mathbf{x}\mathbf{e}_2.$$

Weil $\mathfrak{p}_1$ und $\mathfrak{p}_2$ linksinvariante Subalgebren sind, liegt mit $\mathbf{e}_1$ auch $\mathbf{x}\mathbf{e}_1$ in $\mathfrak{p}_1$; der erste Summand $\mathbf{x}\mathbf{e}_1$ ist also die in $\mathfrak{p}_1$ liegende Komponente $\mathbf{x}_1$ von $\mathfrak{p}_1$, der zweite $\mathbf{x}\mathbf{e}_2$ die in $\mathfrak{p}_2$ liegende Komponente $\mathbf{x}_2$. Wenden wir dies insbesondere auf $\mathbf{x} = \mathbf{e}_1$ an, so sehen wir, daß $\mathbf{e}_1\mathbf{e}_1$ und $\mathbf{e}_1\mathbf{e}_2$ die bzw. in $\mathfrak{p}_1$ und $\mathfrak{p}_2$ liegenden Komponenten von $\mathbf{e}_1$ sind, oder mit andern Worten: es ist $\mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_2 = 0$. In derselben Weise erhält man $\mathbf{e}_2\mathbf{e}_1 = 0$, $\mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_2$. $\mathbf{e}_1$ *und* $\mathbf{e}_2$ *sind also erzeugende Einheiten von* $\mathfrak{p}_1$ *und* $\mathfrak{p}_2$, *deren Produkt in beiden Reihenfolgen verschwindet.*

Mit dieser Überlegung steht in engem Zusammenhang

Satz 4. *Ist* $\mathfrak{p}_1$ *ein invarianter Teilraum, der in dem invarianten Teilraum* $\mathfrak{p}$ *enthalten ist, so kann* $\mathfrak{p}$ *in* $\mathfrak{p}_1$ *und einen zweiten davon linear unabhängigen invarianten Teilraum* $\mathfrak{p}_2$ *zerlegt werden:* $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$.

Beweis. $\mathfrak{e}_1$ sei eine erzeugende Einheit von $\mathfrak{p}_1$. Wenn $\mathfrak{x}$ ganz $\mathfrak{p}$ durchläuft, durchläuft $\mathfrak{x}_1 = \mathfrak{x}\mathfrak{e}_1$ den Teilraum $\mathfrak{p}_1$ und $\mathfrak{x}_2 = \mathfrak{x} - \mathfrak{x}\mathfrak{e}_1$ einen linearen Teilraum $\mathfrak{p}_2$ von $\mathfrak{p}$. Er ist eine links-invariante Subalgebra; denn es gilt

$$\mathfrak{c} \cdot \mathfrak{x}_2 = \mathfrak{c}\mathfrak{x} - (\mathfrak{c}\mathfrak{x})\,\mathfrak{e}_1 = (\mathfrak{c}\mathfrak{x})_2 \qquad [\mathfrak{c} \text{ beliebig}],$$

und $\mathfrak{c}\mathfrak{x}$ liegt wie $\mathfrak{x}$ selber in $\mathfrak{p}$. $\mathfrak{x} = \mathfrak{x}_1 + \mathfrak{x}_2$ ist eine Zerlegung von $\mathfrak{x}$ in zwei zu $\mathfrak{p}_1$ bzw. $\mathfrak{p}_2$ gehörige Komponenten. Um einzusehen, daß $\mathfrak{p}_1$ und $\mathfrak{p}_2$ voneinander unabhängig sind, hat man zu zeigen, daß die Summe von $\mathfrak{x}_1 = \mathfrak{x}\mathfrak{e}_1$ und $\mathfrak{y}_2 = \mathfrak{y} - \mathfrak{y}\mathfrak{e}_1$ nur verschwinden kann, wenn $\mathfrak{x}_1$ und $\mathfrak{y}_2$ einzeln Null sind; $\mathfrak{x}$ und $\mathfrak{y}$ bedeuten hier irgend zwei Größen aus $\mathfrak{p}$. In der Tat folgt aus einer solchen Gleichung $\mathfrak{x}_1 + \mathfrak{y}_2 = 0$ durch hintere Multiplikation mit $\mathfrak{e}_1$ wegen $\mathfrak{x}_1\mathfrak{e}_1 = \mathfrak{x}_1$, $\mathfrak{y}_2\mathfrak{e}_1 = 0$ die Gleichung $\mathfrak{x}_1 = 0$.

Aus Satz 4 geht hervor, daß sich der invariante Teilraum $\mathfrak{p}$ und daher insbesondere $\mathfrak{r}$ selber in unabhängige irreduzible invariante Teilräume zerlegen läßt.

§ 3. Invariante Teilräume im Tensorraum $\mathfrak{R}$.

1. Parallel zu Satz 2 gilt

Satz 5. *Eine Symmetrieklasse $\mathfrak{P}$ von Tensoren besitzt einen erzeugenden idempotenten Symmetrieoperator $\mathfrak{e}$; in dem Sinne, daß $\mathfrak{P}$ zusammenfällt mit der Gesamtheit aller Tensoren von der Form $\mathfrak{e}F$. Für jeden zu $\mathfrak{P}$ gehörigen Tensor F gilt insbesondere $\mathfrak{e}F = F$.*

Beweis. $\mathfrak{P}$ ist bestimmt durch einen invarianten Teilraum $\mathfrak{p}$ des Gruppenraums $\mathfrak{r}$. Die erzeugende Einheit von $\mathfrak{p}$ werde jetzt mit $\hat{\mathfrak{e}}$ bezeichnet. Wenn F ein beliebiger Tensor ist, gehört $\mathfrak{F}\hat{\mathfrak{e}}$ zu $\mathfrak{p}$; wenn F ein Tensor in $\mathfrak{P}$ ist, liegt die Größe $\mathfrak{F}$ in $\mathfrak{p}$ und darum ist $\mathfrak{F}\hat{\mathfrak{e}} = \mathfrak{F}$. Das ist identisch mit den Aussagen, daß im ersten Fall $\mathfrak{e}F$ zu $\mathfrak{P}$ gehört und im zweiten $\mathfrak{e}F = F$ ist. Die letzte Gleichung zeigt, daß jeder $\mathfrak{P}$ angehörige Tensor in der Form $\mathfrak{e}F$ dargestellt werden kann.

Die Hilfsbetrachtung, welche Satz 4 vorausgeschickt wurde, beweist nun auch den letzten Teil von Satz 1 (oder das an ihm, was nicht selbstverständlich ist). Sie liefert nämlich eine Zerlegung $\mathfrak{e} = \mathfrak{e}_1 + \mathfrak{e}_2$ und damit eine Zerlegung jedes zu $\mathfrak{P}$ gehörigen Tensors $F = \mathfrak{e}F$ in zwei Summanden $\mathfrak{e}_1 F$, $\mathfrak{e}_2 F$, die bzw. zu $\mathfrak{P}_1$ uud $\mathfrak{P}_2$ gehören.

Satz 6. *Äquivalente $\mathfrak{p}$ bestimmen äquivalente $\mathfrak{P}$.*

Der *Beweis* beruht auf Satz 3. Sind die beiden invarianten Teilräume $\mathfrak{p}$, $\mathfrak{p}'$ von $\mathfrak{r}$ äquivalent, so lassen sie sich ähnlich aufeinander abbilden mit Hilfe einer Transformation $\mathfrak{x}' = \mathfrak{x}\hat{\mathfrak{b}}$. Dann aber ist $F' = \mathfrak{b}F$ eine ähnliche Abbildung $F \to F'$ von $\mathfrak{P}$ auf $\mathfrak{P}'$. (Es ist zweckmäßig, zugleich die inverse Abbildung $F = \mathfrak{b}'F'$ zu verwenden.)

2. Bis hierhin waren wir ausschließlich mit Symmetrieklassen von Tensoren beschäftigt. Jetzt kommen wir zu der entscheidenden Umkehrung:

Satz 7. *Jeder invariante Teilraum* $\mathfrak{P}$ *von* $\mathfrak{R}$ *ist eine Symmetrieklasse, wird also bestimmt durch einen invarianten Teilraum* $\mathfrak{p}$ *von* $\mathfrak{r}$.

Um $\mathfrak{p}$ zu finden, hat man den kleinsten linearen Teilraum zu konstruieren, der alle Größen $\mathbf{F}$ umfaßt, die aus Tensoren F in $\mathfrak{P}$ entspringen, oder genauer alle Größen $\mathbf{x}$, welche durch den Ausdruck

$$x(s) = \mathbf{s}F(i_1 i_2 \cdots i_f)$$

geliefert werden, wenn für F ein Tensor in $\mathfrak{P}$ und für $i_1 i_2 \cdots i_f$ eine beliebige Indexfolge genommen wird. So kommen wir zu der folgenden Erklärung (K) von $\mathfrak{p}$: $\mathbf{x}$ *gehört zu* $\mathfrak{p}$, *wenn es in der Form*

$$x(s) = \sum_{(i)} c(i_1 \cdots i_f) \cdot \mathbf{s}F(i_1 \cdots i_f) \tag{8}$$

sich darstellen läßt, wo die c *beliebige Zahlen sein dürfen und* F *irgendein Tensor aus* $\mathfrak{P}$ *ist.* Es ist klar, daß dies $\mathfrak{p}$ eine lineare Mannigfaltigkeit ist. Wir haben zu zeigen: a) $\mathfrak{p}$ ist ein invarianter Teilraum von $\mathfrak{r}$. Infolgedessen bestimmt $\mathfrak{p}$ eine Symmetrieklasse $\mathfrak{P}(\mathfrak{p})$ von Tensoren. Es ist weiter zu beweisen: b) $\mathfrak{P} \prec \mathfrak{P}(\mathfrak{p})$; c) $\mathfrak{P}(\mathfrak{p}) \prec \mathfrak{P}$.

a) Behauptung: Wenn s eine feste Permutation ist, gehört mit $\mathbf{x}$ auch die durch $x'(t) = x(st)$ erklärte Größe $\mathbf{x}'$ zu $\mathfrak{p}$. — Es ist allgemein

$$\sum_{(i)} c(i_1 \cdots i_f) \cdot \mathbf{s}F(i_1 \cdots i_f) = \sum_{(i)} \mathbf{s}^{-1} c(i_1 \cdots i_f) \cdot F(i_1 \cdots i_f);$$

darum

$$x'(t) = x(st) = \sum_{(i)} \mathbf{s}^{-1} c(i_1 \cdots i_f) \cdot \mathbf{t}F(i_1 \cdots i_f)$$
$$= \sum_{(i)} c'(i_1 \cdots i_f) \cdot \mathbf{t}F(i_1 \cdots i_f),$$

wo das System c' durch

$$c'(i_{1'} \cdots i_{f'}) = c(i_1 \cdots i_f)$$

erklärt ist.

b) ist trivial: wenn F in $\mathfrak{P}$ liegt, gehört die durch (8) erklärte Größe $\mathbf{x}$ zu $\mathfrak{p}$, welches auch die Zahlen c sein mögen; d. h. die Größe $\mathbf{F}$ mit den Komponenten $\mathbf{s}F$ liegt in $\mathfrak{p}$ oder der Tensor F gehört zu $\mathfrak{P}(\mathfrak{p})$.

c) Wir zeigen: Ist $\hat{\mathbf{e}}$ die erzeugende Einheit von $\mathfrak{p}$ und F ein beliebiger Tensor, so liegt $\mathbf{e}F$ in $\mathfrak{P}$. — Da $\hat{\mathbf{e}}$ zu $\mathfrak{p}$ gehört, muß es aus einem gewissen Tensor E von $\mathfrak{P}$ gemäß einer Gleichung (8):

$$e(s^{-1}) = \sum_{(k)} c_0(k_1 \cdots k_f) \cdot \mathbf{s}E(k_1 \cdots k_f)$$

entstehen; oder es ist

$$e(s) = \sum_{(k)} c_0(k_{1'} \cdots k_{f'}) \cdot E(k_1 \cdots k_f).$$

Die Komponente $F'(i_1 \cdots i_f)$ von $F' = \mathbf{e}F$ ist darum

$$= \sum_{s} e(s) F(i_{1'} \cdots i_{f'}) = \sum_{(k)} c(i_1 \cdots i_f; k_1 \cdots k_f) E(k_1 \cdots k_f)$$

mit

$$c(i_1 \cdots i_f;\ k_1 \cdots k_f) = \sum_s F(i_{1'} \cdots i_{f'})\ c_0(k_{1'} \cdots k_{f'}).$$

Es entsteht F' also aus E durch eine symmetrische Transformation und gehört folglich zu $\mathfrak{P}$. — Liegt F insbesondere in $\mathfrak{P}(\mathfrak{p})$, so ist der nach dem eben Bewiesenen zu $\mathfrak{P}$ gehörige Tensor $\mathbf{e}F$ gleich F.

Zugleich ist damit erkannt:

Satz 8. *In dem invarianten Teilraum $\mathfrak{P}$ von $\mathfrak{R}$ existiert ein primitiver Tensor E, d. h. ein solcher, aus dem alle Tensoren von $\mathfrak{P}$ durch geeignete symmetrische Transformationen entstehen.*

Daraufhin kann die Erklärung von $\mathfrak{p}$ durch die einfachere ersetzt werden: Eine Größe $\mathbf{c}$ gehört dann und nur dann zu $\mathfrak{p}$, wenn sie vermöge der Gleichungen

$$c(s) = \sum_{(i)} c(i_1 \cdots i_f) \cdot \mathbf{s}E(i_1 \cdots i_f)$$

aus einem Zahlsystem $c(i_1 \cdots i_f)$ hervorgeht.

Aus Satz 7 folgt

Satz 9. *Wenn der invariante Teilraum $\mathfrak{p}$ von $\mathfrak{r}$ irreduzibel ist, gilt das Gleiche für den durch $\mathfrak{p}$ bestimmten invarianten Teilraum $\mathfrak{P}$ von $\mathfrak{R}$.*

Der *Beweis* hält sich an die umgekehrte Formulierung: Wenn $\mathfrak{P}$ reduzibel ist, muß $\mathfrak{p}$ reduzibel sein. Es sei also $\mathfrak{P}'$ ein invarianter Teilraum von $\mathfrak{P}$, der weder mit ganz $\mathfrak{P}$ zusammenfällt, noch allein aus dem Tensor 0 besteht. Zu ihm konstruieren wir das zugehörige $\mathfrak{p}'$ nach dem Beweis von Satz 7. Ist $\mathbf{x}$ eine Größe in $\mathfrak{p}'$, so hat es die Form (8), wo F ein Tensor in $\mathfrak{P}'$ ist. Aber F liegt dann in $\mathfrak{P}$, oder die Größe $\mathbf{F}$ und damit $\mathbf{x}$ liegen in $\mathfrak{p}$: $\mathfrak{p}' < \mathfrak{p}$. $\mathfrak{p}'$ bestimmt $\mathfrak{P}'$. Bestünde es nur aus der Größe 0 oder wäre $\mathfrak{p}' = \mathfrak{p}$, so wäre $\mathfrak{P}' = 0$ bzw. $= \mathfrak{P}$.

In Verbindung mit Satz 4 ergibt sich aus Satz 9 die Tatsache:

Satz 10. *$\mathfrak{R}$ läßt sich in irreduzible invariante Teilräume zerlegen.*

§ 4. Äquivalenz in $\mathfrak{r}$ und $\mathfrak{R}$.

1. Die Fälle $n \geqq f$ einerseits, $n < f$ andererseits zeigen einen bemerkenswerten Unterschied, den wir nicht ignorieren können. Wenn die Dimensionszahl $n \geqq f$ ist, sind die $f!$ Größen $\mathbf{s}F$ voneinander linear unabhängig, d. h. es besteht keine Gleichung $\mathbf{a}F = 0$, welche identisch für alle Tensoren F erfüllt ist, außer der trivialen mit dem Koeffizienten $\mathbf{a} = 0$. Denn es sind ja bereits die Komponenten

$$\mathbf{s}F(1, 2, \cdots, f) = F(1', 2', \cdots, f')$$

$f!$ voneinander linear unabhängige Größen. Wenn aber $n < f$ ist, besteht z. B. identisch in F die Gleichung

$$\sum_s \delta_s \cdot \mathbf{s}F = 0,$$

in der $\delta_s = \pm 1$ zu setzen ist, je nachdem s eine gerade oder ungerade Permutation ist. Die im Beweise von Satz 7 durchgeführte Konstruktion von $\mathfrak{p}$ hätten wir also zunächst einmal, statt auf $\mathfrak{P}$, auf die Gesamtheit $\mathfrak{R}$ *aller* Tensoren F der f^{ten} Stufe anwenden sollen. Sie liefert eine links-invariante Subalgebra $\dot{\mathfrak{r}}$, in welcher die zu *jedem* Tensor F gehörige Größe $\mathbf{F}$ enthalten ist. Nur wenn $n \geqq f$ ist, fällt $\dot{\mathfrak{r}}$ mit ganz $\mathfrak{r}$ zusammen. Ein invarianter Teilraum $\mathfrak{p}$ von $\mathfrak{r}$ bestimmt dasselbe $\mathfrak{P}$ wie der Durchschnitt von $\mathfrak{p}$ mit $\dot{\mathfrak{r}}$. Auf Grund dieses Sachverhalts wird man sich (für $n < f$) auf solche $\mathfrak{p}$ beschränken, die in $\dot{\mathfrak{r}}$ enthalten sind. In der Tat führt die Konstruktion im Beweise von Satz 7 stets auf einen $\mathfrak{P}$ bestimmenden Teilraum $\mathfrak{p}$ von $\dot{\mathfrak{r}}$.

Dies mußte vorausgeschickt werden, um die Umkehrung von Satz 6 zu formulieren, die offenbar nur richtig sein kann, wenn wir $\dot{\mathfrak{r}}$ an Stelle von $\mathfrak{r}$ zugrunde legen.

Satz 11. *Der invariante Teilraum $\mathfrak{p}$ bestimmt dasselbe $\mathfrak{P}$ wie der Durchschnitt von $\mathfrak{p}$ mit $\dot{\mathfrak{r}}$. Zwei in $\dot{\mathfrak{r}}$ enthaltene verschiedene oder inäquivalente Teilräume $\mathfrak{p}$ bestimmen zwei verschiedene bzw. inäquivalente $\mathfrak{P}$.*

Im Falle $n \geqq f$ hätte man den Fundamentalsatz 7 auf Grund des folgenden Gedankens beweisen können, der an sich sehr nahe liegt und Einem in der Quantenmechanik durch die Störungstheorie direkt unter die Nase gehalten wird. Man faßt im gegebenen invarianten Teilraum $\mathfrak{P}$ die speziellen Tensoren G ins Auge, von denen alle Komponenten verschwinden, deren Indizes nicht eine Permutation der Ziffern $1, 2, \cdots, f$ sind, und kennzeichnet einen solchen Tensor G durch die Größe $\mathbf{g}$ mit den Komponenten

$$g(s) = G(1', 2', \cdots, f').$$

Die so erhaltenen $\mathbf{g}$ bilden den gesuchten Teilraum $\mathfrak{p}$. Mit dieser Definition wäre Satz 11 (in dem nun die Beschränkung auf $\dot{\mathfrak{r}}$ fortfällt) sehr einfach zu beweisen. Läßt man aber die Einschränkung $n \geqq f$ fallen, so ist erstens nötig, diesen „störungstheoretischen" Gedanken durch den von uns im Beweise von Satz 7 verwendeten zu ersetzen, und zweitens müssen wir über $\dot{\mathfrak{r}}$ soviel wissen, wie im folgenden Satz 12 niedergelegt ist. In ihm gebrauchen wir die Abkürzung

$$S(\mathbf{ab}) = S(\mathbf{ba}) = \sum_s a(s)\, b(s^{-1}).$$

Unter der „Spur" S einer Größe $\mathbf{a}$ verstehen wir nämlich die Komponente $a(\mathfrak{l})$, welche der identischen Permutation $s = \mathfrak{l}$ entspricht. $\mathbf{a}$ und $\mathbf{b}$ liegen „in Involution", wenn die Spur des Produkts $\mathbf{ab}$ verschwindet.

Satz 12. *$\dot{\mathfrak{r}}$ ist eine invariante, d. i. sowohl links- wie rechts-invariante Subalgebra. Eine Größe $\mathbf{c}$ von $\dot{\mathfrak{r}}$, welche mit allen Größen von $\dot{\mathfrak{r}}$ in Involution liegt, ist notwendig $= 0$.*

Beweis. $\mathbf{x}$ gehört zu $\dot{\mathfrak{r}}$, wenn seine Komponenten durch Gleichungen gegeben werden

$$x(s) = \sum_{(i)} c(i_1 \cdots i_f) \cdot \mathbf{s} F(i_1 \cdots i_f),$$

in denen die Zahlen c und F keiner Einschränkung unterworfen sind (während die Indizes i nur die Werte von 1 bis n durchlaufen). Die Tatsache, daß $\dot{\mathfrak{r}}$ linksinvariant ist, kann dem Teil a des Beweises von Satz 7 entnommen werden: die durch $x'(t) = x(st)$ erklärte Größe $\mathbf{x}'$ wird geliefert durch die Gleichungen

$$x'(t) = \sum_{(i)} c'(i_1 \cdots i_f) \cdot \mathbf{t} F(i_1 \cdots i_f)$$

mit den Koeffizienten $c' = \mathbf{s}^{-1} c$. Andererseits gilt für $x''(t) = x(ts)$ offenbar

$$x''(t) = \sum_{(i)} c(i_1 \cdots i_f) \cdot \mathbf{t} F'(i_1 \cdots i_f)$$

mit $F' = \mathbf{s} F$.

$\mathbf{i}$ sei erzeugende Einheit von $\dot{\mathfrak{r}}$. Eine beliebige Größe $\mathbf{x}$ werde zerlegt gemäß der Gleichung

$$\mathbf{x} = \mathbf{x}\mathbf{i} + (\mathbf{x} - \mathbf{x}\mathbf{i}).$$

Erfüllt $\mathbf{c}$ die im Satze genannten Voraussetzungen, so ist mit $\mathbf{c}$ auch $\mathbf{cx}$ eine Größe in $\dot{\mathfrak{r}}$ und darum gilt

$$\mathbf{cx} = \mathbf{cx} \cdot \mathbf{i} = \mathbf{c} \cdot \mathbf{xi}.$$

Die Spur der rechten Seite, des Produkts von $\mathbf{c}$ mit einer in $\dot{\mathfrak{r}}$ gelegenen Größe $\mathbf{xi}$, ist nach Voraussetzung $= 0$, darum ist auch $S(\mathbf{cx}) = 0$. $\mathbf{c} = 0$ ist nunmehr eine unmittelbare Folge der für *alle* $\mathbf{x}$ gültigen Identität $S(\mathbf{cx}) = 0$.

2. Nach dieser Vorbereitung können wir zum Beweise von Satz 11 schreiten. Der durch die Konstruktion (K), Gleichung (8), aus dem invarianten Teilraum $\mathfrak{P}$ von $\mathfrak{R}$ gewonnene invariante Teilraum $\mathfrak{p}$ von $\dot{\mathfrak{r}}$ werde mit $\mathfrak{p}(\mathfrak{P})$ bezeichnet. Es gilt zweierlei zu zeigen: I. Bestimmt der in $\dot{\mathfrak{r}}$ enthaltene invariante Teilraum $\mathfrak{p}$ den invarianten Teilraum $\mathfrak{P}$ von $\mathfrak{R}$, so ist umgekehrt $\mathfrak{p} = \mathfrak{p}(\mathfrak{P})$. II. Sind $\mathfrak{P}_1$, $\mathfrak{P}_2$ zwei äquivalente invariante Teilräume von $\mathfrak{R}$, so sind die zugeordneten $\mathfrak{p}_1 = \mathfrak{p}(\mathfrak{P}_1)$ und $\mathfrak{p}_2 = \mathfrak{p}(\mathfrak{P}_2)$ gleichfalls äquivalent.

I. Die Aussage $\mathfrak{p}(\mathfrak{P}) < \mathfrak{p}$ ist trivial. Denn eine willkürliche Größe $\mathbf{x}$ von $\mathfrak{p}(\mathfrak{P})$ entsteht durch Gleichung (8) aus einem Tensor F in $\mathfrak{P}$. Mit $\mathbf{F}$ liegt dann offenbar auch $\mathbf{x}$ in $\mathfrak{p}$.

$\mathbf{e}$ sei erzeugende Einheit in $\mathfrak{p}$, F ein willkürlicher Tensor,

$$F' = \hat{\mathbf{e}} F = \sum_{s} e(s^{-1}) \cdot \mathbf{s} F;$$

mit Hilfe desselben willkürlichen Koeffizientensystems c mögen nach Formel (8) aus F und F' die Größen $\mathbf{x}$, $\mathbf{x}'$ entspringen. F' ist ein Tensor in $\mathfrak{P}$, folglich gehört nach Definition $\mathbf{x}'$ zu $\mathfrak{p}(\mathfrak{P})$. $\mathbf{x}'$ ist $= \mathbf{xe}$. Das Ergebnis ist dies: liegt $\mathbf{x}$ in $\mathfrak{r}$, so $\mathbf{xe}$ in $\mathfrak{p}(\mathfrak{P})$. Da $\mathfrak{p}$ in $\mathfrak{r}$ enthalten sein sollte, gilt insbesondere: wenn $\mathbf{x}$ in $\mathfrak{p}$ liegt, so $\mathbf{xe} = \mathbf{x}$ in $\mathfrak{p}(\mathfrak{P})$; oder es gilt $\mathfrak{p} \prec \mathfrak{p}(\mathfrak{P})$.

II. In $\mathfrak{P}_1$ wähle man den primitiven Tensor E_1; sein Abbild in $\mathfrak{P}_2$ ist ein primitiver Tensor E_2 von $\mathfrak{P}_2$. Die Größen $\mathbf{c}_1$ und $\mathbf{c}_2$ in den zugehörigen Teilräumen $\mathfrak{p}_1$, $\mathfrak{p}_2$ von $\mathfrak{r}$ werden durch die Gleichungen geliefert

$$c_\alpha(s) = \sum_{(i)} c(i_1 \cdots i_f) \cdot \mathbf{s} E_\alpha(i_1 \cdots i_f) \qquad [\alpha = 1, 2], \tag{10}$$

in der die Koeffizienten $c(i_1 \cdots i_f)$ unabhängig voneinander alle Werte durchlaufen. Die gesuchte ähnliche Abbildung von $\mathfrak{p}_1$ auf $\mathfrak{p}_2$ muß dadurch bewerkstelligt werden, daß man die beiden Größen $\mathbf{c}_1$ und $\mathbf{c}_2$ einander zuordnet, die in dieser Weise demselben Koeffizientensystem $c(i_1 \cdots i_f)$ entspringen. Die Möglichkeit dieser Zuordnung ist aber an die Bedingung gebunden: daß allemal, wenn $\mathbf{c}_1 = 0$ ist, auch $\mathbf{c}_2$ verschwindet. Die Voraussetzung, daß eine E_1 in E_2 überführende ähnliche Abbildung von $\mathfrak{P}_1$ auf $\mathfrak{P}_2$ vorliegt, besagt: Wenn eine symmetrische Transformation den Tensor E_1 in Null überführt, transformiert sie auch E_2 in 0. Das Bindeglied zwischen Voraussetzung und Behauptung ist die Aussage: Wenn für irgend einen Tensor F

$$\sum_s c_1(s^{-1}) \cdot \mathbf{s} F = 0 \tag{11}$$

ist, gilt die entsprechende Gleichung für $\mathbf{c}_2$:

$$\sum_s c_2(s^{-1}) \cdot \mathbf{s} F = 0. \tag{12}$$

In der Tat ist auf Grund der Gleichung

$$c_\alpha(s^{-1}) = \sum_{(k)} \mathbf{s}\, c(k_1 \cdots k_f) \cdot E_\alpha(k_1 \cdots k_f):$$

$$\sum_s c_\alpha(s^{-1}) \cdot \mathbf{s} F(i_1 \cdots i_f) = \sum_{(k)} c(i_1 \cdots i_f;\ k_1 \cdots k_f) \cdot E_\alpha(k_1 \cdots k_f)$$

mit

$$c(i_1 \cdots i_f;\ k_1 \cdots k_f) = \sum_s F(i_{1'} \cdots i_{f'}) \cdot c(k_{1'} \cdots k_{f'}).$$

Die linken Seiten von (11), (12) gehen daher durch *dieselbe* symmetrische Transformation aus E_1 bzw. E_2 hervor.

Insbesondere gilt, wenn $\mathbf{c}_1 = 0$ ist, die Gleichung (12) für einen *beliebigen* Tensor F. Das heißt aber: unter der Voraussetzung $\mathbf{c}_1 = 0$ ist $S(\mathbf{c}_2 \mathbf{x}) = 0$ für alle Größen $\mathbf{x}$ in $\mathfrak{r}$. Außerdem liegt $\mathbf{c}_2$ selber in $\mathfrak{r}$. Die Anwendung des letzten Teiles von Satz 12 vollendet daher den Beweis.

§ 5. In $(\lambda)^f$ enthaltene Darstellungen der linearen Gruppe.

Die Quantentheorie ist allein an der in §§ 3 und 4 aufgedeckten Beziehung zwischen den beiden Gruppen π und σ interessiert. In σ ist die Untergruppe σ_0 der speziellen symmetrischen Transformationen enthalten, welche mit der Gruppe λ aller linearen Transformationen B in n Variablen isomorph ist und darum eine bestimmte Darstellung $(\lambda)^f$ von λ liefert. Ein linearer Teilraum $\mathfrak{P}$ von $\mathfrak{R}$, der gegenüber σ invariant ist, ist auch gegenüber σ_0 invariant; und die Beziehung der Äquivalenz zwischen zwei solchen invarianten Teilräumen bleibt a fortiori bestehen, wenn die Gruppe σ auf die engere σ_0 beschränkt wird. Gilt aber von diesen beiden selbstverständlichen Sätzen auch die Umkehrung? Ja; denn:

Satz 13. *Jede symmetrische Transformation läßt sich durch lineare Kombination geeigneter spezieller symmetrischer Transformationen gewinnen.*

Dies kommt auf den Satz hinaus: ist die lineare Gleichung

$$\sum_{(i,k)} c(i_1 k_1, \cdots, i_f k_f) \cdot x(i_1 \cdots i_f;\ k_1 \cdots k_f) = 0$$

erfüllt für alle x von der Gestalt

$$x(i_1 \cdots i_f;\ k_1 \cdots k_f) = x(i_1 k_1) \cdots x(i_f k_f),$$

so ist sie erfüllt für alle symmetrischen x. Das Paar ik kann als ein einziger Index j geschrieben werden. Dann ist diese Behauptung nichts anderes als die simple, häufig benutzte Tatsache der formalen Algebra: verschwindet eine Form

$$\sum_{(j)} c(j_1 \cdots j_f)\ x(j_1) \cdots x(j_f)$$

für alle Werte der n Variablen $x(j)$, so verschwindet sie mit allen ihren Koeffizienten c — vorausgesetzt, daß c symmetrisch von den f Indizes j abhängt.

Dieses Resultat mit den in §§ 3 und 4 gewonnenen verbindend, erkennen wir:

Satz 14. *Ist* $\mathfrak{p}$ *ein invarianter Teilraum von* $\mathfrak{r}$, *so bilden die Tensoren von der durch* $\mathfrak{p}$ *bestimmten Symmetrie das Substrat einer in* $(\lambda)^f$ *enthaltenen Darstellung* $\mathcal{A}$ *der linearen Gruppe* λ. *Ist* $\mathfrak{p}$ *irreduzibel, so ist auch die Darstellung* $\mathcal{A}$ *irreduzibel. Äquivalenten invarianten Teilräumen* $\mathfrak{p}$ *entsprechen äquivalente Darstellungen* $\mathcal{A}$, *inäquivalenten immer dann inäquivalente, wenn wir uns, wie das natürlich ist, auf die in* $\dot{\mathfrak{r}}$ *enthaltenen Teilräume* $\mathfrak{p}$ *beschränken.*

Die reguläre Darstellung (π) von π bestand darin, daß s die Abbildung $\mathfrak{x}' = s\mathfrak{x}$ im totalen $f!$-dimensionalen Gruppenraum $\mathfrak{r}$ zugeordnet wurde. Im Falle $n < f$ möge hier $\mathfrak{r}$ durch $\dot{\mathfrak{r}}$ ersetzt werden, wodurch eine Darstellung $(\dot{\pi})$ von weniger als $f!$ Dimensionen entsteht. Durch Satz 14 steht zugleich folgendes fest:

Satz 15. *Jeder in* (π) *enthaltenen Darstellung* Π *von* π *entspricht eine eindeutig bestimmte, in* $(\lambda)^f$ *enthaltene Darstellung* Λ *von* λ; *zwischen äquivalenten Darstellungen soll dabei auf beiden Seiten nicht unterschieden werden. Irreduziblem* Π *entspricht ein irreduzibles* Λ. *Die Beziehung wird eineindeutig, wenn wir (im Falle* $n<f$*) die reguläre Darstellung* (π) *durch* $(\dot{\pi})$ *ersetzen.*

Satz 4 führt über Satz 10 zu

Satz 16. *Die Darstellung* $(\lambda)^f$ *von* λ *zerfällt in irreduzible Bestandteile. Erscheint bei der Zerlegung von* $(\dot{\pi})$ *die irreduzible Darstellung* Π *der Permutationsgruppe g-mal, so tritt das zugehörige* Λ *in* $(\lambda)^f$ *gleichfalls g-mal auf.*

§ 6. Verallgemeinerung.

Den Betrachtungen von §§ 3 und 4 kann, *statt der vollen symmetrischen Gruppe* π, *irgendeine Untergruppe, eine Permutationsgruppe* π *zugrunde gelegt werden.* Natürlich ist der Begriff der symmetrischen Transformation ihr anzupassen: von dem Koeffizienten $c(i_1 \cdots i_f; k_1 \cdots k_f)$ wird nur verlangt, daß er ungeändert bleibt, wenn eine *zur Gruppe* π *gehörige* Permutation auf die Subindizes $1, 2, \cdots, f$ ausgeübt wird.

Auch ist zu bemerken, daß *der zugrunde liegende Zahlbereich irgend ein Körper* sein kann im Sinne der abstrakten Algebra; nur darf die „Charakteristik" (in der Steinitzschen Terminologie) nicht eine in der Ordnung der Gruppe π aufgehende Primzahl sein. Um der Beweisführung diesen Grad von Allgemeinheit zu sichern, habe ich hier (im Gegensatz zu der Darstellung in meinem Buch) den Gebrauch konjugiert-komplexer Zahlen vermieden und nur rationale Operationen benutzt. Die Einschränkung hinsichtlich der Charakteristik ist nötig, weil im Beweise von Satz 2 und 3 durch die Ordnung von π dividiert werden muß.

Die Untersuchung von § 5 gestattet ersichtlich keine so weitgehende Verallgemeinerung. Aber man kann eine Permutationsgruppe π von folgender Art ins Auge fassen: die Reihe der Ziffern von 1 bis f sei in mehrere Teilreihen zerlegt, und zu π gehören alle und nur diejenigen Permutationen, welche keine Ziffern zwischen den Reihen austauschen. Es seien z. B. zwei Teilreihen vorhanden; die erste Reihe möge aus den „roten" Ziffern von 1 bis f, die zweite aus den „grünen" Ziffern von 1 bis g bestehen. Es liege ein „roter" m-dimensionaler und ein „grüner" n-dimensionaler Vektorraum vor; wir beschäftigen uns mit Tensoren, die sozusagen mit einem Fuß in dem einen und mit dem andern Fuß in dem andern Raum stehen. Ist der Tensor von f^{ter} Stufe im roten, von g^{ter} Stufe im grünen Raum, so hängt seine allgemeine Komponente $F(i_1 \cdots i_f, j_1 \cdots j_g)$ von f roten, im Bereiche von 1 bis m variierenden und g grünen, den Spielraum von 1 bis n durchlaufenden Indizes ab. Erleiden die Koordinaten x_i im roten Raum die lineare Transformation

$$A: \quad x'_{i'} = \sum_{i=1}^{m} a(i'\,i)\,x_i,$$

die Koordinaten y_j im grünen die lineare Transformation

$$B: \quad y'_{j'} = \sum_{j=1}^{n} b(j'\,j)\,y_j,$$

so erleiden die Tensorkomponenten die Transformation

$$F'(i'_1 \cdots i'_f, j'_1 \cdots j'_g) =$$

$$(13) \qquad \sum_{(i,j)} a(i'_1\,i_1) \cdots a(i'_f\,i_f)\; b(j'_1\,j_1) \cdots b(j'_g\,j_g) \cdot F(i_1 \cdots i_f, j_1 \cdots j_g).$$

Die Mannigfaltigkeit unserer Doppeltensoren ist daher das Substrat einer gewissen Darstellung des vom Paare A, B durchlaufenen direkten Produkts der linearen Transformationsgruppe im roten und grünen Raum. Die Transformationen von der Gestalt (13), welche jetzt die Rolle der „speziellen" symmetrischen Transformationen übernehmen sollen, sind in der Tat symmetrisch mit Bezug auf die Gruppe π derjenigen Permutationen, welche die roten und grünen Ziffern je untereinander beliebig vertauschen.

§ 7. Schlußbemerkungen, insbesondere über das Spin-Problem.

1. Unserer Hauptuntersuchung, §§ 2 bis 5, fügen wir einige Bemerkungen von weniger elementarem und mehr referierendem Charakter hinzu. Der allgemeinen Theorie der endlichen Gruppen ist zu entnehmen, daß bei der Reduktion der regulären Darstellung (π) *jede* irreduzible Darstellung erscheint, und zwar so oft, wie ihr Grad g angibt. Ist $\mathfrak{p}$ ein invarianter Teilraum von $\dot{\mathfrak{r}}$ und $\mathfrak{p}'$ äquivalent zu $\mathfrak{p}$, so kann nach Satz 4 die ähnliche Abbildung von $\mathfrak{p}$ auf $\mathfrak{p}'$ durch eine Gleichung $\mathbf{x}' = \mathbf{x}\mathfrak{b}$ vollzogen werden. Darum gehört, weil $\dot{\mathfrak{r}}$ rechtsinvariante Subalgebra ist, auch ganz $\mathfrak{p}'$ zu $\dot{\mathfrak{r}}$. Demnach tritt jede irreduzible Darstellung, die überhaupt in $(\dot{\pi})$ vorkommt, darin ebenso oft auf wie in (π): so oft, wie ihr Grad angibt. Wir kennzeichnen die einzelnen irreduziblen Darstellungen Π von π durch ihre Charaktere χ; den Grad g und das zugehörige Δ bezeichnen wir daher deutlicher mit $g(\chi)$ und $\Delta(\chi)$. Die Zerlegung von $(\lambda)^f$ erfolgt gemäß der Gleichung:

$$(\lambda)^f = \sum g(\chi) \cdot \Delta(\chi),$$

wobei sich die Summe nur über diejenigen irreduziblen Darstellungen χ erstreckt, die in $(\dot{\pi})$ vorkommen.

Es sollte beachtet werden, daß σ nicht eigentlich eine *Gruppe*, sondern eine *Algebra* ist. Nachdem wir $\mathfrak{R}$ in irreduzible invariante Teilräume zerlegt haben, bestimmen wir unter ihnen einen vollen Satz inäquivalenter irreduzibler Teilräume $\mathfrak{R}_1, \mathfrak{R}_2, \cdots$, indem wir von je g äquivalenten, welche bei

der Zerlegung auftreten, immer einen willkürlich aussuchen. Beziehen wir diese Teilräume je auf ein bestimmtes Koordinatensystem, so wird die symmetrische Transformation C in ihnen durch je eine Matrix $C_1, C_2, \cdots$ dargestellt. Die Korrespondenzen

$$C \to C_1, \quad C \to C_2, \quad \cdots,$$

geben mehrere inäquivalente irreduzible „Darstellungen" der Algebra σ [wobei für eine Algebra der Begriff der Darstellung außer der Erhaltung der Komposition auch die Erhaltung der linearen Beziehungen einschließt: mit $C \to C_1$ ist

$$\alpha C \to \alpha C_1, \quad C + C' \to C_1 + C_1', \quad CC' \to C_1 C_1' \qquad (\alpha \text{ eine Zahl})].$$

Nach einem allgemeinen Theorem von Burnside, Frobenius und Schur folgt daraus, daß für ein variables C die sämtlichen Komponenten der Matrizen $C_1, C_2, \cdots$ voneinander linear unabhängig sind. Aus dem gleichzeitigen Verschwinden der Matrizen $C_1, C_2, \cdots$ folgt rückwärts gemäß unserer Konstruktion, daß das Element C der Algebra σ gleich 0 ist. Das Resultat ist also dies, daß die Algebra σ in mehrere vollständige Matrixalgebren zerfällt. Nach bekannten Sätzen schließt man daraus, daß *jede irreduzible Darstellung der Algebra σ mit einer der hier aufgetretenen äquivalent sein muß,* und ferner, daß *jede Darstellung der Algebra σ in irreduzible zerlegt werden kann.*

Eine Darstellung der Gruppe λ heiße von der Ordnung f, wenn die Komponenten der dem variablen Element $B = \|b(ik)\|$ von λ zugeordneten Matrix ganze rationale Funktionen der $b(ik)$ vom f^{ten} Grade sind. Die Spezialisierung von σ zu σ_0 führt darum auf rein algebraischem Wege zu den Tatsachen: *Jede irreduzible Darstellung f^{ter} Ordnung der Gruppe λ aller linearen Transformationen ist in $(\lambda)^f$ enthalten. Jede Darstellung f^{ter} Ordnung von λ zerfällt in irreduzible.* Dieser Gedankengang ist im wesentlichen von I. Schur angegeben*; man mag in diesen seinen Überlegungen bereits implizite die Idee finden, die *Gruppe* σ_0 der B_f durch die *Algebra* σ aller symmetrischen Transformationen zu ersetzen.

2. Da das Folgende nur für den Leser des V. Kapitels meines Buches über Gruppen und Quanten verständlich ist, verwende ich ohne Erklärung gewisse dort eingeführte Bezeichnungen.

Der Zusammenhang zwischen den Darstellungen Π und Λ von π und λ muß sich in einer einfachen Beziehung zwischen ihren *Charakteren* kundgeben. Die elementare Rechnung liefert die Formel (49) l. c., p. 230, welche zusammen mit der Vollständigkeitsrelation für die primitiven Charaktere χ zu (50) führt. Aus dieser Gleichung ergeben sich, als der Ausdruck der einfachen Eigenschaften der Exponentialfunktion

* In der oben zitierten Arbeit Sitzungsber. Berl. Akad. 1927.

$$a^{m+n} = a^m \cdot a^n \quad \text{und} \quad (ab)^n = a^n \cdot b^n,$$

zwei in § 55 daselbst entwickelte *Reziprozitätstheoreme.*

Zu der Darstellung Π: $s \to U(s)$ von π erhält man eine andere, die *adjungierte* $\widetilde{\Pi}$ durch die Zuordnung $s \to \widetilde{U}(s)$, wenn $\widetilde{U}$ die kontragrediente Transformation zu U ist. Das Schursche Fundamentallemma besagt: Wenn Π, Π' irreduzible Darstellungen sind, so tritt in $\Pi \times \Pi'$ die identische Darstellung einmal oder keinmal auf, je nachdem Π' äquivalent $\widetilde{\Pi}$ ist oder nicht.

Die eindimensionale Darstellung der Permutationsgruppe π, welche dem Element s die Zahl $\delta_s = \pm 1$ zuordnet, werde mit $\{1\}$, die zugehörige Darstellung von λ, deren Substrat die schiefsymmetrischen Tensoren f^{ter} Stufe sind, mit $\{\lambda\}^f$ bezeichnet. Mit einer beliebigen Darstellung Π: $s \to U(s)$ von π ist die *duale* Π^*: $s \to \delta_s \widetilde{U}(s)$ verknüpft. Aus dem angeführten Lemma folgt der

Hilfssatz: Sind Π und Π' irreduzible Darstellungen von π, so enthält $\Pi \times \Pi'$ die Darstellung $\{1\}$ einmal oder keinmal, je nachdem Π' zu Π dual ist oder nicht.

3. Zufolge der Existenz des spins stellt sich, wenn man die dynamische Wirkung des spins vernachlässigt, das gruppentheoretische Problem folgendermaßen. Der zugrunde gelegte Vektorraum hat nicht n, sondern $N = \nu \cdot n$ Dimensionen, und die Variablen sind durch zwei Indizes ι, i unterschieden, deren einer von 1 bis ν, deren anderer von 1 bis n läuft. Im Raum der Tensoren f^{ter} Stufe interessieren jedoch nur diejenigen symmetrischen Transformationen, welche die griechischen Indizes nicht angreifen:

$$(14) \quad F'(\iota_1 i_1, \cdots, \iota_f i_f) = \sum_{(k)} c(i_1 \cdots i_f;\ k_1 \cdots k_f) \cdot F(\iota_1 k_1, \cdots, \iota_f k_f).$$

Im Vektorraum, heißt das, wird die Gruppe λ_N aller linearen Transformationen beschränkt auf die Gruppe der Transformationen von der Art

$$(15) \qquad x'(\iota\, i) = \sum_{k=1}^{n} b(i\, k)\ x(\iota\, k).$$

Eine irreduzible Darstellung von λ_N liefert durch diese Beschränkung eine Darstellung der Gruppe λ_n aller linearen Transformationen $B = \|b(ik)\|$ in n Dimensionen. Sie ist in ihre irreduziblen Bestandteile zu zerlegen.

Das Pauli-Verbot besagt, daß in der Physik diese Aufgabe nur für die spezielle Darstellung $\{\lambda_N\}^f$ zu lösen ist, deren Substrat die schiefsymmetrischen Tensoren f^{ter} Stufe sind.

Es stellt sich als zweckmäßig heraus, in zwei Schritten vorzugehen. Zunächst wird die Gruppe λ_N aller linearen Transformationen im N-dimensionalen Raum eingeschränkt zu der Gruppe $\lambda_\nu \times \lambda_n$, welche aus den Transformationen besteht

$$x'(\iota\, i) = \sum_{\varkappa, k} \beta(\iota\varkappa)\, b(ik)\, x(\varkappa k)$$

und darauf wird für $B = \| \beta(i\varkappa) \|$ die Einheitsmatrix gesetzt. Der erste Schritt wird erledigt durch die Formel

$$\{\lambda_N\}^f = \sum_{\chi} A_\nu(\chi^*) \times A_n(\chi). \tag{16}$$

Die beiden Faktoren sind Darstellungen f^{ter} Ordnung von λ_ν bzw. λ_n, welche den in Evidenz gesetzten zueinander dualen irreduziblen Darstellungen χ^*, χ von π entsprechen. Die Formel ergibt sich, wenn man das zweite der erwähnten Reziprozitätsgesetze mit dem Hilfssatz verbindet. Bezeichnet $h_n(\chi)$ den Grad von $A_n(\chi)$, so liefert nun der zweite Schritt das Resultat: die Darstellung $\{\lambda_N\}^f$ zerfällt im Bereiche der Transformationen (15) gemäß der Gleichung

$$\{\lambda_N\}^f = \sum_{\chi} h_\nu(\chi^*) \cdot A(\chi) \tag{17}$$

in irreduzible Darstellungen f^{ter} Ordnung von $\lambda = \lambda_n$. Die Modifikation, welche die Existenz des spins unter Vernachlässigung seiner dynamischen Einwirkung und das Pauli-Verbot zur Folge haben, besteht also einfach darin, daß *die Multiplizität, mit der das zu χ gehörige Termsystem auftritt, von $g(\chi)$ in $h_\nu(\chi^*)$ sich verwandelt.* Insbesondere kommen jene Termklassen ganz zum Fortfall, für welche $h_\nu(\chi^*) = 0$ ist.† Die dynamische Einwirkung des spins löst diese *Multipletts* in so viele Komponenten auf, als ihre Vielfachheit $h_\nu(\chi^*)$ angibt; und ruft außerdem schwache Interkombinationen zwischen den verschiedenen Termklassen hervor. — Die Dimensionszahl ν des Spinraums ist übrigens $= 2$.

Ich wünschte auf diese Dinge zurückzukommen, weil in meinem Buch der zweite Schritt, die Ersetzung der Matrix B durch 1 nicht explizite erwähnt wurde und infolgedessen die Beschreibung auf p. 259 nicht einwandfrei ist. Für diese Entwicklungen ist übrigens die Ersetzung von σ durch die beschränktere Gruppe σ_0 unwesentlich. Dem physikalischen Problem angemessener wäre es, bei σ zu bleiben und darum statt von der Zerlegung der Darstellung (17) zu sprechen von der Zerlegung der Mannigfaltigkeit $\{\mathfrak{R}_N\}$ der schiefsymmetrischen Tensoren f^{ter} Stufe im N-dimensionalen Raum in Teilräume, welche invariant sind gegenüber der Gruppe der symmetrischen Transformationen von der Art (14).

† Bezeichnen wir unser obiges $\mathfrak{r}$, das ja von der Dimensionszahl n abhängt (und für $n \geqq f$ mit ganz $\mathfrak{r}$ zusammenfällt) der Deutlichkeit halber mit $\mathfrak{r}^n$ und daher $(\dot{\pi})$ mit $(\pi)^n$, so sind das diejenigen Darstellungen χ^*, welche in $(\pi)^\nu$ nicht vorkommen.

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ON THE SIGNATURE OF A QUADRATIC FORM.

By E. T. Browne.*

1. Introduction. A real quadratic form in n variables $x_1, \cdots, x_n$

$$\sum_{i,j}^{1,\cdots,n} a_{ij}\, x_i\, x_j \qquad (a_{ij} = a_{ji}) \tag{1}$$

of rank r can always be reduced by a real non-singular linear transformation to an expression of the type

$$\sum_{i=1}^{r} c_i\, x_i^2. \tag{2}$$

Although the (real) transformation by which (1) is reduced to the form (2) is not unique, it is well known that the number N of negative coefficients in (2) is the same whatever be the particular transformation employed. A similar remark holds for the number P of positive coefficients in (2). Thus, P and N, and therefore their difference $s = P - N$, called the *signature* of the quadratic form, are arithmetic invariants under real non-singular linear transformations. Since $P + N = r$, manifestly s is determined when N and r are known.

Frobenius has shown† that if (1) is of rank r the variables can always be so numbered that no consecutive two of the numbers in the sequence

$$A_0 = 1, \quad A_1 = a_{11}, \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \cdots, \quad A_r = \begin{vmatrix} a_{11}, \cdots, a_{1r} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{r1}, \cdots, a_{rr} \end{vmatrix} \tag{3}$$

are zero and $A_r \neq 0$. (1) is then said to be *regular*, and its matrix *regularly arranged*. Gundelfinger has shown‡ that in this case the number N of negative coefficients in (2) is equal to the number of variations of sign in the sequence (3), where a vanishing term may be given a sign at pleasure.

In the paper of Frobenius it was shown that for certain particular types of forms, for example, recurring forms in which $a_{ij} = a_{i+j-2}$ even if $A_r = 0$ and an arbitrary number of consecutive A's in the sequence (3) vanish, by replacing the vanishing A_r by a certain new determinant A_r', the number of negative coefficients in (2) can always be determined from the new

* Received September 16, 1928.

† Frobenius, *Über das Trägheitsgesetz der quadratischen Formen*, Crelle, vol. 114 (1895), p. 193.

‡ Gundelfinger, *Zur Theorie der quadratischen Formen*, Crelle, vol. 91 (1881), p. 225.

sequence. Moreover, it was shown by Frobenius (and more recently by Franklin)* that if (1) is a general real quadratic form of rank r in which $A_r \neq 0$, N can always be determined when in the sequence (3) not more than two consecutive A's vanish. The interesting demonstration by Franklin was based on certain properties of the secular equation. In this paper we shall show that Gundelfinger's rule and the scheme of rearrangement, mentioned previously as used by Frobenius, furnish a very simple method of arriving at these facts.

2. **Quadratic forms which are not regular.** In the quadratic form (1) of rank r let us suppose that $A_r \neq 0$. Let us suppose also that

$$(4) \qquad \begin{aligned} A_\tau\, A_{\tau+\sigma+1} &\neq 0 \qquad (0<\tau;\ 1 \leqq \sigma \leqq r-\tau-1), \\ A_{\tau+i} &= 0 \qquad (i = 1, \cdots, \sigma). \end{aligned}$$

Since $A_\tau \neq 0$ while $A_{\tau+1} = 0$ there exists a real set $(x_1, \cdots, x_\tau)$ such that

$$(5) \qquad a_{i,\tau+1} = \sum_{j=1}^{\tau} a_{ij}\, x_j \qquad (i = 1, \cdots, \tau+1).$$

If from the elements of the $(\tau+1)$th column of $A_{\tau+2}$ we subtract the sum of the products of the elements of the first τ columns by $x_1, \cdots, x_\tau$ every element in this column is reduced to zero except the last one which is

$$k = a_{\tau+2,\tau+1} - \sum_{j=1}^{\tau} a_{\tau+2,j}\, x_j.$$

Performing the same operation on the $(\tau+1)$th row and expanding, we have

$$(6) \qquad A_{\tau+2} = -k^2 A_\tau.$$

If, therefore, $\sigma = 1$ in (4) so that $A_{\tau+2} \neq 0$, $A_{\tau+2}$ is evidently of opposite sign to A_τ, a fact which is well known. If, however, $\sigma > 1$ so that $A_{\tau+2} = 0$, (5) holds for $i = \tau+2$.

Now holding the first τ columns fixed in $A_{\tau+\sigma+1}$ let us transfer the $(\tau+\alpha)$th column over the $\alpha-1$ columns immediately preceding it and put it in the $(\tau+1)$th place. Proceeding similarly with the rows, we obtain a new sequence of A's which we shall denote by

$$(3) \qquad A_0, A_1, \cdots, A_\tau, A^{(\alpha)}_{\tau+1}, \cdots, A^{(\alpha)}_{\tau+\sigma}, A_{\tau+\sigma+1}, \cdots, A_r.$$

On making use of the conditions (5) for $i = 1, \cdots, \tau+2$, and expanding $A^{(\alpha)}_{\tau+2}$, we have

$$A^{(\alpha)}_{\tau+2} = -k^2_\alpha A_\tau,$$

* Franklin, *A theorem of Frobenius on quadratic forms*, Bull. Amer. Math. Soc., vol. 33 (1927), pp. 447–452.

where

$$k_\alpha = a_{\tau+\alpha,\tau+1} - \sum_{j=1}^{\tau} a_{\tau+\alpha,j}\, x_j.$$

Now $A_{\tau+2}^{(\alpha)}$ cannot be zero for every choice of α $(3 \leqq \alpha \leqq \sigma+1)$ for in that case we would have

$$a_{\tau+\alpha,\tau+1} = \sum_{j=1}^{\tau} a_{\tau+\alpha,j}\, x_j \qquad (\alpha = 3, \cdots, \sigma+1)$$

which in connection with (5) would give $A_{\tau+\sigma+1} = 0$. Hence, for some choice of α $(3 \leqq \alpha \leqq \sigma+1)$ $A_{\tau+2}^{(\alpha)} \neq 0$ and is of *opposite* sign to A_τ. By a similar argument it follows that if in the subsequence

$$A_\tau, A_{\tau+1}^{(\alpha)}, \cdots, A_{\tau+\sigma}^{(\alpha)}, A_{\tau+\sigma+1}$$

two or more consecutive terms vanish, without affecting those terms which do not vanish, the last $\sigma-1$ rows and columns of $A_{\tau+\sigma+1}$ can be so arranged that in the new subsequence

$$A_\tau, A_{\tau+1}^{(\alpha)}, \cdots, A_{\tau+\sigma}^{(\alpha)}, A_{\tau+\sigma+1} \tag{7}$$

no two consecutive A's vanish.

Assuming that the number of variations of sign in the remainder of the sequence (3) can be determined we proceed to the determination of the number ν of variations in the subsequence (7). Since A_τ and $A_{\tau+2}^{(\alpha)}$ have opposite signs, ν evidently satisfies the inequalities

$$1 \leqq \nu \leqq \sigma, \tag{8}$$

ν being even or odd according as A_τ and $A_{\tau+\sigma+1}$ have the same sign or opposite signs.

In the preceding discussion we assumed that $A_\tau \neq 0$ for $\tau > 0$. This restriction may easily be removed. For if $A_1 = A_2 = \cdots = A_\sigma = 0$, we consider the quadratic form

$$x_0^2 + \sum_{i,j}^{1,\cdots,n} a_{ij}\, x_i\, x_j, \tag{9}$$

the sequence (3) corresponding to which is

$$A_0 = 1, A_1 = 1, A_2 = a_{11}, \cdots, A_{r+1} = \begin{vmatrix} a_{11}, \cdots, a_{1r} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{r1}, \cdots, a_{rr} \end{vmatrix}. \tag{3'}$$

The canonical forms of (9) and (1) have exactly the same number of negative coefficients. Moreover, the number of variations of sign in the sequence (3′) is exactly equal to the number of variations in the sequence (3), and in the former $A_\tau \neq 0$ with $\tau > 0$.

We may draw the following immediate conclusions:

If $\sigma = 2$, $\nu = 2$ or 1 according as A_r and A_{r+3} have the same sign or opposite signs.

If $\sigma = 3$ so that $1 \leqq \nu \leqq 3$, if A_r and A_{r+4} have the same sign, ν is even and hence is equal to 2; but if A_r and A_{r+4} have opposite signs, ν is odd and we cannot distinguish between the cases $\nu = 1$ and $\nu = 3$. Indeed, in this latter case the signature of (1) cannot be determined from the sequence (3) alone. Cf. the classical example of Frobenius as quoted by Franklin.*

If $\sigma = 4$ so that $1 \leqq \sigma \leqq 4$, if A_r and A_{r+5} have the same sign, ν is even and we cannot distinguish between the cases $\nu = 2$ and $\nu = 4$; while if A_r and A_{r+5} have opposite signs, ν is odd and we cannot distinguish between the cases $\nu = 1$ and $\nu = 3$.

That all of these cases actually arise is clear from a single example. Thus, consider the form

$$2ax_1x_5 + 2bx_2^2 + 2bx_2x_4 + ax_3^2 + 2bx_4^2$$

the sequence of A's corresponding to which is

$$1, \quad 0, \quad 0, \quad 0, \quad 0, \quad -3a^3b^2.$$

However, on renumbering the variables

$$x_1 = y_5, \quad x_2 = y_1, \quad x_3 = y_2, \quad x_4 = y_3, \quad x_5 = y_4,$$

the form becomes

$$2by_1^2 + 2by_1y_3 + ay_2^2 + 2by_3^2 + 2ay_4y_5,$$

the sequence of A's corresponding to which is

$$1, \quad 2b, \quad 2ab, \quad 3ab^2, \quad 0, \quad -3a^3b^2.$$

The form is now regular and ν can be determined by Gundelfinger's rule. In fact, if $a > 0$ so that A_0 and A_5 have opposite signs, $\nu = 1$ or 3 according as $b > 0$ or $b < 0$; while if $a < 0$ so that A_0 and A_5 have the same sign, $\nu = 2$ or 4 according as $b > 0$ or $b < 0$.

For $\sigma > 4$ it is apparent that the number of cases which may arise is even greater.

We may therefore state the following theorem.

THEOREM 1. *If for a real quadratic form* (1) *of rank r we set up the sequence* (3) *in which* $A_r \neq 0$, *then the number of negative coefficients in the canonical form* (2) *is equal to the number of variations of sign in the sequence* (3), *where*

* Franklin, *loc. cit.*, pp. 451–452.

(i) *if* $A_\tau A_{\tau+2} \neq 0$ *while* $A_{\tau+1} = 0$, *the subsequence* $A_\tau, 0, A_{\tau+2}$ *gives rise to one variation*;

(ii) *if* $A_\tau A_{\tau+3} \neq 0$ *while* $A_{\tau+1} = A_{\tau+2} = 0$, *we assign to the subsequence* $A_\tau, 0, 0, A_{\tau+3}$ *two variations or one variation according as* A_τ *and* $A_{\tau+3}$ *have the same sign or opposite signs*;

(iii) *if* $A_\tau A_{\tau+4} \neq 0$ *while* $A_{\tau+1} = A_{\tau+2} = A_{\tau+3} = 0$, *we assign to the subsequence* $A_\tau, 0, 0, 0, A_{\tau+4}$ *two variations if* A_τ *and* $A_{\tau+4}$ *have the same sign, while if* A_τ *and* $A_{\tau+4}$ *have opposite signs the number of variations is undetermined.*

(iv) *For* $\sigma \geqq 4$, *if* $A_\tau A_{\tau+\sigma+1} \neq 0$ *while* $A_{\tau+i} = 0$ $(i = 1, \cdots, \sigma)$ *the number of variations in the subsequence cannot be determined by the signs of the A's alone.*

3. Recurring forms. The scheme of rearrangement which was employed in the preceding section on the general quadratic form will now be applied to a recurring form, i. e., a form (1) in which

$$a_{ij} = a_{i+j-2}.$$

We first prove the following theorem.

THEOREM 2. *If for a real recurring form* (1) *we have* $A_\tau \neq 0$ $(\tau > 0)$ *while* $A_{\tau+i} = 0$ $(i = 1, \cdots, \sigma)$, *there exist* $\sigma+1$ *real sets* $X^{(k)} = (x_0^{(k)}, \cdots, x_{\tau-1}^{(k)})$ $(k = 1, \cdots, \sigma+1)$ *such that*

$$a_{\tau+k-1+i} = \sum_{j=0}^{\tau-1} a_{i+j}\, x_j^{(k)} \qquad \begin{cases} (k = 1, \cdots, \sigma+1) \\ (i = 0, \cdots, \tau+\sigma-k), \end{cases} \tag{10}$$

and satisfying the conditions

$$\sum_{j=0}^{\tau-1} a_{\tau+i-1+j}\, x_j^{(k)} = \sum_{j=0}^{\tau-1} a_{\tau+i-2+j}\, x_j^{(k+1)} \qquad \begin{cases} (k = 1, \cdots, \sigma) \\ (i = 1, \cdots, \sigma+1). \end{cases} \tag{11}$$

In order to prove this theorem we shall need first to establish two lemmas.

LEMMA 1. *If for a recurring form* (1) *there exists a single set* $X' = (x_0', \cdots, x_{\tau-1}')$ *satisfying*

$$a_{\tau+i} = \sum_{j=0}^{\tau-1} a_{i+j}\, x_j' \qquad (i = 0, \cdots, \tau+\sigma-1), \tag{10_1}$$

there exist σ *additional sets* $X^{(k)}$ *satisfying the conditions* (10) *of the theorem.*

In (10_k) we have on replacing i by $i+1$,

$$\begin{aligned} a_{\tau+k+i} &= \sum_{j=0}^{\tau-1} a_{i+j+1}\, x_j^{(k)} \\ &= \sum_{j=0}^{\tau-2} a_{i+j+1}\, x_j^{(k)} + a_{\tau+i}\, x_{\tau-1}^{(k)} \qquad (i = 0, \cdots, \tau+\sigma-k-1). \end{aligned}$$

In the first term on the right replace j by $j-1$ and in the second term replace $a_{\tau+i}$ by its value from (10_1). We then have

$$(10_{k+1}) \qquad a_{\tau+k+i} = \sum_{j=1}^{\tau-1} a_{i+j}\, x_{j-1}^{(k)} + x_{\tau-1}^{(k)} \sum_{j=0}^{\tau-1} a_{i+j}\, x_j'$$

$$= \sum_{j=0}^{\tau-1} a_{i+j}\, x_j^{(k+1)} \qquad (i = 0, \cdots, \tau+\sigma-k-1),$$

where

$$(12) \qquad x_0^{(k+1)} = x_0'\, x_{\tau-1}^{(k)}; \quad x_j^{(k+1)} = x_j'\, x_{\tau-1}^{(k)} + x_{j-1}^{(k)} \qquad (j = 1, \cdots, \tau-1).$$

Since for $k = 1$, (10_k) becomes (10_1), we have shown that (10_1) implies (10_2); this in turn implies (10_3); and in general (10_k) implies (10_{k+1}) for $k \leqq \tau+\sigma-1$ and therefore for $k \leqq \sigma$. Hence the lemma is proved.

LEMMA 2. *The sets $X^{(k)}$ of Lemma* 1 *satisfy the conditions* (11) *of the theorem.*

For, on replacing in the right hand member of (11) $X^{(k+1)}$ by its value from (12) and using (10_1) with i replaced by $\tau+i-2$ $(i = 1, \cdots, \sigma+1)$ the equality follows at once.

In view of these two lemmas in order to prove the theorem completely we have merely to show the existence of a single set X' satisfying (10_1). That such a set exists for $\sigma = 1$ follows directly from (5) after making obvious changes in notation. To proceed by induction we assume that the theorem is true for σ and prove that it is true for $\sigma+1$.

From the elements of the $(\tau+k)$th column of $A_{\tau+\sigma+1}$ subtract the sum of the products of the elements of the first τ columns by $x_0^{(k)}, \cdots, x_{\tau-1}^{(k)}$ $(k = 1, \cdots, \sigma+1)$. Proceeding similarly with the rows it follows from (10) that $A_{\tau+\sigma+1}$ is reduced to the form

$$\begin{vmatrix} A_\tau, & 0 \\ 0, & B \end{vmatrix}$$

where B is a symmetric determinant of order $\sigma+1$. Indeed, B is recurring, for if b_{ik} denote the element in its ith row and kth column evidently

$$(13) \qquad b_{ik} = b_{ki} = a_{2\tau+i+k-2} - \sum_{j=0}^{\tau-1} a_{\tau+i-1+j}\, x_j^{(k)} \qquad \begin{cases} (k = 1, \cdots, \sigma+1) \\ (i = 1, \cdots, \sigma+1), \end{cases}$$

which in view of (11) equals $b_{i-1,k+1}$. We may therefore write

$$b_{ik} = b_{i+k-2} \qquad (i+k = 2, \cdots, 2\sigma+2).$$

Moreover, from (10_k) with i replaced by $\tau+i-1$,

$$a_{2\tau+i+k-2} - \sum_{j=0}^{\tau-1} a_{\tau+i-1+j}\, x_j^{(k)} = 0 \qquad (i+k \leqq \sigma+1)$$

so that

$$(14) \qquad b_{ik} = 0 \qquad (i+k \leqq \sigma+1).$$

Hence, on expanding $A_{\tau+\sigma+1}$ we have

$$A_{\tau+\sigma+1} = (-1)^{\sigma(\sigma+1)/2}\,(b_\sigma)^{\sigma+1}\,A_\tau, \tag{15}$$

where

$$b_\sigma = a_{2\tau+\sigma} - \sum_{j=0}^{\tau-1} a_{\tau+\sigma+j}\,x_j'.$$

If, therefore, $A_{\tau+\sigma+1}=0$, $b_\sigma=0$ so that (10_1) holds for $i=\tau+\sigma$ and the induction is complete.

COROLLARY 1. *If* $A_\tau A_{\tau+\sigma+1}\neq 0$ $(\tau>0)$ *while* $A_{\tau+i}=0$ $(i=1,\cdots,\sigma)$ *then if* σ *is odd,* A_τ *and* $A_{\tau+\sigma+1}$ *have the same sign or opposite signs according as* σ *is of the form* $4n-1$ *or* $4n+1$.

We may now state a corollary to Theorem 1.

COROLLARY 2. *If in the sequence* (3) *for a real recurring form of rank* r, $A_r\neq 0$ *and three consecutive terms vanish, the subsequence of five terms containing these vanishing terms should be considered as presenting exactly two variations of sign.*

Now let us suppose that $A_\tau A_{\tau+\sigma+1}\neq 0$ while $A_{\tau+i}=0$ $(i=1,\cdots,\sigma)$. In $A_{\tau+\sigma+1}$ let us place the $(\tau+1)$th row and column last. From the elements of the last column of the new determinant subtract the sum of the products of the elements of the first τ columns by $x_0', \cdots, x_{\tau-1}'$. Proceeding similarly with the last row, we have in view of (10)

$$A_{\tau+\sigma+1} = -(a_{2\tau+\sigma} - \sum a_{\tau+\sigma+i}\,x_i')^2\,D_{\tau+\sigma-1}$$

where $D_{\tau+\sigma-1}$ is the principal minor determinant of order $\tau+\sigma-1$ standing in the upper left hand corner of the rearranged determinant $A_{\tau+\sigma+1}$. Evidently $D_{\tau+\sigma-1}\neq 0$ and is of opposite sign to $A_{\tau+\sigma+1}$.

In $D_{\tau+\sigma-1}$ place the $(\tau+1)$th row and column last. Using the multipliers $x_0'', \cdots, x_{\tau-1}''$ and proceeding in a manner similar to that just indicated, we have

$$D_{\tau+\sigma-1} = -(a_{2\tau+\sigma} - \sum a_{\tau+\sigma-1+i}\,x_i'')^2\,D_{\tau+\sigma-3}.$$

This process may be continued using the multipliers $x_0^{(k)}, \cdots, x_{\tau-1}^{(k)}$ $(k=3,\cdots)$ until we arrive at one or the other of the conclusions according as σ is even or odd.

Either $D_{\tau+3} = -c^2 D_{\tau+1}$ $(c\neq 0)$; or $D_{\tau+2} = -d^2 A_\tau$ $(d\neq 0)$. In order to estimate the number of variations of sign due to the vanishing of the σ consecutive A's, we may employ the subsequence consisting of the $\sigma+2$ quantities

$$A_\tau,\quad D_{\tau+1},\quad D_{\tau+2},\cdots,D_{\tau+\sigma-1},\quad D_{\tau+\sigma},\quad A_{\tau+\sigma+1}. \tag{16}$$

Denoting by ε the sign of $A_{\tau+\sigma+1}$ and recalling Corollary 1, it is clear that according as σ is of the form $4s-1$, $4s$, $4s+1$ or $4s+2$ the signs of the numbers in (16) are as follows:

$$\sigma = 4s-1, \qquad \varepsilon\,|\,(\,)-\varepsilon(\,)\varepsilon\,|\,\cdots \text{to } s \text{ terms} \cdots |\,(\,)-\varepsilon(\,)\varepsilon\,|;$$
$$\sigma = 4s, \qquad \pm\varepsilon\,\varepsilon\,|\,(\,)-\varepsilon(\,)\varepsilon\,|\,\cdots \text{to } s \text{ terms} \cdots |\,(\,)-\varepsilon(\,)\varepsilon\,|;$$
$$\sigma = 4s+1, \qquad -\varepsilon(\,)\varepsilon\,|\,(\,)-\varepsilon(\,)\varepsilon\,|\,\cdots \text{to } s \text{ terms} \cdots |\,(\,)-\varepsilon(\,)\varepsilon\,|;$$
$$\sigma = 4s+2, \quad \pm\varepsilon-\varepsilon(\,)\varepsilon\,|\,(\,)-\varepsilon(\,)\varepsilon\,|\,\cdots \text{to } s \text{ terms} \cdots |\,(\,)-\varepsilon(\,)\varepsilon\,|.$$

The number of variations of sign in each of these subsequences is easily estimated and we may summarize the results in the form of a theorem.

THEOREM 3. *If for a real recurring form* (1) *we set up the sequence* (3) *in which* $A_\tau A_{\tau+\sigma+1} \neq 0$ *while* $A_{\tau+i} = 0$ $(i = 1, \cdots, \sigma)$, *then to the subsequence*

$$A_\tau,\ 0,\ 0,\ \cdots,\ 0,\ A_{\tau+\sigma+1}$$

we assign exactly $(\sigma+1)/2$ *variations of sign if* σ *is odd. If* σ *is even of the form* $4s$ $(4s+2)$ *we assign* $\sigma/2$ *or* $\sigma/2+1$ $(\sigma/2+1$ *or* $\sigma/2)$ *variations according as* A_τ *and* $A_{\tau+\sigma+1}$ *are of the same sign or of opposite signs.*

Although Theorem 3 was proved on the assumption that $A_\tau \neq 0$ for $\tau > 0$ the results are still true for $\tau = 0$ as is clear from the discussion in § 2. However, this might have been proved directly. For, let $A_\tau = A_0 = 1$. Then $A_1 = 0$, $A_2 = 0, \cdots, A_\sigma = 0$ imply in turn that $a_0 = 0$, $a_1 = 0, \cdots, a_{\sigma-1} = 0$, and we have

$$A_{\sigma+1} = (-1)^{\sigma(\sigma+1)/2}\, a_\sigma^{\sigma+1} \tag{15'}$$

which is (15) with $\tau = 0$ and $X^{(k)} = 0$. If now we consider each set $X^{(k)}$ as zero and proceed with $A_{\sigma+1}$ in the same manner as we proceeded with $A_{\tau+\sigma+1}$ the result is precisely the same as that previously arrived at. Indeed this is exactly the scheme employed by Frobenius in this special case.

4. Recurring forms for which $A_r = 0$. In the preceding section it was shown that if (1) is a real recurring form of rank r for which $A_r \neq 0$ the signature of the form was completely determined by the sequence of A's alone even when an arbitrary number of consecutive A's vanish. The rule here given for determining the signature is easily identified with that given by Frobenius.

We now consider the case in which $A_r = 0$. Then $A_\tau \neq 0$ $(\tau < r)$ while $A_i = 0$ $(i = \tau+1, \cdots, n)$. Writing $\tau+\sigma+1 = n$ we may adopt the notation of the preceding section. Manifestly the signature of the original form is the same as the signature of the form whose matrix is (12), and hence is the sum of the signatures of the two forms whose matrices are A_τ and B. If ϱ denote the rank of B, evidently $\varrho < \sigma+1$ (since $A_r = 0$). In view of the fact that in the recurring matrix B $b_{ij} = 0$ for $i+j \leqq \sigma+1$, B is of rank exactly σ if $b_\sigma = 0$ while $b_{\sigma+1} \neq 0$. Similarly, B is of rank

$\sigma - 1$ if $b_\sigma = b_{\sigma+1} = 0$ while $b_{\sigma+2} \neq 0$, and in general, B is of rank $\varrho = \sigma + 1 - \mu$ if $b_\sigma = b_{\sigma+1} = \cdots = b_{\sigma-1+\mu} = 0$ while $b_{\sigma+\mu} \neq 0$. In this case manifestly the principal minor determinant B_ϱ of order ϱ standing in the lower right hand corner of B is different from zero, and therefore in view of the manner in which B was built up, it is clear that the determinant of order $r = \tau + \varrho$ formed by bordering A_τ with the last ϱ rows and columns of A is different from zero. Following Frobenius we shall call this determinant A'_r.

In B let us, while maintaining their positions relative to one another, transfer the last ϱ rows and colums over the μ rows and columns preceding them thus bringing the determinant B_ϱ into the upper left hand corner of B. The number N of negative coefficients in the form B is then easily determined by the rule given in Theorem 3 for a recurring form of rank ϱ in which $B_1 = B_2 = \cdots = B_{\varrho-1} = 0$ while $B_\varrho \neq 0$. Since N, which was determined from the sequence

$$1, 0, 0, \cdots \text{to } \varrho - 1 \text{ terms} \cdots, 0, 0, B_\varrho,$$

might have been determined just as well from the sequence

$$A_\tau, 0, 0, \cdots \text{to } \varrho - 1 \text{ terms} \cdots, 0, 0, A_\tau B_\varrho = A'_r,$$

we may formulate the rule as follows.

If for a real recurring form of rank r we set up the sequence (3) in which $A_\tau \neq 0$ $(\tau < r)$ while $A_i = 0$ $(i > \tau)$ and if we adjoin to this sequence $r - \tau - 1$ zeros and in addition a determinant A'_r formed by bordering A_τ with the last $r - \tau$ rows and columns of the matrix of the form

$$(17) \qquad A_0, A_1, \cdots, A_\tau, 0, 0, \cdots \text{to } r - \tau - 1 \text{ terms} \cdots, 0, A'_r$$

then the number N of negative coefficients in the canonical form of (1) is given by the number of variations of sign in the sequence (17) where the number of variations corresponding to σ consecutive zeros in the sequence is determined by Theorem 3.

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ON QUASI-ANALYTIC FUNCTIONS.*

By W. J. Trjitzinsky.

1. Introduction. Quasi-analytic functions are indefinitely differentiable functions which are determined uniquely at every point of the domain of definition when the value of the function and of its successive derivatives of all orders are known at a point of this domain. A different and equivalent way of defining such functions would be by the statement that they possess the property that when the values of the function are known at all points of an interval of the real axis (in the case of functions of a real variable) or at all points of an arc of a curve no matter how small (in the case of functions of a complex variable), the function is known at all points of its domain of existence. At the same time these functions are non-analytic,† that is their formal Taylor's series diverge. Thus, the class C of quasi-analytic functions is an extension of the class of analytic functions and possesses a very essential property in common with the latter class.

For the real variable the first result is due to Denjoy,‡ which, in the form slightly amplified by Carleman,§ can be given as follows:

Let $A_1, A_2, \cdots$ be a sequence of positive constants. Let C_A denote the functions $f(x)$ (defined in the interval (a, b)) for which

$$|f^{(m)}(x)| < h^m A_m$$

h being a finite constant independent of m and x, and $m = 1, 2, \cdots$, $a \leqq x \leqq b$. Then every function of class C_A is completely determined by the knowledge of $f(x)$ and of all its derivatives at a single point of (a, b), provided that

$$\sum \frac{1}{\sqrt[m]{A_m}}$$

diverges.

Thus, for quasi-analyticity it is sufficient that the function be non-analytic and that at the same time it should belong to a class of functions C_A. By a *Denjoi kth class* of quasi-analytic functions we consider those of the class C_A, where $A_m = (m \cdot \log m \cdot \log_2 m \cdots \log_k m)^m$ with $k \geqq 1$.‖

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* Received November 24, 1928.

† By excluding analytic functions from the class of quasi-analytic functions we exclude what would be trivial from the point of view of this paper.

‡ Comptes Rendus, t. 173 (1921), p. 1329.

§ See *Les fonctions quasi-analytique*, by T. Carleman, Paris, 1926. This treatise will be referred to in the sequel as (C.).

‖ $\log m = \log_1 m$, $\log \log m = \log_2 m$, $\log \log \log m = \log_3 m$, etc.

On account of convenience of classification we shall consider for the most part quasi-analytic functions belonging to some Denjoy class. A Denjoy kth class will be denoted by C_{A_k}.

Borel's monogenic functions,* whose discovery antedates that of quasi-analytic functions of a real variable implied in Denjoys theorem (sometimes called Denjoy-Carleman's theorem), are quasi-analytic in a complex domain C from which a certain point set of measure zero has been excluded. It will be shown in this paper that Borel's functions can be made to belong to the first Denjoy class of quasi-analytic functions by a certain specification of the domain of existence of the function. Further, series of the form

$$\sum c_n/(x - a_n)$$

which are quasi-analytic when the c_n satisfy certain conditions specified by Borel and Carleman and which form a particular case of Borel's monogenic functions, are considered; conditions are found under which such series belong to the Denjoy kth class; these conditions admit less restrictive specifications of the c_n than those of Borel and Carleman. An important problem is that of representation of quasi-analytic functions when the value of the function and of its successive derivatives are given at a point. A representation of this kind, which was given by Carleman,† will be extended. Necessary and sufficient conditions will be given in order that a sequence of polynomials should converge to a quasi-analytic function. Finally, a new representation of a class of functions will be given.

2. **Borel's monogenic functions.** In developing his theory of monogenic non-analytic functions of a complex variable,‡ Borel defines domains C as follows. Let A_n be an enumerable set of points everywhere dense in a portion of the plane. Let r_n be such that

$$r_n > 0, \quad \sum_{n=1}^{\infty} r_n \tag{1}$$

converges, and

$$\sum_{i=n+1}^{\infty} r_i < \frac{r_n}{4} \qquad (n = 1, 2, \cdots). \tag{2}$$

About every A_n a circle $S_n^{(h)}$ of radius $r_n^{(h)}$, which is determined in a definite manner and is such that

* *Leçons sur les fonctions monogènes uniformes d'une variable complex*, by E. Borel, Paris, 1917. This monograph will be referred to in the sequel as (B).

† See (C.), p. 72.

‡ See (B.)

$$\frac{r_n}{2^{h+1}} < r_n^{(h)} < \frac{r_n}{2^h}, \tag{3}$$

can be drawn so that non of these circles intersect each other, for a fixed positive integer h, and for $n = 1, 2, \cdots$. When from a closed region R of the plane containing all of the points A_n, the interiors of the circles $S_n^{(h)}$ $(n = 1, 2, \cdots)$ have been excluded, the remaining point set is denoted by $C^{(h)}$. Its frontier consists of all those circumferences $S_n^{(h)}$ which are exterior to each other and of the frontier of R; $C^{(h)}$ *is perfect,* and consists of the *points of its frontier and the limiting points of the frontier*; these latter points are said to be interior to $C^{(h)}$. $C^{(h)}$ is interior* to $C^{(h+1)}$ $(h = 1, 2, \cdots)$. The point set C consists of points P, each one of which belongs to $C^{(h)}$ for all $h > h_1$, where h_1 depends on P. *C is not perfect* since the A_n are limiting points of C, but do not belong to C; *it consists of the region R from which a point set of measure zero and of the power of the continuum has been excluded.* This latter point set contains the points A_n $(n = 1, 2, \cdots)$ and is determined, as can be seen from the preceding, by a set of numbers satisfying conditions (1), (2). A domain C belongs to the class (C) of "domains of Cauchy" if r_n satisfy (1), (2) and

$$\lim_{n\to\infty} \frac{n}{\log_3 \frac{1}{r_n}} = 0. \tag{4}$$

If $C^{(p)}$ and C are given and C belongs to the class (C), then "reduced domains" $\Gamma^{(p)}$ and Γ are determined by means of the numbers ϱ_n in the same way as $C^{(p)}$ and C were determined by means of r_n; the ϱ_n are such that

$$\frac{1}{\varrho_n^2} < \log_2 \frac{1}{r_n}, \tag{5}$$

$$\lim_{n\to\infty} n^\alpha \varrho_n = 0 \qquad \text{(for } n = 1, 2, \cdots, \text{ and for any } \alpha). \tag{6}$$

$\Gamma^{(p)}$ is contained in $C^{(p)}$ and Γ in C, and C contains points not belonging to Γ.

A function $f(z)$ is *monogenic* in a domain C if

1. *It is continuous in C.*

2. *It possesses at every point M of C a unique and continuous derivative with respect to $z = x + iy$.*†

* I. e., every point of $C^{(h)}$ belongs to $C^{(h+1)}$; in the sequel a set R_1 will be said to be interior to another set R_2 when every point of R_1 is a point of R_2.

† M belongs to $C^{(p)}$. Let M' be any other point of $C^{(p)}$. Then, if the limit $\frac{f(M')-f(M)}{\overline{MM'}}$, when $\overline{MM'} = z - z' \to 0$, exists for all p, it is independent of p and is the derivative at M. Continuity in C means continuity in every $C^{(p)}$ interior to C.

For monogenic functions the following representation, which amounts to a generalization of Cauchy's theorem, is given

$$(7) \qquad f(x) = \frac{1}{2\pi i}\int_K \frac{f(z)\,dz}{z-x} - \sum_n \frac{1}{2\pi i}\int_{S_n^{(p)}} \frac{f(z)\,dz}{z-x}.$$

K denotes a simple curve all of whose points belong to $C^{(p)}$, x is interior to K and is a point of "a reduced domain" $I^{(p)}$. The summation refers to the circumferences $S_n^{(p)}$ which limit $C^{(p)}$.

Using this representation, Borel shows that monogenuity in a domain C implies indefinite differentiability. Expanding each one of the terms of (7) in a series of Mittag-Leffler polynomials, along a line, whose points belong to a certain domain $I'^{(p)}$ which is interior to $I^{(p)}$, Borel shows that monogenic functions representable by means of (7) possess the property of quasi-analyticity.

We propose now to find expressions dominating the absolute values of the successive derivatives $f^{(m)}(x)$ defined by (7). Since it is known that (7) can be differentiated term by term any number of times, we have

$$(8) \qquad \frac{1}{m!}f^{(m)}(x) = \frac{1}{2\pi i}\int_K \frac{f(z)\,dz}{(z-x)^{m+1}} - \sum_n \frac{1}{2\pi i}\int_{S_n^{(p)}} \frac{f(z)\,dz}{(z-x)^{m+1}}.$$

Let

$$\frac{1}{2\pi i}\int_{S_n^{(p)}} \frac{f(z)\,dz}{(z-x)^{m+1}} = \varphi_n(m).$$

The length of the paths of integration will be denoted by $2\pi r_n$, and M denotes the maximum of $|f(z)|$ in $C^{(p)}$. Without any sensible error* $|z-x|$ can be replaced by $|A_n - x|$. This is due to the fact that with A_n, as a center there are two circles, one of radius $r_n^{(p)}$, the other of radius $\varrho_n^{(p)}$, and $r_n^{(p)}$ is very small compared with $\varrho_n^{(p)}$. On the smaller circle we have z, and x is exterior to the larger circle. Since $\varrho_n^{(p)}$ is not greater than $|A_n - x|$, it follows that

$$\varphi_n(m) < \frac{M r_n^{(p)}}{(\varrho_n^{(p)})^{m+1}}.$$

But $\varrho_n/2^{p+1} < \varrho_n^{(p)} < \varrho_n/2^p$, so that $1/\varrho_n^{(p)} < 2^{p+1}/\varrho_n$. From this using (5) we have

$$(9) \qquad \varphi_n(m) = M r_n 2^{(p+1)(m+1)-p} \cdot \log_2^{\frac{m+1}{2}}\left(\frac{1}{r_n}\right) \qquad (n = 1, 2, 3, \ldots).$$

Since each term in the second member of (7) is an analytic function of x, neglecting the first term we have without loss of generality

* See (B.), p. 142.

$$(10)\qquad \left|\frac{f^{(m)}(x)}{m!}\right| < \sum_{n=1}^{\infty} \varphi_n(m) < d \cdot 2^{(p+1)m} \cdot \sum_{n=1}^{\infty} r_n \log_2^{\frac{m+1}{2}} \left(\frac{1}{r_n}\right).$$

As a consequence of (4)

$$\frac{n}{\log_3\left(\frac{1}{r_n}\right)} = \frac{1}{\varphi(n)} \qquad (\varphi(n) \underset{n\to\infty}{\to} \infty),$$

$$\frac{1}{r_n} = e^{e^{e^n \cdot \varphi(n)}}.$$

So that

$$(11)\qquad \sum_{n=1}^{\infty} r_n \log_2^{\frac{m+1}{2}} \left(\frac{1}{r_n}\right) < \sum_{n=1}^{\infty} e^{-e^{e^n \cdot \varphi(n)}} \cdot e^{m \cdot n \cdot \varphi(n)} = \sum_{n=1}^{\infty} e^{-\left(e^{e^n \varphi(n)} - m \cdot n \cdot \varphi(n)\right)}.$$

For a fixed m the last member of (11) converges, and thus the following theorem can be stated.

THEOREM I. *Borel's monogenic functions defined in a domain C are such that for all x in a reduced domain $\Gamma^{(p)}$, the successive derivatives satisfy the inequalities*

$$(12)\qquad |f^{(m)}(x)| < d \cdot m^m \cdot 2^{(p+1)m} \cdot \sum_n r_n \cdot \log_2^{\frac{m+1}{2}} \left(\frac{1}{r_n}\right) = d \cdot m^m \cdot 2^{(p+1)m} \cdot \psi(m) \qquad (m = 1, 2, \cdots)$$

where d is independent of m and finite for every finite p. The r_n define C, and $\Gamma^{(p)}$ is defined by $\varrho_n^{(p)}$ satisfying (5), (6).

Restricting r_n so that r_n approaches zero monotonically the condition (4) is found to be satisfied by

$$(13)\qquad \frac{1}{r_n} = e^{e^{e^{\varphi(n) \cdot n}}}$$

where $\varphi(n)$ approaches ∞ monotonically. When r_n is given by (13), conditions (1) are also satisfied.

We substitute (13) in (2) and multiply both sides by $1/r_{n+1}$, thus obtaining

$$(14)\qquad 1 + \frac{1}{e^{\left[e^{e^{\varphi(n+2)\cdot(n+2)}} - e^{e^{\varphi(n+1)\cdot(n+1)}}\right]}} + \cdots < \frac{1}{4} e^{\left[e^{e^{\varphi(n+1)\ (n+1)}} - e^{e^{\varphi(n)\cdot n}}\right]}.$$

The left member is bounded and the second increases without limit. Hence (2) is satisfied. Thus we have proved the

THEOREM 2. *The numbers r_n given by* (13), *where $\varphi(n)$ approaches ∞ monotonically with n, are suitable for the determination of domains C, of the class (C), for which monogenic functions can be defined. When r_n is restricted to approach zero monotonically, the necessary and sufficient condition that* (1), (2), (4) *be satisfied is that r_n be given by* (13).

When the r_n are known to approach zero monotonically, inequalities (12) of Theorem 1 can be simplified. We have

$$(15) \qquad \psi(m) = \sum_{n=1}^{\infty} e^{v_n(m)}, \qquad v_n(m) = \frac{(m+1)}{2} n\varphi(n) - e^{e^{n\cdot\varphi(n)}}.$$

For every m there exists a value of $n = n_m$ such that $v_n(m) < 0$ for all $n > n_m$. Since $v_n(m)$ approaches $-\infty$ rapidly and monotonically when n increases on from the value $n = n_m + 1$, and since every term in $\sum_{n=n_m+1}^{\infty} e^{v_n(m)}$ is less than unity, it follows that

$$\sum_{n=n_m+1}^{\infty} e^{v_n(m)} < h_1 \qquad (h_1 \text{ independent of } m).$$

On the other hand

$$\sum_{n=1}^{n=n_m} e^{v_n(m)} < n_m e^{\frac{m+1}{2}\cdot n_m\cdot\varphi(n_m)},$$

so that

$$(16) \qquad \psi(m) < h\cdot n_m \cdot e^{\frac{m+1}{2}\cdot n_m\cdot\varphi(n_m)} \qquad (h \text{ independent of } m).$$

Now it can be shown that $v_n(m) < 0$ for $n = \log_2 m$, if $\log\varphi(\log_2 m)/\log m$ is bounded for all m. Assuming that this inequality holds we have:

$$\frac{m+1}{2}\cdot\log_2 m\cdot\varphi(\log_2 m) < e^{e^{\log_2 m\,\varphi(\log_2 m)}},$$

$$\log\frac{m+1}{2} + \log_3 m + \log\varphi(\log_2 m) < e^{\log_2 m\,\varphi(\log_2 m)},$$

$$\log\frac{m+1}{2}\cdot\left[1 + \frac{\log_3 m}{\log\frac{m+1}{2}} + \frac{\log\varphi(\log_2 m)}{\log\frac{m+1}{2}}\right] < e^{\log_2 m\,\varphi(\log_2 m)},$$

$$\log_2\frac{m+1}{2} + \eta < \log_2 m\cdot\varphi(\log_2 m),$$

where η is bounded.

$$\log_2\frac{m+1}{2}\cdot\left[1 + \frac{\eta}{\log_2\frac{m+1}{2}}\right] < \log_2 m\cdot\varphi(\log_2 m),$$

$$(17) \qquad \log_3\frac{m+1}{2} + \varepsilon_1 < \log_3 m + \log\varphi(\log_2 m).^*$$

(17) holds; then, since all of the steps by which it was derived are reversible, it follows that for $n = \log_2 m$

* ε_p denotes here and in the following a bounded function of m approaching zero when m approaches ∞.

$$v_n(m) < 0 \tag{18}$$

(a fortiori (18) holds for $n \geqq \log_2 m$). Thus, we can take $n_m = [\log_2 m]$ (the nearest integer to $\log_2 m$ less than $\log_2 m$). The inequality (16) then becomes

$$\psi(m) < h \cdot \log_2 m \cdot e^{\frac{m+1}{2} \cdot \log_2 m \cdot \varphi(\log_2 m)}.$$

We restate now Theorem 1 for a more special class of domains C.

THEOREM 3. *Borel's monogenic functions defined in a domain C are such that for all x in a reduced domain $\Gamma^{(p)}$ the successive derivatives satisfy the inequalities*

$$|f^{(m)}(x)| < d \cdot m^m \cdot 2^{(p+1)m} \cdot \log_2 m \cdot e^{\frac{m+1}{2} \log_2 m \cdot \varphi(\log_2 m)} \tag{19}$$

$$(m = 1, 2, \cdots)$$

where d is independent of m and finite for every finite p; the r_n, which define C, are restricted to approach zero monotonically and are such that when expressed in the form $r_n = e^{-e^{n\varphi(n)}}$, $\varphi(n)$ will be $< e^{ke^n}$ (k independent of n); $\Gamma^{(p)}$ is defined by $\varrho_n^{(p)}$ satisfying* (5), (6).

Inequalities (19) are satisfied in particular, when x is on any line interior to $\Gamma^{(p)}$ (all the points of the line belong to $\Gamma^{(p)}$); these inequalities do not insure that the Denjoy inequalities hold (see introduction). (The latter inequalities are known to be sufficient for quasi-analyticity of an indefinitely differentiable function on the segment of the real axis). We inquire now whether the domain of definition of Borel's monogenic functions can be so specified that on every line interior to its reduced domain $\Gamma^{(p)}$, the Denjoy inequalities hold, so that the function will belong to one of the Denjoy classes of quasi-analytic functions on every line interior to $\Gamma^{(p)}$. At the same time quasi-analyticity of a class of monogenic functions would be proved without the use of Mittag-Leffler's expansion (which is Borel's method).

In defining ϱ_n let us replace (6) by its more particular case (without loss of generality we take $r_n < 1$)

$$\frac{1}{\varrho_n^2} < \log_3 \left(\frac{1}{r_n}\right). \tag{20}$$

In order that condition (5)

$$\lim_{n \to \infty} n^\alpha \varrho_n = 0 \qquad \text{(any fixed } \alpha\text{)}$$

should still hold we find that, if

$$r_n = e^{-e^{n \cdot \varphi(n)}},$$

* Consequence of the condition $\log \varphi(\log_2 m)/\log m < k$.

it is sufficient that $\varphi(n)$ should approach ∞ monotonically and faster than any power of n. Thus, we assume the r_n are such that

$$r_n = e^{-e^{e^n \cdot \varphi(n)}}, \quad \varphi(n) > n^\alpha \qquad (n = 1, 2, \cdots, \text{ for any } \alpha), \tag{21}$$

and the ϱ_n are defined by

$$\frac{1}{\varrho_n^2} < \log_3\left(\frac{1}{r_n}\right), \quad \lim n^\alpha \varrho_n = 0 \qquad (\text{any fixed } \alpha). \tag{22}$$

Repeating the reasoning by which (9) was derived we now have

$$\varphi_n(m) < M r_n 2^{(p+1)(m+1)-p} \cdot \log_3^{\frac{m+1}{2}}\left(\frac{1}{r_n}\right) \qquad (n = 1, 2, \cdots),$$

so that inequality (12) of Theorem 1 becomes

$$|f^{(m)}(x)| < d \cdot m^m \cdot 2^{(p+1)m} \sum_n r_n \cdot \log_3^{\frac{m+1}{2}}\left(\frac{1}{r_n}\right), \quad \text{or}$$

$$|f^{(m)}(x)| < d \cdot m^m \cdot 2^{(p+1)m} \cdot \sum_{n=1}^{\infty} e^{-e^{e^n \varphi(n)}} \cdot n^{\frac{m+1}{2}} \cdot \varphi^{\frac{m+1}{2}}(n) \tag{23}$$

$$= d \cdot m^m \cdot 2^{(p+1)m} \cdot \psi(m),$$

where d is independent of m, and $m = 1, 2, \cdots$.

In $\psi(m) = \sum_n v_n(m)$, where

$$v_n(m) = e^{-e^{e^n \varphi(n)}} \cdot [n\,\varphi(n)]^{\frac{m+1}{2}},$$

it is observed that for every m there exists a value of $n = n_m$ such that $v_n(m) < 1$ for all $n > n_m$. Since the $v_n(m)$ approach zero rapidly and monotonically when n increases from the value $n = n_m + 1$; and since every term in $\sum_{n=n_m+1}^{\infty} v_n(m)$ is less than unity it can be seen that

$$\sum_{n=n_m+1}^{\infty} v_n(m) < h_1 \qquad (h_1 \text{ independent of } m).$$

Also we have

$$\sum_{n=1}^{n=n_m} v_n(m) < n_m \cdot [n_m \varphi(n_m)]^{\frac{m+1}{2}},$$

so that

$$\psi(m) < h \cdot n_m [n_m \varphi(n_m)]^{\frac{m+1}{2}} \qquad (h \text{ independent of } m). \tag{24}$$

For simplicity we take $\varphi(n) = e_g^n$ $(g \geqq 1)$.* The inequality $v_n(m) < 1$ can be written in the form

* $e^n = e_1^n$, $e^{e^n} = e_2^n, \cdots$; $e_0^n = \log_0 n = n$, $e_{-s}^n = \log_s n$.

$$[n\varphi(n)]^{\frac{m+1}{2}} < e^{e^{n\varphi(n)}},$$

$$\left(\frac{m+1}{2}\right)[\log n + \log\varphi(n)] < e^{e^{n\varphi(n)}},$$

$$\left(\frac{m+1}{2}\right)\log\varphi(n)\cdot\left[1+\frac{\log n}{\log\varphi(n)}\right] < e^{e^{n\varphi(n)}},$$

$$\log\left(\frac{m+1}{2}\right) + \log_2\varphi(n) + \eta < e^{n\varphi(n)}, \tag{25}$$

where η is bounded and may approach zero. Substituting $\varphi(n) = e_g^n$ in (25) we have

$$\log\frac{m+1}{2} + e_{g-2}^n + \eta < e^{ne_g^n}. \tag{26}$$

We seek to satisfy (26) with $n = \log_s m$. Substituting, we have

$$\log\frac{m+1}{2} + \log_{s-g+2} m + \eta < e^{\log_s m\cdot\log_{s-g} m}. \tag{27}$$

Suppose $s-g+2 \geqq 1$, then $\log_{s-g+2} m/\log\frac{m+1}{2}$ is bounded or it approaches zero. Thus from the inequality

$$\log\frac{m+1}{2}\left[1+\frac{\log_{s-g+2} m+\eta}{\log\frac{m+1}{2}}\right] < e^{\log_s m\cdot\log_{s-g} m}$$

it follows that

$$\log_2\frac{m+1}{2} + \eta_1 < \log_s m\cdot\log_{s-g} m \qquad (\eta_1 \text{ bounded}),$$

and further

$$\log_3\frac{m+1}{2} + \varepsilon_1 < \log_{s+1} m + \log_{s-g+1} m. \tag{28}$$

This latter inequality is satisfied when

$$s = 2+g \qquad (g \geqq 1). \tag{29}$$

By virtue of (29) the inequality

$$s-g+2 \geqq 1$$

is also satisfied.

Thus we can take as an admissible value of n_m $n_m = [\log_s m]$ (nearest integer to $\log_s m$ less than $\log_s m$, and $s = g+2$).

If in (27) we suppose $s-g+2<1$, then in place of (28) we can derive from (27) by a sequence of reversible operations the following inequality

$$\log_{s-g+4} m + \varepsilon_2 < \log_{s+1} m + \log_{s-g+1} m$$

which gives $s \leqq g-2$. Since we are interested in the smallest value of $\log_s m$ we take $s = g-2$. However, the better value for our purposes, is $s = g+2$.

From (24) it follows that

$$\psi(m) < h \log_{g+2} m \cdot [\log_{g+2} m \log_2 m]^{\frac{m+1}{2}} < h(\log_2 m \cdot \log_{g+2} m)^m$$

where h is independent of m and $g \geqq 1$. Inequalities (23) become

$$(30) \qquad |f^{(m)}(x)| < d \cdot (2^{(p+1)} \cdot m \log_2 m \cdot \log_{g+2} m)^m < (h_1 \cdot m \cdot \log m)^m$$

where $m = 1, 2, 3, \cdots$, and h is independent of m, but dependent on p.

Thus the following theorem has been proved

THEOREM 4. *Borel's monogenic functions belong to the 1-st Denjoy class of quasi-analytic functions* (C_{A_1}) *(and therefore to all classes of rank higher than one) along every line interior to a reduced domain* $\Gamma^{(p)}$ *of the domain* $C^{(p)}$, *provided that* $C^{(p)}$ *is defined by*

$$r_n = e^{-e^{n\varphi(n)}} \qquad (\varphi(n) = e_g^n,\ g \geqq 1),$$

and $\Gamma^{(p)}$ *is defined by* ϱ_n *such that* $1/\varrho_n^2 < \log_s (1/r_n)$ $(n = 1, 2, \cdots)$ *and* $\lim_{n\to\infty} n^\alpha \varrho_n = 0$ (α *any fixed number*).

Thus we make no restrictions on the A_n, but specify merely the nature of the point set of measure zero. This latter point set contains the points A_n and it is excluded from the domain of definition. In the "Cauchy" domains specified in this way, Borel's monogenic functions belong to the Denjoy classes of quasi-analytic functions. For a class of monogenic functions this is a new proof of quasi-analyticity.

3. **On the series of the form** $\sum c_n/(x-a_n)$. Some of the properties of series of the form

$$(1) \qquad \sum \frac{c_n}{x-a_n}$$

are mentioned by Carleman,* for instance:

"*Functions given by* (1), *when quasi-analytic, constitute a particular case of Borel's monogenic functions.*"

"*If the* $c_n \neq 0$, *the set of points of divergence of* (1) *has the power of the continuum in every domain where the* a_n *are everywhere dense.*"

A condition was given by Carleman† (less restrictive than the one given by Borel)

$$(2) \qquad |c_n| < e^{-n^{1+\varepsilon}}, \qquad (\varepsilon > 0)$$

* See (C.), pp. 97–99.

† Comptes Rendus, t. 174, 27 février 1922.

which secures quasi-analyticity of (1) (when the a_n are everywhere dense in a portion of the plane). Another condition given by Carleman is

$$\sum \frac{\log_2 \left|\frac{1}{c_n}\right|}{\log \left|\frac{1}{c_n}\right|} \tag{3}$$

converges.

We procede to prove the following theorem

THEOREM 5. *Let the* $a_n = x_n + iy_n$ *be an enumerable set of points,* $y_n \neq 0$; *let the* x_n *be everywhere dense in the interval* (a, b), *and let* $|y_n|$ *approach zero monotonically so that*

$$|y_n| \geqq \frac{1}{\varphi(n)} \tag{4}$$

where $\varphi(n)$ *approaches infinity monotonically, and*

$$\varphi(n) \leqq h \log n \cdot \log_2 n \cdots \log_k n \tag{5}$$

(k *a fixed integer and* h *a constant independent of* n). *The* c_n *are such that*

$$\sum_{n=1}^{\infty} |c_n| \, \varphi^{n+1}(n) \tag{6}$$

converges. With these conditions on c_n *and* a_n,

$$F(x) = \sum \frac{c_n}{x - a_n} \tag{7}$$

represents, on the interval of the real axis $(a \leqq x \leqq b)$, *a quasi-analytic function belonging to a Denjoy class of quasi-analytic functions of rank not higher than the k-th. The condition on* c_n *given by* (6) *is less restrictive than Carleman's condition* (2).

Indefinite differentiability term by term of (7) follows from the uniform convergence $(a \leqq x \leqq b)$ of the series

$$\frac{(-1)^m F^{(m)}(x)}{m!} = \sum_n \frac{c_n}{(x - a_n)^{m+1}} \qquad (m = 1, 2, \cdots) \tag{8}$$

which will be incidentally seen to be the case when we procede to derive expressions independent of x which dominate $|F^{(m)}(x)|$.

We have $|x - a_n| = |x - x_n - iy_n| \geqq y_n$, so that

$$\left|\frac{1}{(x - a_n)^{m+1}}\right| \leqq \left|\frac{1}{y_n}\right|^{m+1} \qquad (a \leqq x \leqq b); \tag{9}$$

and from (8) we have

$$|F^{(m)}(x)| < m! \sum_n |c_n| \left|\frac{1}{y_n}\right|^{m+1}. \tag{10}$$

Using (4), (10) gives

$$|F^{(m)}(x)| < m! \cdot \sum_{n=1}^{\infty} |c_n|\, \varphi^{m+1}(n). \qquad (m = 1, 2, \cdots). \tag{11}$$

Letting

$$|c_n\, \varphi^{n+1}(n)| = d_n \qquad (d_n \geqq 0) \tag{12}$$

it is observed that, by (6), $\sum_{n=1}^{\infty} d_n$ converges and $d_n < g$ (g independent of n and m). Substituting (12) in (11)

$$|F^{(m)}(x)| < m! \sum_{n=1}^{\infty} d_n\, v_n(m) \qquad (m = 1, 2, \cdots)$$

where

$$v_n(m) = \varphi^{m-n}(n).$$

Further,

$$m! \sum_{n=1}^{\infty} d_n\, v_n(m) = m! \,[L_n(m) + R_n(m)]$$

where

$$L_n(m) = \sum_{n=1}^{n=m} d_n\, v_n(m), \qquad R_n(m) = \sum_{n=m+1}^{\infty} d_n\, v_n(m).$$

By considering the form of $R_n(m)$

$$R_n(m) = \frac{d_{m+1}}{\varphi(m+1)} + \frac{d_{m+2}}{\varphi^2(m+1)} + \cdots < \frac{1}{\varphi(m+1)} \cdot (d_{m+1} + d_{m+2} + \cdots)$$

and in virtue of the fact that $\varphi(m+1)$ approaches infinity with m and that $(d_{m+1} + d_{m+2} + \cdots)$ is bounded (and approaches zero when $m \to \infty$), it follows that

$$R_n(m) < l \qquad (l \text{ independent of } m \text{ and } n)$$

We have then, since

$$L_n(m) = \sum_{n=1}^{n=m} d_n\, v_n(m) < g \cdot m\, \varphi^m(m),$$

the inequality

$$|F^{(m)}(x)| < m! \,[g\, m\, \varphi^m(m) + l] < g_1\, m^{m+\frac{3}{2}} \cdot \varphi^m(m)^*$$
$$(m = 1, 2, \cdots, g \text{ a constant independent of } m).$$

By (5) it follows that

$$|F^{(m)}(x)| < g_1\, m^{m+\frac{3}{2}} \cdot \varphi^m(m) < (g_2 \cdot m \cdot \log m \cdot \log_2 m \cdots \log_k m)^m \tag{13}$$

where g_2 is any number $> g_1$. This proves that under the suppositions of Theorem 5, $F(x)$ is quasi-analytic of a Denjoy class, of rank not exceeding the k-th.

* For m sufficiently large $m! < (2\pi e)^{\frac{1}{2}}\, m^{m+\frac{1}{2}}$.

We procede to demonstrate that condition (6) of Theorem 5 is less restrictive than the condition (2) of Carleman. In fact, let

$$|c_n| \leqq e^{-n\cdot\log n} = n^{-n}. \tag{14}$$

From this inequality, together with (5), it follows that

$$\sum_{n=1}^{\infty} |c_n|\,\varphi^{n+1}(n) < \sum_{n=1}^{\infty} h^{n+1}\cdot(\log n\cdot\log_2 n\cdots\log_k n)^{n+1}\cdot e^{-n\cdot\log n} = \sum_{n=1}^{\infty} u_n,$$

and further

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{h^{1+\frac{1}{n}}\cdot(\log n\cdot\log_2 n\cdots\log_k n)^{1+\frac{1}{n}}}{e^{\log n}}$$

$$= \lim_{n\to\infty} h^{1+\frac{1}{n}}\cdot(\log n\log_2 n\cdots\log_k n)^{1+\frac{1}{n}}/n = 0.$$

Thus (6) is satisfied whenever (14) or (2) holds. But (14) is a condition on the $|c_n|$ less restrictive than (2). Hence, generally speaking, (6) is less restrictive than (2).

To show that (6) is satisfied by c_n, which do not satisfy Carleman's condition (3), let

$$|c_n| = e^{-n\log n} = n^{-n} \qquad (n = 1, 2, \cdots) \tag{15}$$

which by the preceding satisfy (6). On substituting (15) in (3) it is found that

$$\sum_n \frac{\log_2\left|\frac{1}{c_n}\right|}{\log\left|\frac{1}{c_n}\right|} = \sum_n \frac{\log n + \log_2 n}{n\log n} = \sum_n\left[\frac{1}{n} + \frac{\log_2 n}{n\log n}\right]$$

which diverges. Thus (3) does not hold, while (6) does.

This completes the proof of the theorem.

Less restrictive conditions on the c_n, than those of Carleman, should be expected, since in Theorem 5 there are additional specifications on a_n, which are not assumed in Carleman's conditions (2), (3) for quasi-analyticity.

The theorem can be generalised by replacing (5) by

$$\varphi(m) \leqq h\frac{\alpha_m}{m} \qquad \left(\text{necessarily } \lim_{m\to\infty}\frac{\alpha_m}{m} = \infty\right)$$

where the $A_m = \alpha_m^m$ define a class C_A of quasi-analytic functions; (13) then becomes

$$|F^{(m)}(x)| < g_1 m^{m+\frac{3}{2}}\varphi^m(m) < g_1 m^{\frac{3}{2}}(h\alpha_m)^m < R\cdot k^m\cdot A_m \qquad (k > h).$$

The generalised theorem can be stated as follows

THEOREM 6. *Let $F(x)$ be defined as in Theorem* 5, *except that the condition* (5) *will now be replaced by*

$$\varphi(m) \leqq h \cdot \frac{\alpha_m}{m}$$

where the $A_m = \alpha_m^m$ *define a class* C_A *of quasi-analytic functions. Then* $F(x)$ *is quasi-analytic on the interval* (a, b) *and is of the class* C_A.

4. **An extension of Carleman's method of representation.** Carleman* has given a representation of quasi-analytic functions of class C_A, i. e., such that

$$|f^{(n)}(x)| < Rk^n A_n \qquad (n = 1, 2, \cdots), \tag{1}$$

when the values of the functions and of its successive derivatives are known at a point (say $x = 0$). He lets

$$f^{(n)}(0) = c_n \cdot n!. \tag{2}$$

An assumption is made that the c_n are such that there exists a quasi-analytic function so that (2) holds.

Letting

$$A_n = \alpha_n^n, \tag{3}$$

it is possible to find numbers β_n such that

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0,$$

and such that the numbers $B_n = \beta_n^n$ define a class of quasi-analytic functions C_B.

The integral

$$J_n(f) = \sum_{m=0}^{m=n} \frac{1}{(\beta_m)^{2m}} \cdot \int_0^1 [f^{(m)}(x)]^2 \cdot dx \tag{4}$$

is a minimum for an expression of the form

$$f_n(x) = \sum_{m=0}^{n-1} c_m \cdot \omega_{n,m}(x) \tag{5}$$

where

$$f^{(m)}(0) = c_m \cdot m! \qquad (m = 0, 1, \cdots, (n-1)).$$

Further, Carleman shows that

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (0 \leqq x \leqq a) \tag{6}$$

where $f(x)$ is a quasi-analytic function of class C_A (provided that $\sum_n c_n x^n$ does not converge, so that $f(x)$ is not analytic) and such that

$$f^{(m)}(0) = c_m \cdot m! \qquad (m = 0, 1, 2, \cdots).$$

* See (C.), pp. 65–73.

Letting

$$F(t) = f(x \cdot t) \qquad (0<x<a) \tag{7}$$

it follows that

$$F^{(m)}(0) = x^m \cdot c_m \cdot m! \qquad (m = 0, 1, \cdots). \tag{8}$$

Since $F(t)$, as function of t, is quasi-analytic developement (6) can be applied so that

$$F(t) = f(x \cdot t) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \omega_{n,m}(t) \cdot c_m \cdot x^m. \tag{9}$$

Letting $t = 1$, (9) becomes

$$f(x) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \omega_{n,m}(1) \cdot c_m x^m \tag{10}$$

which converges uniformly in the interval $(0, a)$, so that Carleman's theorem* can be stated as follows:

Given a sequence of constants $A_0, A_1, A_2, \cdots, A_n, \cdots$ defining a class C_A of quasi-analytic functions, there exist numbers $\omega_{n,m}$ depending only on A_n such that, if there exists a quasi-analytic function $f(x)$ for which

$$f^{(m)}(0) = c_m \cdot m!,$$

such a function admits the representation

$$f(x) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \omega_{n,m} \cdot c_m \cdot x^m.$$

The constants $\omega_{n,m}$ are given by

$$\omega_{n,m} = \omega_{n,m}(1)$$

where $\omega_{n,m}(x)$ are the functions involved in the series (5) *which minimises* (4).

We shall now extend this result by taking for granted all the results before (7). This latter assumption will be replaced by the following

$$F(t) = f[\varphi(x, t)] \qquad (0<x<a) \tag{11}$$

where

$$\varphi(x, t) = x \cdot t + t(t-1)\, p(x).\dagger \tag{12}$$

Let $p(x)$ in (12) be so restricted that $F(t)$, as function of t, is quasi-analytic. Then (6) can be applied to the developement of $F(t)$ as a function of t. The numbers $F^{(n)}(0)$ have to be computed. Let

$$F^{(n)}(0) = \bar{c}_n(x) \cdot n!. \tag{13}$$

* See (C.), p. 72.

† In place of $p(x)$ we can take a suitable function $p(x, t)$. This however would complicate the extension.

Using (11) and the properties of $\varphi(x, t)$, as given by (12), the following expression for $F^{(n)}(t)$ suggests itself:

$$(14)\quad F^{(n)}(t) = \sum_{r=0}^{n} u_{n,r} \cdot n(n-1)\cdots(n-r+1) \cdot f^{(n-r)}(\varphi(x,t)) \cdot \left[\frac{\partial \varphi(x,t)}{\partial t}\right]^{n-2r} \cdot p^r(x),$$

where $u_{n,r} = (n-r)!/r!(n-2r)!$, $n(n-1)\cdots(n-r+1) = 1$ for $r = 0$, and $r_n = n/2$ for n even, and $r_n = (n-1)/2$ for n odd.

The above expression is inferred from computing $F^{(n)}(t)$ for a few successive values of n $(n = 1, 2, \cdots)$.

We prove (14) by induction. Differentiating (14), it follows that

$$(15)\quad F^{(n+1)}(t) = \sum_{r=0}^{r=r_{n+1}} v_{n,r} f^{(n+1-r)}(\varphi(x,t)) \cdot \left[\frac{\partial \varphi(x,t)}{\partial t}\right]^{n+1-2r} \cdot p^r(x)$$

where

$$v_{n,r} = u_{n,r} \cdot n \cdot (n-1)\cdots(n-r+1) + u_{n,r-1} \cdot n \cdot (n-1)\cdots(n-r+2) \cdot 2 \cdot (n-2r+2).$$

The induction would be complete, if the relation

$$(16)\quad v_{n,r} = u_{n+1,r} \cdot (n+1) \cdot n \cdots (n-r+2)$$

could be proved.

Assuming that (16) holds, and substituting the expressions for $u_{n,r}$ from (14) in the both members of (16), we are led to consider

$$n!\left[\frac{1}{r!\,(n-2r)!} + \frac{2(n-2r+2)}{(r-1)!\,(n-2r+2)!}\right] = \frac{(n+1)!}{r!\,(n-2r+1)!}.$$

This identity is easily verified. Thus (14) holds for all $n = 1, 2, \cdots$.

Letting $t = 0$ in (14) and noting that

$$\bar{c}_n(x) = F^{(n)}(0)/n!, \quad u_{n,r} = (n-r)!/r!\,(n-2r)!, \quad \varphi(x, 0) = 0,$$

$$\frac{\partial \varphi(x,t)}{\partial t} = x + (2t-1)\,p(x) = x - p(x) \text{ for } t = 0,$$

we have

$$(17)\quad \begin{aligned} \bar{c}_n(x) &= \sum_{r=0}^{r=r_n} f^{(n-r)}(0)\, \frac{(x-p(x))^{n-2r}}{(n-2r)!} \cdot \frac{p^r(x)}{r!} \\ &= \sum_{r=0}^{r=r_n} \frac{(n-r)! \cdot c_{n-r}}{r! \cdot (n-2r)!} \cdot (x-p(x))^{n-2r} \cdot p^r(x) \end{aligned}$$

where $r_n = n/2$ if n is even, and $r_n = (n-1)/2$ if n is odd.

When $F(t)$ is developed by means of (6), t replaces x in $\omega_{n,m}(x)$ and the $\bar{c}_n(x)$ replace the c_n, so that

$$(18)\quad F(t) = f[\varphi(x,t)] = \lim_{n\to\infty} \sum_{m=0}^{n-1} \bar{c}_m(x) \cdot \omega_{n,m}(t).$$

We now let $t = 1$. Then since

$$F(1) = f[\varphi(x, 1)] = f(x),$$

it follows that

$$f(x) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \bar{c}_m(x) \cdot \omega_{n,m} \qquad (\omega_{n,m} = \omega_{n,m}(1)). \tag{19}$$

Thus when $p(x)$ is suitably restricted, (19) converges in the interval $(0, a)$.

Thus the following theorem has been proved.

THEOREM 7. *Given a set of constants $A_0, A_1, \cdots, A_n, \cdots$ defining a class C_A of quasi-analytic functions, there exist numbers $\omega_{n,m}$ depending on the A_n (and independent of c_n, $p(x)$) such that, if there exists a quasi-analytic function $f(x)$ of class C_A, for which*

$$f^{(m)}(0) = c_m \cdot m! \qquad (m = 0, 1, 2, \cdots),$$

such a function admits the representation

$$f(x) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \omega_{n,m} \cdot \bar{c}_m(x),$$

$$\bar{c}_m(x) = \sum_{r=0}^{r_m} \frac{(m-r)! \cdot c_{m-r}}{r!\,(m-2r)!} (x - p(x))^{m-2r} \cdot p^r(x)$$

($r_m = m/2$ for m even and $= (m-1)/2$ for m odd, $p(x)$ a suitable analytic function).

The numbers $\omega_{n,m}$ are identical with the corresponding numbers defined in Carleman's theorem quoted before.*

Theorem 7 reduces to Carleman's theorem for $p(x) \equiv 0$.

In the proof of Carleman's theorem uniformity of convergence of the second member of (19) can be established. In the more general case, considered above, we leave uniformity of convergence unproved.

If it is desirable to obtain a representation of $f^{(i)}(x)$, it cannot be asserted that

$$f^{(i)}(x) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \omega_{n,m}\, \bar{c}_m^{(i)}(x)$$

is a valid representation. It is possible to procede otherwise. Since $f_1(x) = f^{(i)}(x)$ is a quasi-analytic function of class C_A and is such that

$$f_1^{(m)}(0) = f^{(m+i)}(0) = c'_m \cdot m! = c_{m+i}(m+i)!$$

it follows that Theorem 7 can be applied giving the following theorem

THEOREM 8. *The successive derivatives of a quasi-analytic function $f(x)$, represented by means of Theorem 7, can be given by*

* See (C.), p. 72.

$$f^{(i)}(x) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \omega_{n,m}\, \bar{c}_{m,i}(x) \qquad (i = 1, 2, \cdots),$$

(20)

$$\bar{c}_{m,i}(x) = \sum_{r=0}^{r_m} \frac{(m-r+i)!\cdot c_{m-r+i}}{r!\cdot(m-2r)!}\,(x-p(x))^{m-2r}\cdot p^r(x).^*$$

If $p(0) = 0$, by noting that $\bar{c}_{0,i}(0) = i!\, c_i$ and $\bar{c}_{m,i}(0) = 0$ $(m>0,$ $i = 0, 1, 2, \cdots)$ it is easily verified that (20) gives

(21) $$f^{(i)}(0) = i!\, c_i.$$

We have, in fact,

$$f^{(i)}(0) = \lim_{n\to\infty} \omega_{n,0}\, i!\, c_i.$$

But

$$\lim_{n\to\infty} \omega_{n,m} = 1,^\dagger$$

so that (21) follows.

Suppose $p(c) = 0$ $(0 \leqq c \leqq a)$, then

$$\bar{c}_{m,i}(c) = \frac{(m+i)!}{0!}\cdot\frac{c_{m+i}}{m!}\cdot c^m$$

and

$$f^{(i)}(c) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \omega_{n,m}\,\frac{(m+i)!\cdot c_{m+i}}{m!}\cdot c^m$$

which for $i = 0$ becomes

$$f(c) = \lim_{n\to\infty} \sum_{m=0}^{n-1} \omega_{n,m}\cdot c_m\cdot c^m.$$

This the following corollary results

COROLLARY 1. *At any point $x = c$ $(0 \leqq c \leqq a)$ for which $p(c) = 0$, the representations of a quasi-analytic function $f(x)$ given by Theorems 7, 8, reduce to that of Carleman.*

Any representation of indefinitely differentiable functions $f(x)$, $a \leqq x \leqq b$, for which initial values $c_n\cdot n!$ (values of the function and of its successive derivatives) at a point of (a, b) are given, is a representation of a quasi-analytic function (which does not reduce to the case of an analytic function at least at $x = 0$), provided that the c_n are such that $\sum_n c_n x^n$ diverges for all values of x. In addition the representation should be so specified that $|f^{(n)}(x)| < k^n A_n$, A_n defining a class C_A of quasi-analytic functions.

Representations of indefinitely differentiable functions with an assigned set of initial values have been given by de la Vallée Poussin‡ and by

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* $\bar{c}_{m,0}(x) = \bar{c}_m(x)$, $(m-r)!\cdot c'_{m-r} = c_{m-r+i}\cdot(m-r+i)!$

† See (C.), p. 73.

‡ The Rice Institute Pamphlet, vol. 12 (1925), No. 2, pp. 164–172.

myself.* So far no one succeeded in specifying representations of this kind in the aforesaid way. This would be a natural procedure.

5. **Sequences of polynomials.** Suppose that $n!\,c_n = f^{(n)}_{(0)}$ $(n = 0, 1, \cdots)$ are the values at $x = 0$ of a quasi-analytic function $f(x)$ of class C_A, and of its successive derivatives. Let the $\omega_{n,i}$ be the set of constants defined by Carleman's theorem (see preceding section of this paper) which corresponds to the class C_A. It is known that the sequence of polynomials $[f_n(x)]$, where

$$f_n(x) = \omega_{n,0}\,c_0 + \omega_{n,1}\,c_1\,x + \cdots + \omega_{n,n-1}\,c_{n-1}\,x^{n-1}, \tag{1}$$

converges to the function $f(x)$ mentioned above.

If any other sequence of polynomials $[\bar{f}_n(x)]$, where

$$\bar{f}_n(x) = \bar{\omega}_{n,0}\,c_0 + \bar{\omega}_{n,1}\,c_1\,x + \cdots + \bar{\omega}_{n,n-1}\,c_{n-1}\,x^{n-1}, \tag{2}$$

should converge to $f(x)$ for, say $|x|<1$, it is necessary and sufficient that the set of polynomials $y_n(x)$, where

$$\begin{aligned} y_n(x) &= f_n(x) - \bar{f}_n(x) \\ &= (\omega_{n,0} - \bar{\omega}_{n,0})\,c_0 + (\omega_{n,1} - \bar{\omega}_{n,1})\,c_1\,x + \cdots + (\omega_{n,n-1} - \bar{\omega}_{n,n-1})\,c_{n-1}\,x^{n-1}, \end{aligned} \tag{3}$$

should converge to zero for $|x|<1$.

We recall one of the more specialized theorems of Kojima's:†

"*The necessary and sufficient conditions that the sequence* (y_n) *defined by*

$$y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

is convergent to zero, whenever the sequence (x_n) *converges to zero are that*

$$\lim_{n\to\infty} a_{ni} = 0 \qquad (i = 1, 2, \cdots), \tag{4}$$

and

$$|a_{n1}| + \cdots + |a_{nn}| < M \quad (n = 1, 2, \cdots)." \tag{5}$$ (Toeplitz's theorem.)

If in (3) we let x be such that $|x|<1$ and $x^{n-1} = x_n$ then it is seen that the condition that the sequence (x_n) converges to zero is satisfied.

Letting

$$(\omega_{n,i-1} - \bar{\omega}_{n,i-1})\,c_{i-1} = a_{n,i}, \tag{6}$$

it follows that the theorem just quoted is applicable. Condition (4) is equivalent to the condition that $\bar{\omega}_{n,i}$ be such that

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* *Functions with assigned initial Values,* Amer. Journ. Math., 51 (1929), pp. 189–204.

† *On Generalized Toeplitz's Theorems* ....., Tohoku Math. Journ., 12 (1917), pp. 291–326. Also see p. 123, Smail's "*History and Synopsis of the Theory of Summable Infinite Processes*". University of Oregon Publications, vol. 2 (1925), No. 8.

$$(7) \qquad a_{n,i+1} = (\omega_{n,i} - \bar{\omega}_{n,i})\, c_i = g_i(n) \qquad (i = 0, 1, \cdots)$$

where $\lim_{n\to\infty} g_i(n) = 0$ $(i = 0, 1, 2, \cdots)$, i. e.

$$\bar{\omega}_{n,i} = \omega_{n,i} - \frac{g_i(n)}{c_i}.$$

Conditions (5) can then be expressed in the form

$$|g_0(n)| + |g_1(n)| + \cdots + |g_{n-1}^{(n)}| < M \qquad (n = 1, 2, \cdots).$$

We have thus the following theorem

THEOREM 9. *Let* $n!\, c_n = f_{(0)}^{(n)}$ $(n = 0, 1, 2, \cdots)$ *be the values at* $x = 0$ *of a quasi-analytic function* $f(x)$ *of class* C_A, *and of its successive derivatives. Let the* $\omega_{n,i}$ *be the set of constants associated with the class* C_A *by means of Carleman's theorem.**

In order that the sequence of polynomials $[\bar{f}(x)]$ *defined by*

$$(8) \qquad \bar{f}_n(x) = \bar{\omega}_{n,0} \cdot c_0 + \bar{\omega}_{n,1}\, c_1 \cdot x + \cdots + \bar{\omega}_{n,n-1} \cdot c_{n-1} \cdot x^{n-1}$$

be convergent to $f(x)$ *for all real* x *such that* $|x| < 1$, *it is necessary and sufficient that* $\bar{\omega}_{n,i}$ *be of the form*

$$(9) \qquad \bar{\omega}_{n,i} = \omega_{n,i} - \frac{g_i(n)}{c_i} \qquad \left(\lim_{n\to\infty} g_i(n) = 0, \quad i = 0, 1, 2, \cdots\right)$$

and

$$(10) \qquad |g_0(n)| + |g_1(n)| + \cdots + |g_{n-1}(n)| < M\dagger \qquad (n = 1, 2, \cdots).$$

6. A representation of a class of functions. De la Vallée Poussin‡ has shown that a Fourier's series represents a quasi-analytic or analytic function, when the coefficients of the series satisfy certain inequalities.

We propose to give a different representation embodied in the following theorem.

THEOREM 10. *Consider functions of the form,*

$$(1) \qquad F(x) = \sum_{n=1}^{\infty} \frac{d_n f(m_n x)}{m_n^n}$$

$(\sum |d_n|$ *convergent,* $m_n > 1$ *and monotonically approaches* ∞ *with* n*), where* $f(x)$ *is an indefinitely differentiable function such that*

$$(2) \qquad |f(x)|, \quad |f^{(n)}(x)| < g$$

* See section (4) of this paper.

† Throughout this section M is independent of n.

‡ Comptes Rendus (5 mars 1923), also see The Rice Institute Pamphlet, vol. 12 (1925), Nr. 2, pp. 154–7. Necessary and sufficient conditions are given by Carleman, see (C.), p. 95.

(for all real x, $n = 1, 2, \cdots$, g independent of x and n). These functions are such that for all points of the real axis the successive derivatives satisfy the inequalities

$$|F^{(i)}(x)| < (h m_i)^i \tag{3}$$

($h > 1$ and independent of x and i).

In particular, when the A_n ($A_n = m_n^n$) define a class C_A of quasi-analytic functions, (1) *represents a quasi-analytic or analytic function at every point of the real axis.*

On account of the conditions (1), (2), the series (1) is indefinitely differentiable term by term

$$|F^i(x)| = |\sum d_n m_n^{i-n} f^{(i)}(m_n x)| < g \sum |d_n| m_n^{i-n} = g[L_i + R_i]$$
$$\text{(all real } x,\ i = 1, 2, \cdots),$$

L_i and R_i are given by

$$L_i = \sum_{n=1}^{n=i} |d_n| m_n^{i-n} < g_2 \cdot i \cdot m_i^i \qquad (g_2 \text{ independent of } i),$$

$$R_i = \frac{|d_{i+1}|}{m_{i+1}} + \frac{|d_{i+2}|}{m_{i+2}^2} + \cdots < \frac{1}{m_{i+1}} \cdot \sum_i d_i < g_1 \quad (g_1 \text{ independent of } i).$$

Thus

$$|F^{(i)}(x)| < g[g_2 i m_i^i + g_1] < h_1 i m_i^i < (h m_i)^i \qquad (h > 1).$$

This proves the theorem.

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A CERTAIN POINT-TO-LINE TRANSFORMATION IN SPACE OF FOUR DIMENSIONS.*

By B. C. Wong.

1. Let three three-dimensional quadric varieties Q, Q', Q'' be given in 4-space. To any point P we make correspond the line p of intersection of the three polar hyperplanes with respect to Q, Q', Q''. Corresponding to the ∞^4 points in S_4 we have a system Δ of ∞^4 lines in S_4. It is the purpose of this paper to describe some of the features of this transformation, which we designate by the symbol T. It will be found that in Δ are ∞^4 systems each consisting of ∞^3 lines which are trisecants of a certain surface F^6 and are the transforms of the points of an S_3. Such a system, designated by Σ, will be projected on to an S_3 into a complex of lines of order 8. We shall find that the ∞^4 F^6's have in common a curve J^{10}, which is the locus of the singular points of the transformation and that the locus of the singular planes is a hypersurface of order 15. The congruence Γ of lines which lie in an S_3 and are the transforms of the points of the corresponding F^6 will be found to be of order 3, class 6, rank 5. We shall also discuss the ruled hypersurfaces and surfaces, especially those of the third order, whose lines belong to Δ.

2. We note that any S_3 through P meets the line p in a point P'. P and P' are corresponding points of the well-known involutorial cubic transformation in S_3 whose fundamental quadric surfaces are the intersections of S_3 with Q, Q', Q''.

3. It is not difficult to infer that, by the transformation T, the points of any line l in S_4 go into the lines of a two-dimensional locus F^3 of order 3 whose intersection with an S_3 is a twisted cubic curve. If l is a line of Δ, i. e., a line which is the image of a point L, then F^3 is a cone with vertex at L. The points of a plane transform into the rulings of a cubic hypersurface V_3^3 whose intersection with a general S_3 is a general cubic surface. Corresponding to the points of an S_3 are ∞^3 lines in S_4 forming a system Σ. Of these ∞^3 lines ∞^1 lie in S_3 itself forming a ruled octic surface F^8. The generators of F^8 are the trisecants of a sextic curve C^6 which is the Jacobian curve of the three quadrics of intersection of S_3 with Q, Q', Q''. From this we infer that all the ∞^3 lines of Σ are the trisecants of a two-dimensional manifold F^6 of order 6. F^6 is the locus of points whose images are the ∞^2 lines of the system Δ that lie in S_3. Let Γ denote the congruence of these ∞^2 lines.

* Received October 18, 1927; in revised from June 8, 1928.

4. The properties of F^6 are known.* As its points are in a one-to-one correspondence with the points of a plane σ with 10 base-points P_i $[i = 0, 1, \cdots, 9]$, the fundamental curves being quartic curves, there is a one-to-one correspondence between the lines of Γ and the points of σ. Let T' denote the former correspondence and TT' the latter.

5. The locus of the singular points of the transformation T which have for images planes, the singular planes of T, instead of lines is a curve J^{10} of order 10 which is the Jacobian curve of Q, Q', Q''. All the ∞^4 F^6's determined by the ∞^4 3-spaces of S_4 have this curve in common. Its points are in one-to-one correspondence with the points of a plane quintic c^5 which may be made to lie in the plane σ, passing through the 10 basepoints P_i.

6. The transform of a unisecant of J^{10} is composed of a singular plane and a quadric surface and that of a bisecant is composed of two singular planes and a third plane which is that of a pencil of lines corresponding to the points of the bisecants.

7. Every singular plane is such that every line in it is a line of Δ and that every point in it transforms into a line incident with J^{10}. Hence the V_3^8 corresponding to such a plane is a cubic hypercone with its vertex on J^{10}.

8. The locus of the singular planes is a hypersurface of order 15. To show this, it is necessary to consider the transformation T' or the transformation TT'. By T' every fundamental quartic curve in σ yields a 3-space sextic curve C^6 whose points by T' go into the rulings of an octic surface F^8. Letting m be the order of the transformation TT', n that of the variety into which each of the 10 points P_i in σ transforms, μ that of a curve c^μ in σ to be transformed, and N that of the surface F^N corresponding to c^μ, we have the relation

$$\mu m - n \sum \alpha_i = N,$$

where α_i is the number of times c^μ passes through P_i. In the case just mentioned, $\mu = 4$, $\sum \alpha_i = 10$, $N = 8$, and we have, therefore,

$$4m - 10n = 8.$$

To determine n, consider the lines p_i on F^6 which are the images by T' of the points P_i in σ. Each of these lines p_i meets S_3 in a point on J^{10}, and hence its transform by T is a cubic surface made up of a quadric surface lying entirely in S_3 and a singular plane meeting S_3 in a line on

* Veronese, *Behandlung der projektischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Prinzip des Projizierens und Schneidens*, Math. Ann., vol. 19, pp. 159–234, paragraph 58. Also B. C. Wong, *On the correspondence between space sextic curves and plane quartics in four-space*, Bull. Amer. Math. Soc., vol. 32, pp. 156–158 and *On certain two-dimensional varieties of order $r(r-1)/2$ in r-space and certain curves traced on them*, Annals of Math. (2), vol. 27, pp. 330–336.

the quadric surface. Therefore, $n = 2$ and $m = 7$. Hence the order of TT' is 7, i. e., every line in σ yields a ruled surface of order 7 whose generators are lines of the congruence Γ in S_3 and each of the points P_i yields a quadric surface. The formula connecting μ and N now becomes

$$7\mu - 2\sum \alpha_i = N.$$

Putting $\mu = 5$, $\sum \alpha_i = 10$, we have $N = 15$, i. e., the transform by TT' of the quintic c^5 is an F^{15} in S_3. But the generators of F^{15} are the intersections of S_3 with the singular planes which form the hypersurface we are considering. Hence the hypersurface is of order 15.

9. Let a given S_3 meet J^{10} in the 10 points $A_i\,[i = 0, 1, \cdots, 9]$. The corresponding singular planes a_i are the 10 planes meeting the F^6 whose trisecants are the transforms of the points of S_3 in the 10 cubic curves γ_i^3. The points of the 45 lines $A_i A_k$ except A_i and A_k go into the lines of the 45 pencils of lines whose planes a_{ik} meet F^6 in the conics $\varkappa_{ik}^2$. Each γ_i^3 transforms by T into a cubic cone having its vertex at A_i and passing through the 9 remaining points, and each $\varkappa_{ik}^2$ transforms into a ruled cubic surface having the line $A_i A_k$ for double line.

10. The conics $\varkappa_{ik}^2$ are the transforms by T' of the lines $P_i P_k$ in σ and the cubics γ_i^3 are those of the cubics c_i^3 each through 9 of the base-points and not through the ith. From the second formula in Paragraph 8 we see that the transforms by TT' of the lines $P_i P_k$ and those of the cubics c_i^3 are cubic surfaces in S_3. Each of the lines $P_i P_k$ meets the quintic c^5 in three points besides P_i and P_k and each of the cubics c_i^3 meets c^5 in six points in addition to the base-points. Hence the planes of $\varkappa_{ik}^2$ meet J^{10} each in three points and those of γ_i^3 each in six points. We infer, therefore, that each singular plane is incident with J^{10} six times and each point of J^{10} is on six singular planes. Hence the hypersurface, J_8^{15}, of the singular planes contains J^{10} sextuply and it also has on it a double twodimensional manifold F^{85} of order 85 which is the locus of the intersections of the planes of the hypersurface or the locus of points whose transforms are the bisecants of J^{10}. Every singular plane meets J_8^{15} in a line and a curve of order 13 which has six quintuple points on J^{10} and is of genus 6, the same as that of the quintic c^5 in σ.

11. Now we determine the number of lines of the system Δ that lie in any given plane. For convenience, take the plane, π, through A_0, A_1, A_2, three of the 10 points A_i in S_3. It meets the cubic cone with vertex at A_0 in the lines $A_1 A_0$, $A_2 A_0$ and a third line p_0; the cone with vertex at A_1 in $A_0 A_1$, $A_2 A_1$ and p_1; and the cone with vertex at A_2 in $A_0 A_2$, $A_1 A_2$, and p_2. It also meets the cubic surfaces with $A_0 A_1$, $A_1 A_2$, $A_2 A_0$ as double lines in these double lines and three others each belonging to one of the three surfaces. The three lines are evidently identical with p_0, p_1, p_2.

Hence, the six lines A_0A_1, A_1A_2, A_2A_0, p_0, p_1, p_2 are the only lines of Δ lying in π. We infer, therefore, that a plane in general position contains six lines of Δ.

12. There are, however, planes in each of which more than six lines of Δ lie, and they are of three systems, J, B, C. The first of these, J, consisting of all the singular planes in each of which every line is a line of Δ has already been considered. The second system, B, consists of ∞^2 planes. Each of the ∞^4 F^6's determined by the ∞^4 3-spaces of S_4 meets 45 of these planes in conics, and each plane contains a pencil of lines of Δ whose vertex is, in general, not on J^{10}. Eeach line of this pencil is the transform of a point on a bisecant of J^{10}. If the bisecant is in a singular plane a, the vertex of the pencil is at a point A of J^{10} to which a corresponds. But as a meets J^{10} in 6 points and therefore contains 15 bisecants, A is on 15 planes each with a pencil of lines of Δ at this point. The planes of this sort constitute an ∞^1-system which is contained in the ∞^2-system B.

13. The third system, C, also an ∞^1-system, is obtained in the following manner. The six points of J^{10} in any plane of J determine five by five six conics. The transform of each of these conics consists of six planes of which five belong to J and the sixth contains a pencil of lines of Δ at a point of J^{10}, the lines being the images of the points of the conic. Every point of J^{10} is on six such planes.

14. The cubic hypercone with vertex at a point of J^{10} corresponding to a singular plane is such that it contains 27 planes of which 6 belong to J, 6 to C, and 15 to B. Any S_3 not through the vertex meets the hypercone in a general cubic surface with 27 lines.

15. As a general plane contains six lines of Δ and meets J_3^{15} in a curve of order 15 with 85 nodes, the corresponding V_3^3 has six conical points which are the vertices of two-dimensional cubic cones entirely contained in it and has ∞ of its rulings incident with J^{10} forming a two-dimensional locus of order 45. Eighty-five generators of this locus and hence of V_3^3 are bisecants of J^{10}. Any general S_3-section of V_3^3 is a general cubic surface whose 27 lines can be readily identified.

16. From what has hitherto been said we find that a general point P of S_4 is the vertex of a cone F^3 of lines of Δ. If P is on J_3^{15}, F^3 degenerates into a plane pencil and a quadric cone; if P is on the double manifold F^{85} of J^{15}, F^3 is composed of three pencils two of which lie in singular planes. The points of J^{10} are vertices of three-dimensional cones.

17. Consider now the system Σ of the ∞^3 trisecants of F^6 corresponding to the points of a given S_3. It contains ∞^3 cubic hypersurfaces corresponding to the ∞^3 planes of S_3 and ∞^4 two-dimensional cubic varieties corresponding to the ∞^4 lines in S_3. Among the latter ∞^2 are cones with

vertices on F^6. As there are 10 bundles of lines at A_i in S_3 each containing a cubic cone of lines of Δ, there are in Σ 10 ∞^2-families of quadrics, each containing and ∞^1-family of cones having their vertices on a plane cubic of F^6 and forming a hypersurface of order 9. In Σ are also found 45 pencils of lines.

18. The projection Σ' of Σ from a general point in S_4 on to an S_3 is a complex of lines. This complex is evidently of order 8. The complex curve in any plane of S_3 is of class 8, order 20, deficiency 3, and has 132 nodes, 36 cusps, 6 triple tangents, and no inflexions. Every complex cone is of order 8 and has 7 nodal elements. The locus of the vertices of these cones that degenerate is a sextic surface, the projection of F^6, with a double septimic curve. Each of these degenerate cones is composed of a cubic and a quintic cone. The cones with vertices on the double septimic curve are each composed of two cubics and a quadric. Other properties of the complex can be inferred without difficulty.

19. It remains to add a few words concerning the congruence Γ of lines in S_3 which are the transforms of the points of F^6. It is of order 3, class 6 and rank 5. Its focal surface is of order 14 and class 20. The surfaces of the congruence can be obtained by means of the transformation TT' from the curves of the plane σ. By assigning proper values to m and $\Sigma\alpha_i$ in the formula

$$N = 7m - 2\Sigma\alpha_i$$

we obtain surfaces of Γ of any desired order. The case $N = 1$ does not exist, for there are no singular planes in S_3 which is assumed to be general. The case $N = 2$ is exceptional but we know that there exist 10 quadric surfaces coming from the 10 base-points P_i in σ. For $N = 3$ we have $m = 1$, $\Sigma\alpha_i = 2$ giving rise to the 45 cubic surfaces having the lines $A_i A_k$ for double lines and $m = 3$, $\Sigma\alpha_i = 9$ giving rise to the 10 cubic cones at A_i. And so on.

20. It is of interest to notice the ∞^4-family of octic surfaces resulting from putting $m = 4$, $\Sigma\alpha_i = 10$. The fundamental quartic curves in σ yield by T' ∞^4 3-space sextic curves. Let $C^{6'}$ be one of these sextics. It transforms by T into a trisecant surface of another sextic C^6 which is the intersection of S_3 and the F^6 corresponding to the S_3 containing $C^{6'}$. The special or degenerate cases are numerous and can be obtained readily. Note that this family is not the only family of octic surfaces in Γ, for there are three others given by $m = 2$, $\Sigma\alpha_i = 3$; $m = 6$, $\Sigma\alpha_i = 17$; $m = 8$, $\Sigma\alpha_i = 24$. The surfaces of these families are, however, no trisecant surfaces of sextic curves.

THE UNIVERSITY OF CALIFORNIA.

ON THE FIRST CASE OF FERMAT'S LAST THEOREM.*

By H. S. Vandiver.

If

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in rational integers x, y and z are prime to the odd prime p then

$$B_{\frac{p-i}{2}} f_i(t) \equiv 0, \qquad f_{p-1}(t) \equiv 0 \pmod{p},$$
$$i = 3, 5, \cdots, p-2, \qquad -t = x/y, y/x, x/z, z/x, y/z, z/y, \tag{1a}$$

and

$$f_i(t) = \sum_{k=1}^{p-1} k^{i-1} t^k,$$

which are known as the Kummer criteria. The six values of t given are incongruent modulo p unless $t^2 - t + 1 \equiv 0 \pmod{p}$ in which case only two are incongruent and $t = -1$, in which case only three are incongruent. Pollaczek† proved that $t^2 - t + 1 \equiv 0 \pmod{p}$ is inconsistent with (2) leaving the important special case $t = -1$ to be disposed of.

1. In the present paper I obtain a result (Theorem 1) concerning this case and some other criteria concerning (1) without the assumption $t = -1$.

Assume that $x \equiv y \pmod{p}$. Let $\alpha = e^{2i\pi/p}$. Assume that the second factor of the class number of the field $k(\alpha)$ is prime to p, then

$$(x + \alpha^k y)(x + \alpha^{-k} y) = \eta_k \omega_k^p \tag{1b}$$

where η_k is a real unit and ω_k is an integer in $k(\alpha)$. If n is odd then

$$(x+y)^n \left[\frac{d^n \log (x + e^v y)}{d v^n} \right]_{v=0}$$

is divisible by $x - y$ (Frobenius).‡

We will employ Furtwängler's law of reciprocity

$$\left\{ \frac{\omega}{\theta} \right\} \left\{ \frac{\theta}{\omega} \right\}^{-1} = \left\{ \frac{\theta, \omega}{\mathfrak{p}} \right\} \tag{2}$$

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* Received December 11, 1928.

† Wiener Bericht, Abt. IIa., 126 (1917), 45–49.

‡ Berlin. Sitzungsberichte, 1910, p. 829, relation (11) and p. 843, relation (1).

where ω and θ are two integers in $k(\alpha)$ with (ω) and (θ) prime to each other and to $\mathfrak{p} = (1-\alpha)$. Let $(\theta) = (\mathfrak{a}\mathfrak{a}_{-1})^g$ where $\mathfrak{a}$ is any prime ideal in the field $k(\alpha)$ and prime to ω and $\mathfrak{p}$, and $\mathfrak{a}_{-1}$ is the ideal obtained from $\mathfrak{a}$ by the substitution (α/α^{-1}). Assume that $\mathfrak{a}\mathfrak{a}_{-1}$ belongs to the exponent g. Since the second factor of the class number of the field $k(\alpha)$ is prime to p then g is not divisible by p. Set

$$\omega = \alpha^a (x + \alpha^k y) \tag{2a}$$

where a is chosen so that ω is semi-primary, so that $a = -k/2$. Hence

$$\left\{\frac{\theta}{\omega}\right\} = 1.$$

Hence (2) gives

$$\left\{\frac{\omega}{\theta}\right\} = \left\{\frac{\theta, \omega}{\mathfrak{p}}\right\}. \tag{3}$$

By definition we also have

$$\left\{\frac{\theta, \omega}{\mathfrak{p}}\right\} = \alpha^P,$$

where

$$P = -l_1(\theta)\frac{N(\omega)-1}{p} + l_1(\omega)\frac{N(\theta)-1}{p} + \sum_{s=2}^{p-2}(-1)^{s-1} l^s(\theta)\, l^{p-s}(\omega), \tag{4}$$

and

$$l_s(\theta) = \left[\frac{d^s \log \theta}{d v^s}\right]_{v=0},$$

and if

$$\theta = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{p-2}\alpha^{p-2},$$

then

$$\theta(e^v) = a_0 + a_1 e^v + a_2 e^{2v} + \cdots + a_{p-2} e^{v(p-2)}.$$

Since, as already noted we have $(x+y)^n \cdot l^n(\omega)$ is divisible by $x-y$ if n is odd then this expression is divisible by p if $x \equiv y \pmod p$. We also have $l_n(\theta) \equiv 0 \pmod p$ for n odd. We also have

$$\frac{N(\omega)-1}{p} = \frac{\frac{x^p+y^p}{x+y}-1}{p} = \frac{x^p+y^p-(x+y)}{p(x+y)}$$

since it is known that if y is divisible by p it is also divisible by p^2 and also $x^p \equiv x \pmod{p^2}$ and we find

$$\frac{N(\omega)-1}{p} \equiv 0 \pmod p.$$

Hence $P \equiv 0 \pmod{p}$. Hence

$$\left\{\frac{\omega}{\mathfrak{a}\,\mathfrak{a}_{-1}}\right\} = 1$$

and we infer that $\omega\,\omega_{-1}^{p-1}$ is the p-th power of an integer in $k(\alpha)$. Hence we may write

(3a) $$(x+\alpha y)\,(x+\alpha^{-1}y)^{p-1} = \alpha^{-2a}\,\gamma^p$$

where γ is an integer in the field $k(\alpha)$. Multiplying together this equation and (1a) we have

$$(x+\alpha^k y)^2\,(x+\alpha^{-k}y)^p = \varepsilon_k'(\delta')^p$$

where δ is an integer and ε_k is a unit in $k(\alpha)$. This gives easily

(4) $$x+\alpha^k y = \varepsilon_k\,\delta_k^p.$$

We also have from (1)

(4a) $$x+y = \delta_0^p.$$

In (4) put $k=1$ and -1 in turn and subtract. We have

$$(\alpha-\alpha^{-1})y = \varepsilon_1\,\delta_1^{'p} - \varepsilon_{-1}\,\delta_{-1}^p.$$

Using the original equation together with this equation for $k=1$ and subtracting we have

$$(\alpha^k-\alpha)y = \varepsilon_k\,\delta_k^p - \varepsilon_1\,\delta_1^p.$$

Division of the last two equations gives a relation which may be put in the form

(5) $$(\alpha^k-\alpha^{-1})\,\varepsilon_1\,\delta_1^p - (\alpha^k-\alpha)\,\varepsilon_{-1}\,\delta_{-1}^p = (\alpha-\alpha^{-1})\,\varepsilon_k\,\delta_k^p.$$

Setting $-k$ for k in this relation we obtain

(6) $$(\alpha^{-k}-\alpha^{-1})\,\varepsilon_1\,\delta_1^p - (\alpha^{-k}-\alpha)\,\varepsilon_{-1}\,\delta_{-1}^p = (\alpha-\alpha^{-1})\,\varepsilon_{-k}\,\delta_{-k}^p.$$

In equation (4) we have by taking the residues modulo $(1-\alpha)^2$

(7) $$\varepsilon_k = \alpha^{ky/x+y}\,\eta_k = \alpha^{k/2}\,\eta_k$$

where η_k is a real unit in the field $k(\alpha)$.

2. Consider $q = 1+cp$, $c<p$, and q prime. Then an ideal divisor of (q) does not remain unaltered by substitution (α/α^{-1}). Hence by a result due to Pollaczek* if $\mathfrak{q}$ is a prime ideal divisor of q (where $\mathfrak{q}_{-1}$ is obtained from $\mathfrak{q}$

* Math. Zeitschr., 21 (1924), Theorem VI.

by the substitution (α/α^{-1}) is such that $(\mathfrak{q}\mathfrak{q}_{-1})^g$ is a principal ideal with g prime to p. Hence the reasoning by which (3a) was obtained gives using (2a) and noting that $x+\alpha y$ is prime to q, by Furtwängler's theorem,*

$$\left\{\frac{x+\alpha y}{\mathfrak{q}}\right\} = \left\{\frac{\alpha}{\mathfrak{q}}\right\}\left\{\frac{x+\alpha^{-1}y}{\mathfrak{q}}\right\}$$

without the assumption that the second factor of the class number of $k(\alpha)$ is prime to p. From this we have immediately

$$(x+\alpha y)^c \equiv \alpha^c (x+\alpha^{-1}y)^c \pmod{\mathfrak{q}},$$

whence

$$x^c + \alpha y c x^{c-1} + \cdots + \alpha^c y^c \equiv \alpha^c x^c + c x^{c-1}\alpha^{c-1} y + \cdots + y^c \pmod{\mathfrak{q}}.$$

This holds also if we set $\alpha = 1$ and also holds if α is replaced by any of its powers. Take $\alpha = \alpha^0, \alpha, \alpha^2, \cdots, \alpha^{p-1}$, in turn, multiply the second congruence by α^{p-c+k} and in general the i-th congruence by $\alpha^{(i-1)(p-c+k)}$ and add the results, we obtain, if $k = (c+2)/2$, since $xyz \not\equiv 0 \pmod{q}$,

$$x^{c-k} y^{c-k} (x^2 - y^2) \equiv 0, \qquad x \equiv \pm y \pmod{q}.$$

The congruence $x \equiv -y \pmod{q}$ is impossible by Furtwängler's theorem, and $x \equiv y \pmod{q}$ gives from (1)

$$2y^p + z^p \equiv 0 \pmod{q}$$

or

$$2^c \equiv 1 \pmod{q}.$$

The above results together with those obtained in another paper by the writer† may be combined into the following:

THEOREM 1. *If* (1) *is satisfied in integers prime to each other and to the odd prime* p *and* $x \equiv y \pmod{p}$ *then we have*

1. *The congruence* $2^{p-1} \equiv 1 \pmod{p^4}$ *holds.*
2. $x \equiv y \pmod{p^3}$.
3. *There exists no prime* q *in the set* $q = 1 + cp$, $c = 2, 4, \cdots, p-1$, *such that* $2^c \not\equiv 1 \pmod{q}$.

3. We now proceed also to an extension of Theorem IV in a previous paper of the writer's.‡ Using the relation (2) without the assumption that $x \equiv y \pmod{p}$ but the other symbols being defined as previously where $P \not\equiv 0 \pmod{p}$, but, if $x + \alpha^k y \equiv \omega$, takes the form

* Wiener Bericht, Abt. IIa, 121 (1912), p. 589-592.

† Trans. Amer. Math. Soc., 15 (1914), p. 203.

‡ Proc. Nat. Acad. Sci., vol. 12 (1926), p. 109.

$$-\sum_s k^{p-2s}\left[\frac{d^{p-2s}\log(x+e^v y)}{dv^{p-2s}}\right]_{v=0}\left[\frac{d^{2s}\log(\theta(e^v))}{dv^{2s}}\right]_{v=0}$$

$$\equiv \frac{-kyg}{x+y}I(\alpha)+gI(x+\alpha^k y) \pmod p, \qquad s=1,2,\cdots,\frac{p-3}{2}.$$

And

$$\left\{\frac{\gamma}{\mathfrak{a}\,\mathfrak{a}_{-1}}\right\} = \alpha^{I(\gamma)},$$

where γ is an integer in $k(\alpha)$. Multiply both sides of this congruence by k^{2t-1} and let k range over the integers $1, 2, \cdots, p-2$, and add the resulting congruences; we obtain

$$\left[\frac{d^{p-2t}\log(x+e^v y)}{dv^{p-2t}}\right]_{v=0}\left[\frac{d^{2t}\log\theta(e^v)}{dv^{2t}}\right]_{v=0} \equiv gk^{2t-1}\sum_{k=1}^{p-1} I(x+\alpha^k y) \pmod p$$

for $t<(p-1)/2$.

4. Consider the relation (1 b). Replace this as usual by an identity in e^v and put $k=1$ so that we have

$$(8) \qquad (x+\alpha^v y)(x+\alpha^{p-v}y) = \eta(e^v)\,\omega^p(e^v)+X\frac{e^{vp}-1}{e^v-1},$$

where X is a polynomial in e^v. To proceed further we express η_k in a particular form. Set

$$\varepsilon_t = \varepsilon^{P(p-1)}$$

where P is the symbolic power

$$(1-s)(r^2-s)(r^4-s)\cdots(r^{2t-2}-s)(r^{2t+2}-s)\cdots(r^{p-3}-s)$$

with s denoting the substitution (α/α^{-1}). We may show that the set of units

$$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{\mu-1}$$

where $\mu=(p-1)/2$ forms a system of independent real units in the field $k(\alpha)$. Hence we may write

$$(9) \qquad \eta = \varepsilon_1^{N_1}\varepsilon_2^{N_2}\cdots\varepsilon_{\mu-1}^{N_{\mu-1}}$$

where the N's are rational fractions and assume* that the denominators of the N's are prime to p also in the above expressions we have

$$\varepsilon = \sqrt{\frac{(1-\alpha^r)(1-\alpha^{-r})}{(1-\alpha)(1-\alpha^{-1})}}$$

* This includes the assumption that the second factor of the class number of $k(\alpha)$ is prime to p, which may be shown as in Hilbert, Bericht. Deut. Math. Verein., 1894, p. 435.

where r is a primitive root of p. Using this form of η in (8) we obtain since

$$\left[\frac{d^{2i}\log\frac{e^{ax}-1}{e^x-1}}{dx^{2i}}\right]_{x=0} = (-1)^{i+1}\frac{B_i}{2i}(a^{2i}-1),$$

$$\left[\frac{d^{2n}\log(x+e^v y)}{dv^{2n}}\right]_{v=0} \equiv N_n\left(\frac{p-1}{2}\right)(-1)^{N+1}\frac{B_n}{2n}(r^{2n}-1).$$

Hence if B_n is determined modulo p we may use the Kummer criteria (1 a)

$$B_a\left[\frac{d^{p-2a}\log(x+e^v y)}{dv^{p-2a}}\right]_{v=0} \equiv 0 \pmod p$$

for $a = 1, 2, \cdots, \mu-1$. We may apply this to the last relation by multiplying both members by the second factor in the congruence with $n = a$ and obtain

$$\left[\frac{d^{2n}\log(x+e^v y)}{dv^{2n}}\right]_{v=0}\left[\frac{d^{p-2n}\log(x+e^v y)}{dv^{p-2n}}\right]_{v=0}$$

$n = 1, 2, 3, \cdots, \mu-1$. But in the Kummer criteria just mentioned we might have used the set (x, z) in lieu of (x, y) and similarly for the four other combinations obtained from x, y, z. Hence we will obtain another derivation without the use of Furtwängler's law of reciprocity, but under the assumption that the denominators of the N's in (9) are prime to p, of the following criteria for the first case of Fermat's last theorem.*

$$f_{p-n}(t)f_n(1-t) \equiv 0, \qquad f_n(t)f_{p-n}(t) \equiv 0 \pmod p,$$

where $n = 2, 3, \cdots, p-2$, and

$$f_n(t) = \sum_{i=1}^{p-1} i^{n-1}t^i;$$

$$-t = \frac{x}{y}, \frac{y}{x}, \frac{x}{z}, \frac{z}{x}, \frac{y}{z}, \frac{z}{y}.$$

5. Suppose p is congruent to 1 modulus 3 and consider the cyclotomic subfield generated by $\alpha+\alpha^s+\alpha^{s^2}$, where

$$s^3 \equiv 1 \pmod p,$$

$s \not\equiv 1 \pmod p$. Assume that the class number of this field is prime to p then consider the first case of Fermat's last theorem and we shall have

* Vandiver, Proc. Nat. Acad. Sci., 11 (1925), p. 291.

$$(x+\alpha y)(x+\alpha^s y)(x+\alpha^{s^2} y) = \eta\,\omega^p,$$

where η is a unit in $k(\alpha)$, and ω is an integer in $k(\alpha)$. We may write

$$\eta = \alpha^t \varepsilon,$$

where ε is a real unit in $k(\alpha)$.

Replacing the above relation by an identity in e^v, taking logarithms of both members and differentiating 3 times with the substitution $v = 0$ we obtain

$$\left[\frac{d^3 \log(x+e^v y)}{dv^3}\right]_{v=0} \equiv 0 \pmod{p}.$$

Replacing the set (x, y) by (x, z), etc., we find that this is impossible. Hence Fermat's relation does not hold in case 1 if

$$p \equiv 1 \pmod{3}$$

and the class number of the field defined by $\alpha+\alpha^s+\alpha^{s^2}$ is prime to p. It is easy to verify that analogous results will hold in case $p \equiv 1 \pmod{5}$ and also $p \equiv 1 \pmod{7}$.

AN ALGORITHM FOR TRANSFORMING KUMMER CRITERIA IN CONNECTION WITH FERMAT'S LAST THEOREM.*

By H. S. Vandiver.

If

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in integers prime to the odd prime p it is known that

$$B_n f_{p-2n}(t) \equiv 0, \qquad f_{p-1}(t) \equiv 0 \pmod{p} \tag{2}$$

where $-t \equiv x/y, y/x, x/z, z/x, y/z, z/y$ modulo p,

$$f_a(t) = \sum_{i=1}^{p-1} i^{a-1} t^i \quad (a>1,\ n = 1, 2, \cdots, (p-3)/2),$$

and $B_1 = 1/6$, $B_2 = 1/30$, etc., are the numbers of Bernoulli. The congruences involving the B's are known as the Kummer criteria for the solution of (1).

In the present paper we shall apply the theory of power characters in the field $k(\alpha)$ where $\alpha = e^{2i\pi/p}$ to (1) and obtain results which are essentially transformations of the formulae (2). This method yields various combinations of (2) which I have not yet been able to derive by direct methods and some of these will be considered here. Using the assumption mentioned concerning (1) we have†

$$\prod_{v=1}^{k-1} \prod_{r=1}^{\left[\frac{vp}{k}\right]} (x + \alpha^{[1:r]} y)^v = \alpha^{\frac{-2kyq(k)}{x+y}} \omega^{2p} \tag{3}$$

where k is an integer greater than 1 and less than p, ω is an integer in the field $\Omega(\alpha)$, $q(k) = (k^{p-1} - 1)/p$, and $[1 : r]$ is the least positive solution of $rx \equiv 1 \pmod{p}$. Take p-th power characters in (3) with respect to $x + \alpha\beta y$ in the field $\Omega(\alpha\beta)$, we have if $\mu = (p-1)/2$, $\beta = e^{2i\pi/n}$, n is a prime $\not\equiv 1 \pmod{p}$,

$$\left\{\frac{\alpha y}{x+\alpha\beta y}\right\}_n^{\mu(k-1)} \prod_r \left\{\frac{-\beta + \alpha^{[1:r]-1}}{x+\alpha\beta y}\right\}_n = \left\{\frac{\alpha}{x+\alpha\beta y}\right\}_n^{\frac{-2kyq(k)}{x+y}} \tag{4}$$

which holds provided $x + \alpha\beta y$ is prime to $1-\alpha$ in the field $\Omega(\alpha\beta)$. Now

$$\left\{\frac{\alpha}{x+\alpha\beta y}\right\}_n = \alpha^{\frac{N(x+\alpha\beta y)-1}{p}}$$

* Received December 11, 1928.

† Annals of Math. 21 (1919), p. 78.

where N denotes the norm in the field $\Omega(\alpha\beta)$, also

$$N(x+\alpha\beta y) = \frac{(x^{np}+y^{mp})(x+y)}{(x^n+y^n)(x^p+y^p)}.$$

But $x^{np}+y^{np} \equiv x^n+y^n \pmod{p^2}$ and $x^p+y^p \equiv x+y \pmod{p^2}$ so that $N(x+\alpha\beta y) \equiv 1 \pmod{p^2}$ and

$$\left\{\frac{\alpha}{x+\alpha\beta y}\right\}_n = 1. \tag{5}$$

We also have

$$\left\{\frac{y}{x+\alpha\beta y}\right\}_n^{n-1} = \left\{\frac{y}{\frac{x^n+\alpha^n y^n}{x+\alpha y}}\right\}_n.$$

Let

$$\frac{x^n+\alpha^n y^n}{x+\alpha y} = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_s$$

where the $\mathfrak{p}$'s are prime ideals in the field $\Omega(\alpha)$, and since all these ideals are of first degree in $\Omega(\alpha\beta)$

$$\frac{x^n+\alpha^n y^n}{x+\alpha y} = \mathfrak{q}_1 \mathfrak{q}_2 \cdots \mathfrak{q}_{s(n-1)}$$

the $\mathfrak{q}$'s in the field $\Omega(\alpha\beta)$. Let the norm of $\mathfrak{p}_a$ be q_a, then in field $\Omega(\alpha)$ we have

$$y^{\frac{q_a-1}{p}} \equiv \left[\frac{y}{\mathfrak{p}_a}\right] \pmod{\mathfrak{p}_a}.$$

Now if, in the field $\Omega(\alpha\beta)$ we have $\mathfrak{p}_a = \mathfrak{q}_1^{(a)} \mathfrak{q}_2^{(a)} \cdots \mathfrak{q}_{n-1}^{(a)}$ then

$$y^{\frac{q_a-1}{p}} \equiv \left\{\frac{y}{\mathfrak{q}_r^{(a)}}\right\}_n \pmod{\mathfrak{q}_r^{(a)}}$$

so that

$$\left\{\frac{y}{\mathfrak{q}_r^{(a)}}\right\}_n = \left\{\frac{y}{\mathfrak{p}_a}\right\},$$

and

$$\left\{\frac{y}{\mathfrak{p}_a}\right\}_n = \left\{\frac{y}{\mathfrak{p}_a}\right\}^{n-1},$$

hence

$$\left\{\frac{y}{x+\alpha\beta y}\right\}_n = \left\{\frac{y}{\frac{x^n+\alpha^n y^n}{x+\alpha y}}\right\}$$

and

$$\left\{\frac{y}{\frac{x^n+y^n\alpha^n}{x+\alpha y}}\right\} = \left\{\frac{y}{x^n+y^n\alpha^n}\right\}\left\{\frac{y}{x+\alpha y}\right\}^{-1},$$

and by Eisenstein's law of reciprocity, since $\left\{\frac{\alpha}{y}\right\} = 1$,

$$\left\{\frac{y}{x^n+y^n\alpha^n}\right\} = \left\{\frac{x^n+y^n\alpha^n}{y}\right\} = \left\{\frac{x^n}{y}\right\} = 1.$$

Similarly $\left\{\frac{y}{x+\alpha y}\right\} = 1$. Hence, since $n-1 \not\equiv 0 \pmod{p}$,

$$\left\{\frac{y}{x+\alpha\beta y}\right\}_n = 1. \tag{6}$$

By means of (4), (5) and (6), we can write

$$\prod_r \left\{\frac{1-\beta\alpha^{1-[1:r]}}{x+\alpha\beta y}\right\}_n = 1. \tag{7}$$

By means of $x+y = v^p$ we also have

$$\left\{\frac{1-\alpha\beta}{x+\alpha\beta y}\right\}_n = 1. \tag{8}$$

In (3) substitute α^m for α, we find as before

$$\prod_r \left\{\frac{1-\beta\alpha^{1-m[1:r]}}{x+\alpha\beta y}\right\}_n = 1. \tag{9}$$

Now set, if $\left\{\frac{\theta}{x+\alpha\beta y}\right\} = \alpha^{I(\theta)}$,

$$\begin{aligned}
D_0 &= I(1-\alpha\beta) + I(1-\alpha^2\beta) + \cdots + I(1-\alpha^{p-1}\beta),\\
D_1 &= I(1-\alpha\beta) + 2I(1-\alpha^2\beta) + \cdots + (p-1)I(1-\alpha^{p-1}\beta),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
D^{p-2} &= I(1-\alpha\beta) + 2^{p-2}I(1-\alpha^2\beta) + \cdots + (p-1)^{p-2}I(1-\alpha^{p-1}\beta).
\end{aligned}$$

Adding, we get from (8)

$$D_0 + D_1 + \cdots + D_{p-2} \equiv 0 \pmod{p}. \tag{10}$$

Similarly we obtain

$$D_0 + b^{p-2}D_1 + b^{p-3}D_2 + \cdots + bD_{p-2} \equiv -I(1-\alpha^b\beta) \pmod{p}, \tag{11}$$

for $b \not\equiv 0$. We shall now show that (11) may also be used if $b = 0$. For in this case it reduces to

$$I(1-\beta^p) - I(1-\beta) \equiv -I(1-\beta) \pmod{p},$$

and this holds since if we have $\prod_{i=0}^{p-1}(x+\alpha^i y) = -z^p$, then $\left\{\frac{\prod_{i=0}^{p-1}(x+\alpha^i y)}{x+\alpha\beta y}\right\}_n = 1$,

and this gives

$$\left\{\frac{1-\beta^p}{x+\alpha\beta y}\right\}_n = 1. \tag{11a}$$

Apply (11) to (9) we have

$$-\sum_r I(1-\alpha^{1-m[1:r]}\beta)$$

$$\equiv \mu(k-1)D_0+\sum_r(1-m[1:r])^{p-2}D_1+\cdots+\sum_r(1-m[1:r])D_{p-2} \tag{12}$$

$$\equiv 0 \pmod p.$$

Expand the right hand member according to powers of m, we have

$$m^{p-2}[1:r]^{p-2}D_1+(p-2)\sum_r m^{p-3}[1:r]^{p-3}+\sum_r m^{p-3}[1:r]^{p-3}$$

+ terms of lower degree than $p-3 \equiv 0 \pmod p$. Multiply through by m and let m range over the integers $1, 2, \cdots, p-1$ and add the resulting congruences, we have

$$\sum_r [1:r]^{p-2}D_1 \equiv 0 \pmod p.$$

But we have

$$\sum_r [1:r]^{p-2} \equiv \frac{(1-k)b_1}{2} \pmod p.$$

Hence we have, if $k \neq 1$, $\sum_r [1:r]^{p-2} \not\equiv 0 \pmod p$ and therefore

$$D_1 \equiv 0 \pmod p. \tag{13}$$

We also can derive more complicated criteria by considering other powers of m. For example, from (12), by collecting coefficients of m, we get

$$\sum_r [1:r](2D_1+3D_2+\cdots+(p-1)D_{p-2} \equiv 0 \pmod p. \tag{14}$$

We can select k so that $[1:r] \equiv 0 \pmod p$, and this may be written after division by $\sum_r [1:r]$,

$$(p-2)D_1-\frac{(p-2)(p-3)}{2}D_2+\binom{p-2}{3}D_3+\cdots+D_{p-2} \equiv 0 \pmod p. \tag{15}$$

Similarly from (12) by considering coefficients of m^2, we have

$$\sum_r [1:r]^2\left(\binom{p-2}{2}D_1+\binom{p-3}{2}D_2+\cdots+\binom{2}{2}D_{p-3}\right) \equiv 0 \pmod p.$$

But here we have $\sum_r [1:r]^2 \equiv 0 \pmod p$. In (12), coefficients of m^3, m^4, etc., give in general

$$\sum_r [1:r]^n\left(\binom{p-2}{n}D_1+\binom{p-3}{n}D_2+\cdots+D_{p-(n+1)}\right) \equiv 0 \pmod p.$$

But we have

$$\sum_r [1:r]^n \equiv \frac{(1-k^{n+1})\,b_{n+1}}{(n+1)\,k^n} \pmod p,$$

$$b_{2a+1} = 0, \qquad b_{2a} = (-1)^{a-1} B_a,$$

and if we let k be a primitive root of p, then $k^{n+1}-1 \not\equiv 0 \pmod p$, and therefore

$$b_{n+1}\left(\binom{p-2}{n} D_1 + \binom{p-3}{n} D_2 + \cdots + D_{p-(n+1)}\right) \equiv 0 \pmod p.$$

Another expression for D_s is given by the writer,* and employing a scheme similar to the one explained in the present paper for deducing (35) from (34) we have the D's expressed in terms of x and y and the above congruence yields another proof of the criterion†

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{[p/3]^2} \equiv 0 \pmod p,$$

for the solution of (1) in integers not zero and prime to the odd prime p.

2. To proceed further we examine the power character

$$\left\{\frac{\dfrac{1-\beta^s}{1-\beta}}{\mathfrak{p}}\right\}$$

where $\mathfrak{p}$ is an ideal in the field $\Omega(\theta)$, θ is a primitive mth root of unity, β is a primitive nth root of unity, α is a primitive pth root of unity, n and p being odd primes and $n \not\equiv 1 \pmod p$. Also $pn = m$. Consider

$$x + \theta y = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_s$$

where each $\mathfrak{p}$ is a prime ideal in $\Omega(\theta)$. We have

$$\psi_1(\theta) = \sum \theta^{-bh+(a+b)\operatorname{ind}(g^h+1)} = \prod \mathfrak{p}_t(\theta^j)$$

where since all the $\mathfrak{p}$'s are of the first degree then the rational prime q_i is the norm of $\mathfrak{p}_i$, h ranges over the integers $0, 1, \cdots, q_1 - 1$ excepting $(q_1-1)/2$. We also have $jt \equiv 1 \pmod m$, t ranging over all the integers less than and prime to m, satisfying

$$|al| + |bl| > m \tag{15a}$$

where $|w|$ represents the least positive residue of w, modulo m.

Set

$$q_\mu = 1 + r_\mu m.$$

* Amer. Journ. Math., 47 (1925), p. 147, relation (12).

† Annals of Math. (2), 26 (1924), p. 88–94.

Also, g is a primitive root of p_1 and $\text{ind}(g^h+1)$ is defined as the integer r in the set $1, 2, q_1-1$ which satisfies the relation $g^h+1 \equiv g^r \pmod{q_1}$.

We shall assume that

$$a+b \not\equiv 0 \pmod{n}, \tag{16}$$

$b = nt$, t prime to p. In addition we shall assume that g and θ are connected by the relation

$$g^\nu \equiv \theta \pmod{\mathfrak{p}_1}.$$

In a similar way we set up the functions $\Psi_2(\theta), \Psi_3(\theta), \cdots, \Psi_s(\theta)$ and multiplying the functions together we have

$$\prod_s \Psi_s(\theta) = \varepsilon \prod_j (x+\theta^j y),$$

where ε is a unit in $\Omega(\theta)$. Following Kronecker this unit must be $\pm\theta^c$ where c is an integer. We may then write the identity

$$\prod_s \Psi_s(\beta e^v)^{2n} = e^{2vnc} \prod_j (x+e^{vj}\beta^j y)^{2n} + VW \tag{17}$$

where $V = (e^{vp}-1)/(e^v-1)$. Since $(x^p-1)/(x-1)$ is irreducible in the field $\Omega(\beta)$, W is a polynomial in e^v with coefficients in $\Omega(\beta)$. This may be written

$$\frac{\prod_s \Psi_s^{2n}(\beta e^v)}{e^{2vnc}\prod_j (x+e^{vj}\beta^j y)^{2n}} = 1+VW_1$$

where

$$W_1 = \frac{W}{e^{2vnc}\prod_j (x+e^{vj}\beta^j y)^{2n}}.$$

Taking logarithms and differentiating as to V we have

$$2n\sum_s \frac{d\log \Psi_s(\beta e^v)}{dv} - 2nc - 2n\sum_j \frac{d\log(x+e^{vj}\beta^j y)}{dv} = \frac{d(VW_1)}{1+VW_1}. \tag{17a}$$

Now assume that $(x+\beta^j y)$ is prime to the ideal (p), then

$$\left[\prod_j (x+e^{vj}\beta^j y)\right]_{v=0} \not\equiv 0 \pmod{p}.$$

Also

$$[V]_{v=0} \equiv \left[\frac{dV}{dv}\right]_{v=0} \equiv 0 \pmod{p}$$

and

$$\left[\frac{d(VW_1)}{dv}\right]_{v=0} \equiv 0 \pmod{p}.$$

Let β take on all the primitive roots of $x^n = 1$ as values, we have

$$(18) \qquad \sum_i \sum_s \frac{d \log \psi_s(\beta^i e^v)}{dv} - \sum_i \sum_j \left[\frac{d \log (x + e^{vj} \beta^{ij} y)}{dv}\right]_{v=0} \equiv (n-1)\, c \pmod p.$$

Now examine

$$\left[\frac{d \log \psi(\beta^i e^v)}{dv}\right]_{v=0}$$

for any one of the ψ's. By definition, this becomes, for i arbitrary

$$(19) \qquad \frac{1}{\psi(\beta)} \sum_h (-bh + (a+b) \operatorname{ind} (g^h + 1))\, \beta^{-bh + (a+b) \operatorname{ind}(g^h+1)}.$$

Now set $g^h + 1 \equiv g^{h'} \pmod{\mathfrak{p}}$, whence

$$\operatorname{ind} (g^h + 1) \equiv h' \pmod p,$$

and

$$\operatorname{ind} (g^{h'} - 1) \equiv h \pmod p,$$

h' ranging over the values $1, 2, \cdots, p-2$. $\psi(\beta)$ then becomes

$$\sum_{h'} \beta^{-b \operatorname{ind}(g^{h'}-1) + (a+b)h'}.$$

Since $b \equiv 0 \pmod n$ and $a + b \not\equiv 0 \pmod n$, we have

$$\psi(\beta) = \frac{\beta^{q-1} - 1}{\beta - 1} - 1$$

so (19) becomes, if we write h for h'

$$-\sum_h (-b \operatorname{ind} (g^h - 1) + (a+b)\, h)\, \beta^{(a+b)h}.$$

This may be written

$$b \sum \operatorname{ind} (g^h - 1)\, \beta^{(a+b)h} - \sum (a+b)\, h\, \beta^{(a+b)h}.$$

Set $(a+b)\, h = k$, then the second part is

$$\sum_{k=1}^{q-2} k \beta^k \equiv 0 \pmod p,$$

since this is the expression for

$$\frac{d}{dv}\left(\frac{e^{v(q-1)} - 1}{e^v - 1}\right)$$

with $e^v = \beta$. Hence the sum reduces to

$$b \sum \operatorname{ind} (g^h - 1)\, \beta^{ah}.$$

Let β range over the primitive roots of $x^n \doteq 1$, then

$$(20) \qquad \begin{aligned} & b\sum_{i=1}^{n-1}\sum_{h}\operatorname{ind}(g^h-1)\,\beta^{iah} \\ & \equiv -b\sum\operatorname{ind}(g^h-1)+bn\sum_{k=1}^{p\nu-1}\operatorname{ind}(g^{kn}-1) \pmod p. \end{aligned}$$

To reduce this consider the product

$$(g-1)(g^2-1)\cdots(g^{q-2}-1).$$

We have for z arbitrary

$$\frac{z^{q-1}-1}{z-1} \equiv (z-g)(z-g^2)\cdots(z-g^{q-2}) \pmod{\mathfrak{p}}.$$

Let $z=1$, we have

$$-(q-1) \equiv (g-1)(g^2-1)\cdots(g^{q-2}-1) \pmod{\mathfrak{p}},$$

or

$$\operatorname{ind}(1) \equiv 0 \equiv \sum_{h}\operatorname{ind}(g^h-1) \pmod{\mathfrak{p}}.$$

Similarly

$$\frac{z^{p\nu}-1}{z-1} \equiv (z-g^n)(z-g^{2n})\cdots(z-g^{n(p\nu-1)}) \pmod{\mathfrak{p}},$$

and for $z=1$

$$\frac{-q+1}{n} \equiv -p\nu \equiv (g^n-1)(g^{2n}-1)\cdots(g^{n(p\nu-1)}-1) \pmod{\mathfrak{p}}.$$

Hence

$$\begin{aligned} \operatorname{ind}(-1)+\operatorname{ind}(q-1)-\operatorname{ind} n &\equiv \sum_{k=1}^{p\nu-1}\operatorname{ind}(g^{kn}-1) \pmod p \\ -\operatorname{ind} n &\equiv \sum_{k=1}^{p\nu-1}\operatorname{ind}(g^{kn}-1) \pmod p. \end{aligned}$$

Hence (20) becomes

$$(21) \qquad b\sum_{i=1}^{n-1}\sum_{p}\operatorname{ind}(g^h-1)\,\beta^{iah} \equiv -bn\ \operatorname{ind} n \pmod p.$$

We shall now obtain another relation involving c. Multiply (17a) by β^{-1} and let β range over all its values we have

$$(22) \qquad \begin{aligned} & \sum_i\sum_s \beta^{-i}\left[\frac{d\log \Psi_s(\beta^i e^v)}{dv}\right]_{v=0} - \sum_i\sum_j \beta^{-i}\left[\frac{d\log(x+e^{vj}\beta^{ij}y)}{dv}\right]_{v=0} \\ & \equiv -c \pmod p. \end{aligned}$$

Examine

$$\sum_i \beta^{-i}\left[\frac{d\log \Psi(\beta^i e^v)}{dv}\right]_{v=0}.$$

For i arbitrary this is

$$\frac{1}{\psi(\beta)} \sum_h (-bh + (a+b) \operatorname{ind}(g^h+1)) \beta^{-1-bh+(a+b)\operatorname{ind}(g^h+1)}$$

and since $\psi(\beta) = -1$, writing k for $(a+b)h$ and $g^h+1 \equiv g^{h'} \pmod{\mathfrak{p}}$, the above transforms, if we write h for h', into

$$-\sum_h (-b \operatorname{ind}(g^h-1) + (a+b)h) \beta^{(a+b)h-1}.$$

If we put k for $(a+b)h$, this may be written

$$b \sum_h \beta^{ah-1} \operatorname{ind}(g^h-1) + \sum_k k\beta^{k-1}$$

and we have $\sum_k k\beta^{k-1} \equiv 0 \pmod{p}$. Using this and putting $a \equiv 1 \pmod{n}$, the original expression becomes

$$\sum_i \beta^{-i} \left[\frac{d \log \psi(\beta^i e^v)}{dv}\right]_{v=0} \equiv b \sum_{i=1}^{n-1} \sum_h \beta^{i(ah-1)} \operatorname{ind}(g^h-1)$$
$$\equiv bn \sum_H \operatorname{ind}(g^H-1) \equiv bn \sum_H \operatorname{ind}(g^H-1),$$

where H ranges over the integers $< q-1$ of the form 1 (mod n). We have

$$z^\nu - 1 \equiv (z-1)(z-g^m) \cdots (z-g^{(\nu-1)m}).$$

Put $z = g^{-j}$ and multiply both sides by $g^{j\nu}$. We have

$$1 - g^{j\nu} \equiv (g^j-1)(g^{m+j}-1) \cdots (g^{(\nu-1)m+j}-1) \pmod{\mathfrak{p}}.$$

Hence

$$\sum_H \operatorname{ind}(g^H-1) \equiv \sum_r \operatorname{ind}(1-g^{\nu r}),$$

where r ranges over the integers $< m$ of the form 1 (mod n). And since $g^\nu \equiv \alpha\beta \pmod{\mathfrak{p}}$ we also have

$$\sum_H \operatorname{ind}(g^H-1) \equiv \sum_r \operatorname{ind}(1-(\alpha\beta)^r).$$

The integers r are congruent, mod p, to $0, 1, \cdots, p-1$, in some order so that

$$\sum_r \operatorname{ind}(1-(\alpha\beta)^r) \equiv \sum_{r_1=0}^{p-1} \operatorname{ind}(1-\alpha^{r_1}\beta) \equiv \operatorname{ind}(1-\beta^p),$$

so that the original sum takes the form $nb \operatorname{ind}(1-\beta^p)$, or

$$b \sum_{i=1}^{n-1} \sum_h \beta^{i(h-1)} \operatorname{ind}(g^h-1) \equiv bn \operatorname{ind}(1-\beta^p). \tag{23}$$

If we use the extended notation

$$\text{ind}\,(\omega, \mathfrak{p}_\mu) \text{ for } \omega = g_\mu^{\text{ind}\,(\omega, \mathfrak{p}_\mu)} \pmod{\mathfrak{p}_\mu}$$

where g_μ is a primitive root of $q_\mu = 1 + \nu_\mu m$, $\mathfrak{p}_\mu$ being one of the prime ideal divisors of $x + \theta y$, we may write (21) and (23) in the form

$$b \sum_{i=1}^{n-1} \sum_h \text{ind}\,(g_\mu^h - 1)\beta^{ih} \equiv bn \,\text{ind}\,(n, \mathfrak{p}_\mu) \pmod{p}$$

$$b \sum_{i=1}^{n-1} \sum_h \beta^{-i} \,\text{ind}\,(g_\mu^h - 1)\beta^{ih} \equiv bn \,\text{ind}\,(1 - \beta^p, \mathfrak{p}_\mu) \pmod{p}.$$

From (18) and (19) we then obtain using $(x + \theta y) = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_s$

$$\sum_i \sum_s \left[\frac{d \log \psi_s(\beta^i e^v)}{dv}\right]_{v=0} \equiv bn \sum_{\mu=1}^{s} \text{ind}\,(n, \mathfrak{p}_\mu) \pmod{p}, \tag{24}$$

and we shall write

$$\equiv bn \,\text{ind}\,(n, x + \theta y) \pmod{p}.$$

Similarly

$$\sum_i \sum_s \beta^{-i} \left[\frac{d \log \psi_s(\beta^i e^v)}{dv}\right]_{v=0} \equiv bn \,\text{ind}\,(1 - \beta^p, x + \theta y) \pmod{p}. \tag{25}$$

We shall also write

$$\left\{\frac{n}{x + \alpha\beta y}\right\}_n = \alpha^{I(n, x + \alpha\beta y)}. \tag{26}$$

Now by definition

$$\omega^{\frac{q-1}{p}} \equiv \left\{\frac{\omega}{\mathfrak{p}}\right\}_n \pmod{\mathfrak{p}}.$$

If we write $\omega^{\frac{q-1}{p}} = \alpha^t$, and $\omega^n \equiv g^{n \,\text{ind}\,\omega} \pmod{\mathfrak{p}}$, then

$$\omega^{\frac{q-1}{pn} \cdot n} \equiv g^{\frac{q-1}{pn} \cdot n \,\text{ind}\,\omega} \pmod{\mathfrak{p}},$$

and

$$\omega^{\frac{q-1}{p}} \equiv \theta^{n \,\text{ind}\,\omega} \pmod{\mathfrak{p}},$$

whence $n \,\text{ind}\,\omega \equiv t \pmod{p}$; so that

$$n \,\text{ind}\,(n, x + \theta y) \equiv I(n, x + \theta y),$$
$$n \,\text{ind}\,(1 - \beta^p, x + \theta y) \equiv I(1 - \beta^p, x + \theta y),$$

and (24) and (25) become

$$\sum_i \sum_s \left[\frac{d \log \psi_s(\beta^i e^v)}{dv}\right]_{v=0} \equiv b\, I(n, x + \theta y), \tag{27}$$

$$(28)\qquad \sum_i \sum_s \beta^{-i}\left[\frac{d\log \psi_s(\beta^i e^v)}{dv}\right]_{v=0} \equiv bI(1-\beta^p, x+\theta y).$$

From (18), (22), (27), and (28) we get by addition, noting that $(n-1)$ was assumed prime to p,

$$(29)\qquad \begin{aligned} &-\frac{b}{n-1} I(n, x+\theta y) + bI(1-\beta^p, x+\theta y) \\ &\equiv \frac{1}{n-1}\sum_i \sum_j \left[\frac{d\log(x+e^{vj}\beta^{ij}y)}{dv}\right]_{v=0} \\ &+ \sum_i \sum_j \beta^{-i}\left[\frac{d\log(x+e^{vj}\beta^{ij}y)}{dv}\right]_{v=0} \pmod p. \end{aligned}$$

We now proceed to the investigation of $\left\{\frac{n}{x+\theta y}\right\}_n$. The ideal $(x+\theta y)$ we shall assume for the present is prime to (p). This ideal is prime to (n) since $n \not\equiv 1 \pmod p$.

$$\left\{\frac{n}{x+\alpha\beta^k y}\right\}_n = \left\{\frac{n}{x+\alpha\beta y}\right\}_n,$$

$$\left\{\frac{n}{x+\theta y}\right\}_n^{n-1}\left\{\frac{n}{x+\alpha y}\right\}_n = \left\{\frac{n}{x^n+\alpha^n y^n}\right\}_n.$$

By the method whose explanation follows (5) of this article we have

$$\left\{\frac{n}{x+\alpha\beta y}\right\}_n = \left\{\frac{n}{\frac{x^n+\alpha^n y^n}{x+\alpha y}}\right\} = \left\{\frac{n}{x^n+\alpha^n y^n}\right\}\left\{\frac{n}{x+\alpha y}\right\}^{-1},$$

where the symbol $\left\{\frac{\omega}{\varepsilon}\right\}$ represents a power character in $\Omega(\alpha)$. By Eisenstein's law of reciprocity, we have

$$\left\{\frac{n}{x^n+\alpha^n y^n}\right\} = \left\{\frac{\alpha}{n}\right\}^{\frac{-ny^n}{x^n+y^n}}\left\{\frac{x^n+\alpha^n y^n}{n}\right\}$$

since $x^n \equiv x$, $y^n \equiv y \pmod n$, the right hand member becomes

$$\left\{\frac{\alpha}{n}\right\}^{\frac{-ny^n}{x^n+y^n}}\left\{\frac{x+\alpha^n y}{n}\right\} = \left\{\frac{\alpha}{n}\right\}^{\frac{-ny^n}{x^n+y^n}+\frac{ny}{x+y}}\left\{\frac{n}{x+\alpha y}\right\}^n,$$

so that

$$\left\{\frac{n}{x+\alpha\beta y}\right\} = \left\{\frac{n}{x+\alpha y}\right\}^{n-1}\left\{\frac{\alpha}{n}\right\}^{\frac{nxy(x^{n-1}-y^{n-1})}{(x+y)(x^n+y^n)}},$$

or

$$I(n, x+\theta y) \equiv (n-1)\, I(n, x+\alpha y) + n \cdot \frac{n^{p-1}-1}{p} \cdot \frac{xy(x^{n-1}-y^{n-1})}{(x+y)(x^n+y^n)} \pmod{p}. \tag{30}$$

Substituting this in (29) we have

$$\begin{aligned} &-\frac{b}{n-1}\left((n-1)\, I(n, x+\alpha y)\right) + n \cdot \frac{n^{p-1}-1}{p} \cdot \frac{xy(x^{n-1}-y^{n-1})}{(x+y)(x^n+y^n)} \\ &\qquad + b\, I(1-\beta^p, x+\theta y) \\ &\equiv \frac{1}{n-1} \sum_i \sum_j \left[\frac{d(\log x + e^{vj}\beta^{ij} y)}{dv}\right]_{v=0} \\ &\quad + \sum_i \sum_j \beta^{-i} \left[\frac{d \log (x + e^{vj}\beta^{ij} y)}{dv}\right]_{v=0} \pmod{p}. \end{aligned} \tag{31}$$

In general substitute $-\frac{y}{x} = t$, (31) becomes after using, from (11a),

$$I(n, x+\alpha y) \equiv I(1-\beta^p, x+\theta y) \equiv 0 \pmod{p}$$

and if $b = n$, $a = 1$

$$\begin{aligned} &-\frac{n^2}{n-1}\, q(n) \frac{-(1-t^{n-1})\, t}{(1-t)(1-t^n)} \\ &\equiv -\frac{1}{n-1} \sum j \sum_{i=1}^{n-1} \frac{\beta^i t}{1-\beta^i t} - \sum_i \sum_j \frac{j\beta^{ij-i} t}{1-\beta^{ji} t} \pmod{p}. \end{aligned} \tag{31a}$$

If we now consider

$$X^m + Y^m + Z^m = 0 \tag{31b}$$

with $m = np$ and X, Y, and Z rational integers prime to each other and to m, then

$$X + \theta^s Y = \varepsilon i^m, \qquad s = 0, 1, \cdots, m-1, \tag{32}$$

where ε is a unit and i an ideal in $k(\theta)$. Using (30) and (32) we may deduce a simple criterion for the solution of (31b) under the conditions mentioned. By employing n-th powers in place of p-th powers in (30) we obtain a similar criterion with n and p transposed. The relation (31a) was derived under the assumption that $(x+\theta y)$ is prime to (p). This ideal will be prime to (p) unless $x^n + y^n \equiv 0 \pmod{p}$. Hence if we multiply (31a) by $1-t^n$, the relation holds for all primes for n such that $n \not\equiv 1 \pmod{p}$, that is

$$\begin{aligned} &\frac{n^2}{n-1}\, q(n) \frac{1-t^{n-1}}{1-t} \\ &\equiv \frac{-\sum j}{n-1} \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} \beta^{i(s+1)} t^s - \sum_j \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} j\, \beta^{(s+1)(ij-i)} t^s. \end{aligned} \tag{32a}$$

Let A_k be the sum

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}+\cdots+\frac{1}{\mu_e},$$

where $\mu_1, \mu_2, \cdots, \mu_e$ are the integers of the form $k \pmod n$ which satisfy $\mu+|n\mu|>pn$. We then have

$$\sum_{s=0}^{n-1}\sum_{v=1}^{n-1}\sum_{i=1}^{n-1} A_v \beta^{siv-i} t^s = \sum_j \sum_{i=1}^{n-1}\sum_{s=1}^{n-1} j \beta^{sij-i} t^s.$$

Carrying through the summation with respect to i, the sum reduces to

$$-\sum_{k=1}^{n-1} A_k \sum_{s=0}^{n-1} t^s + n \sum_{v=1}^{n-1} A_v t^{\left(\frac{1}{v}\right)-1},$$

where $\left(\frac{1}{v}\right)$ is the least positive solution of $vn \equiv 1 \pmod n$. Put

$$S = \sum_l \frac{[l+|nl|-|(n+1)l|]\, l^{sp}}{pn}$$

where l ranges over all the integers less than pn of the form $\epsilon \pmod n$. The coefficient of l^{ps} is zero or 1 according as $l+|nl| \lesseqgtr pn$. Multiplying out this reduces to

$$pnS = \sum_l l^{ps+1} + \sum_l |nl|\, l^{ps} - \sum_l |(n+1)l|\, l^{ps}.$$

We note that $|(n+1)l|$ runs through all the values l, modulo pn. So if we set

$$|(n+1)l| = H, \quad (n+1)l = H+pnV,$$

then

$$l = \frac{H}{n+1}+\frac{pnV}{n+1},$$

and

$$l^{ps} \equiv \left(\frac{H}{n+1}\right)^{ps} \pmod{p^2},$$
$$\equiv (n+1)^{p(p-1)-ps} H^{ps} \pmod{p^2}.$$

Hence

$$\sum_l |(n+1)l|\, l^{ps} \equiv \sum_H (n+1)^{p(p-1)-ps} H^{ps+1} \pmod{p^2}$$

H ranging over all the integers less than and prime to pn of the form $\epsilon \pmod n$. We have also if in $\sum_l |nl|\, l^{ps}$ we put $l = d+pf$ then

$$|nl| = |n(d+pf)| = |nd| = nd$$

and

$$l^{ps} \equiv d^{ps} \pmod{p^2}.$$

Now d ranges over all the integers $1, 2, \cdots, p-1$, so that we have

$$\sum_l |n\,l|\,l^{ps} \equiv n\sum_{d=1}^{p-1} d^{ps+1} \pmod{p^2},$$

so we can then write

$$pnS \equiv (1-(n+1)^{p(p-1)-ps})\sum H^{ps+1} + n\sum_{d=1}^{p-1} d^{ps+1} \pmod{p^2}.$$

To reduce the expression $\sum H^{ps+1}$ consider the symbolic relation

$$(x+1+b)^n - (x+b)^n = nx^{n-1} \quad (n>0)$$

where $(b+1)^n = b_n$, $b_s = b^s$, $n>1$.

Put $\frac{x}{q}$ for x and $k = n$, we have

$$\left(\frac{x}{q}+1+b\right)^k - \left(\frac{x}{q}+b\right)^k = k\left(\frac{x}{q}\right)^{k-1}$$

or

$$(x+q+qb)^k - (x+qb)^k = kqx^{k-1}$$

whence

$$\begin{aligned}
(\varepsilon+pn+nb)^{ps+2} - (\varepsilon+(p-1)n+nb)^{ps+2} &= n(ps+2)(\varepsilon+(p-1)n)^{ps+1},\\
(\varepsilon+(p-1)n+nb)^{ps+2} - (\varepsilon+(p-2)n+nb)^{ps+2} &= n(ps+2)(\varepsilon+(p-2)n)^{ps+1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(\varepsilon+n+nb)^{ps+2} - (\varepsilon+nb)^{ps+2} &= n(ps+2)\varepsilon^{ps+1}.
\end{aligned}$$

Addition gives

$$\frac{(\varepsilon+pn+nb)^{ps+2} - (\varepsilon+nb)^{ps+2}}{n(ps+2)} = \sum_H H^{ps+1}.$$

Put $s = p-2$; then

$$\begin{aligned}
(\varepsilon+pn+nb)^{ps+2} &= (\varepsilon+nb+pn)^{(p-1)^2+1}\\
&= (\varepsilon+nb)^{(p-1)^2+1} + np\,((p-1)^2+1)\,(\varepsilon+nb)^{(p-1)^2}\\
&\quad + \frac{n^2p^2((p-1)^2+1)(p-1)^2}{2}(\varepsilon+nb)^{p(p-2)}\\
&\quad + \cdots.
\end{aligned}$$

Now on the right hand side all terms after the third can be expressed as fractions whose numerators are divisible by p^2 and denominators prime to p, by the von Staudt-Clausen theorem. Also

$$\begin{aligned}
(\varepsilon+nb)^{p(p-2)} &= (\varepsilon^{p-2} + (p-2)\varepsilon^{p-3}nb + \cdots + (nb)^{p-2})^p\\
&= \varepsilon^{(p-2)p} + (p-2)\varepsilon^{(p-3)p}(nb)^p + \cdots + (nb)^{p(p-2)} + pM,
\end{aligned}$$

where M is a polynomial in b. If any term in M involves $b^{s(p-1)}$ then on substitution of $b_{s(p-1)}$ the fraction pb has a denominator prime to p. Hence $(\varepsilon+nb)^{p(p-2)}$ can be expressed as a fraction whose denominator is prime to p. We can then write

$$\frac{(\varepsilon+pn+nb)^{(p-1)^2+1}-(\varepsilon+nb)^{(p-1)^2+1}}{n((p-1)^2+1)} \equiv \sum H^{ps+1} \tag{33}$$

$$\equiv p(\varepsilon+nb)^{(p-1)^2} \pmod{p^2}.$$

We now have

$$(nb+\varepsilon)^{(p-1)^2} \equiv n^{(p-1)^2}\left(b+\left[\frac{\varepsilon}{n}\right]\right)^{(p-1)^2} \pmod{p},$$

where $\left[\frac{\varepsilon}{n}\right]$ is the least solution of $xn \equiv \varepsilon \pmod{p}$. But

$$\frac{\left(b+\left[\frac{\varepsilon}{n}\right]\right)^{(p-1)^2}-b^{(p-1)^2}}{(p-1)^2}$$

$$\equiv 1^{(p-1)^2-1}+2^{(p-1)^2-1}+\ldots+\left(\left[\frac{\varepsilon}{n}\right]-1\right)^{(p-1)^2-1} \pmod{p},$$

or

$$\left(b+\left[\frac{\varepsilon}{n}\right]\right)^{(p-1)^2} \equiv b^{(p-1)^2}+\sum_{s=1}^{\left[\frac{\varepsilon}{n}\right]-1}\frac{1}{s} \pmod{p},$$

so that (33) gives

$$\sum H^{ps+1} \equiv n^{(p-1)^2} p b_{(p-1)^2} + p\sum_{s=1}^{\left[\frac{\varepsilon}{n}\right]-1}\frac{1}{s} \pmod{p^2}.$$

From a previous relation we have since

$$\sum_{d=1}^{p-1} d^{(p-1)^2} \equiv p b_{(p-1)^2} \pmod{p^2},$$

$$pnS \equiv (1-(n+1)^p)\left(p b_{(p-1)^2} n^{(p-1)^2} + p\sum_{s=1}^{\left[\frac{\varepsilon}{n}\right]-1}\frac{1}{s}\right) + np b_{(p-1)^2} \pmod{p^2},$$

$$\equiv p b_{(p-1)^2}(n^{(p-1)^2}-n^{(p-1)^2}(n+1)^p+n) - np\sum_{s=1}^{\left[\frac{\varepsilon}{n}\right]-1}\frac{1}{s} \pmod{p^2},$$

or

$$Sn \equiv \frac{n^{(p-1)^2}(n+1)^p-n^{(p-1)^2}-n}{p} - n\sum_{s=1}^{\left[\frac{\varepsilon}{n}\right]-1}\frac{1}{s} \pmod{p}.$$

Similarly if $S_1 = \sum \nu^{-1}$ where ν ranges over all the integers prime to pn and less than pn and satisfying $\nu + |n\nu| > |(n+1)\nu|$ we find similarly

$$S_1 \equiv \frac{n(n+1)^p + n^{(p-1)^2} - n^2 - n^{(p-1)^2}(n+1)^p}{pn} \pmod p,$$

$$\equiv \frac{(n^2-1)\,q(n+1) + n\,q(n)}{n} \pmod p;$$

$$\frac{n(n+1)^p + n^{(p-1)^2} - n^2 - n^{(p-1)^2}(n+1)^p}{pn(1-n)}$$

$$- \frac{n(n+1)^p + n^{(p-1)^2} - n^2 - n^{(p-1)^2}(n+1)^p}{pn}$$

$$+ \frac{n^{(p-1)^2+1}(n+1)^p - n^{(p-1)^2+1} - n^2}{pn}$$

$$\equiv \frac{-n^2(n^{(p-1)^2}-1)}{p(1-n)} \pmod p.$$

Also

$$n^{(p-1)^2} \equiv 1 + (p-1)\,pq(n) \pmod{p^2},$$

or

$$\frac{n^{(p-1)^2}-1}{p} \equiv -q(n) \pmod p.$$

The relation (32a) can then be written

$$\frac{n^2 q(n)}{n-1}\,\frac{1-t^{n-1}}{1-t}$$

$$(33a) \equiv \left(\frac{t^{n-1}-t}{t-1} - (n-1)\,t^{n-1}\right)\frac{\sum j}{n-1} + \left(\frac{t^{n-1}-1}{t-1} + t^{n-1}\right)\sum j$$

$$- n\sum_{v=1}^{n-1} A_v\, t^{\left(\frac{1}{v}\right)-1}.$$

These congruences for $n = 2$ and 3 yield new derivations of the criteria $2^{p-1} \equiv 3^{p-1} \equiv 1 \pmod{p^2}$ for the solution of (1) in case I.

Frobenius* obtained (33a) for any n directly from (2) after many transformations. For comparison use the reduced form of A_v.

3. In (3) take power characters in the field $\Omega(\alpha\beta)$ with respect to $(x+\beta y)$. This gives

$$\left\{\frac{\beta+\alpha}{x+\beta y}\right\}^{\sum[1:r]} = \left\{\frac{\alpha}{x+\beta y}\right\}^{\frac{-kyq(k)}{x+y}}$$

and since

$$\sum[1:r] \equiv -ky(k) \pmod p,$$

* Berlin. Sitzungsberichte, 1914, p. 675–6.

we have

$$\left\{\frac{-\beta+\alpha}{x+\beta y}\right\} = \left\{\frac{\alpha}{x+\beta y}\right\}^{\frac{y}{x+y}},$$

and we obtain easily

$$\left\{\frac{1-\alpha\beta}{x+\beta y}\right\} = \left\{\frac{\alpha}{x+\beta y}\right\}^{\frac{x}{x+y}}.$$

We also have by other considerations*

$$\sum_{\gamma=1}^{k}\sum_{s=1}^{n-1}\beta^{-s}\,\frac{d_0^{p-1}\log \psi_\gamma(\beta^s e^v)}{dv^{p-1}} \tag{34}$$
$$\equiv -I(\alpha, x+\beta y)+I(\alpha\beta-1, x+\beta y) \pmod p,$$

where $x+\beta y = \mathfrak{q}_1\,\mathfrak{q}_2\cdots\mathfrak{q}_k$, the $\mathfrak{q}$'s being prime ideals in $\Omega(\theta)$. The ψ function which occurs here is defined as before with reference to $\mathfrak{q}$ with the corresponding norms q'. Hence

$$\prod_l (x+\beta^{[1:l]}y) = \theta^t \prod_{\gamma=1}^{k}\psi_\gamma(\theta).$$

and if $p^2-1\equiv 0 \pmod n$, then

$$\prod_l (x+\beta^{[1:l]}y)^{p^i-1} = \alpha^{-t}\prod_{\gamma=1}^{k}\psi_\gamma(\theta)^{p^i-1}.$$

This gives

$$\frac{\prod_l (x+\beta^{[1:l]}y)^{p^i-1}}{e^{-tv}\prod_{\gamma=1}^{k}\psi_\gamma(\beta e^v)^{p^i-1}} = 1+VW,$$

where $V=(e^{vp}-1)/(e^v-1)$, and W is a rational function of e^v with coefficients in $\Omega(\beta)$. Setting $v=0$, we obtain

$$\frac{\prod_l (x+\beta^{[1:l]}y)^{p^i-1}-1}{p} = W(0).$$

Also

$$-\sum_l \frac{d_0^{p-1}\log(x+\beta^{[1:l]}y)}{dv^{p-1}} = 0$$
$$\equiv -\sum_\gamma \frac{d^{p-1}\log\psi_\gamma(e^v\beta)}{dv^{p-1}} + \frac{d_0^{p-1}(VW)}{1+V_0W(0)} \pmod p,$$

* Amer. Journ. Math., l. c., p. 147.

or

$$-\sum_{\gamma}\left[\frac{d_0^{p-1}\log \Psi_\gamma(e^v\beta)}{dv^{p-1}}\right]_{v=0} \equiv W(0) \pmod p,$$

and

$$(35)\quad \sum_{s=1}^{n-1}\sum_{\gamma=1}^{k}\frac{\beta^{-s}\, d_0^{p-1}\log \Psi_\gamma(e^v\beta^s)}{dv^{p-1}} \equiv \sum_s \beta^{-s}\frac{\prod_l (x+\beta^{g[1:l]}y)^{p^{l-1}}-1}{-p} \pmod p.$$

To reduce the right hand member of this relation we note that if we set $t = -y/x$ that this member reduces, since $x^{p^l-1} \equiv 1 \pmod{p^2}$, to the negative of

$$\frac{\sum_{s=1}^{n-1}\varrho^{-s}\prod_h (1-\varrho^{sh}t)^{p^l-1}+1}{p} \pmod p,$$

if we write ϱ for β and $hl \equiv 1 \pmod p$. We have, since $t^p \equiv t \pmod{p^2}$,

$$(1-\varrho^h t)^p \equiv 1-\varrho^{hp}t+pf(\varrho^h t) \pmod p,$$

where

$$f(r) = r+2^{p-2}r^2+\cdots+(p-1)^{p-2}r^{p-1}.$$

Hence

$$(1-\varrho^h t)^{p^2} \equiv (1-\varrho^{hp}t)^p \pmod{p^2},$$
$$\equiv 1-\varrho^{hp^2}t+pf(\varrho^{hp}t) \pmod{p^2},$$

and ultimately

$$(1-\varrho^h t)^{p^l} \equiv 1-t\varrho^h+pf(\varrho^{hp^{l-1}}t) \pmod{p^2},$$

or

$$(1-\varrho^h t)^{p^l-1} \equiv 1+p\frac{f(\varrho^{hp^{l-1}}t)}{1-\varrho^h t} \pmod{p^2}.$$

Hence

$$\sum_s \frac{\varrho^{-s}\prod_h (1-\varrho^{sh}t)^{p^l-1}+1}{p} \equiv \sum_h\sum_s \varrho^{-s}\frac{f(t\varrho^{hp^{l-1}s})}{1-\varrho^{hs}t}.$$

Now changing ϱ into ϱ^p, we have, modulo p,

$$(36)\qquad \sum_s \frac{\varrho^{-s}f\left(t\varrho^{\frac{h}{p}s}\right)}{1-\varrho^{hs}t} \equiv \sum_s \frac{\varrho^{sp}f(t\varrho^{-sh})}{1-\varrho^{-hsp}t}.$$

Frobenius* gave the formula

$$x^n F(nH')+x^{n-1}F(nH'+1)+\cdots+xF(nH'+n-1) = \frac{x(1-x^n)}{1-x}F(H),$$

* Berlin. Sitzungsberichte, 1910, p. 847.

where $(H')^n = H^n(x^m)$, F is a polynomial in H', and H is defined as in Frobenius, l. c., p. 826.

Put $\varrho^{-1}x$ for x, $f = (H')^{p-2}$ and multiply by ϱ^{-k}, we have

$$\varrho^{-k}x^n(nH')^{p-2} + \varrho^{-k+1}x^{n-1}(nH'+1)^{p-2} + \cdots + \varrho^{-k}x(nH'+n-1)^{p-2}$$
$$\equiv \varrho^{-k}\frac{1-x^n}{1-\varrho^{-1}x}H^{p-2}\varrho^{-1}x \pmod{p}.$$

Letting ϱ range over all the distinct roots of $x^n = 1$, we have

$$nx^{n-k}(nH'+k)^{p-2} = -\sum_{\varrho}(1-x^n)\varrho^{-k-1}\frac{xR_{p-2}(\varrho^{-1}x)}{(1-\varrho^{-1}x)^{p-1}}$$
$$= -\sum\frac{(1-x^n)\varrho^{-k-1}x(1-\varrho^{-1}x)R_{p-2}(\varrho^{-1}x)}{1-\varrho^{-p}x^p}$$

where R is defined as in Frobenius, l. c., p. 826, but since (Frobenius, l. c., p. 843)

$$(1-\varrho^{-1}x)R_{p-2}(\varrho^{-1}x) \equiv \frac{f(\varrho^{-1}x)}{x\varrho^{-1}} \pmod{p},$$

then

$$nx^{n-k}(nH'+k)^{p-2} \equiv \sum_{\varrho}\frac{(1-x^n)\varrho^{-k}f(\varrho^{-1}x)}{1-\varrho^{-p}x^p} \pmod{p}.$$

Which may be applied to (36) if x is an integer t.

CERTAIN EXPANSIONS INVOLVING DOUBLY INFINITE SERIES.*

By Morgan Ward.

1. **Introduction.** The present paper extends the results of a previous article† on the same subject. If

$$P(x, y) = \sum_{r,s=0}^{\infty} P_{rs}\, x^r\, y^s, \qquad Q(x, y) = \sum_{r,s=0}^{\infty} Q_{rs}\, x^r\, y^s$$

are power series in x and y, I showed in R how to express the coefficients of P/Q, $\exp Q$, $\log Q$ as simple determinants in the coefficients P_{rs}, Q_{rs} and here I shall apply these results to the expansion of

$$[P_1(x, y)]^{m_1} \cdot [P_2(x, y)]^{m_2} \cdots [P_t(x, y)]^{m_t}$$

where $m_1, m_2, \cdots, m_t$ are any real numbers.

The analogous expansion problem for singly infinite series has been solved by Mangeot‡; David§ and Segar‖ had previously dealt with the simple case of a single power series raised to an arbitrary power.

2. **Notation.** In this paper, if P, Q, P_i, etc. are power series in x, y the corresponding coefficients are denoted by P_{rs}, Q_{rs}, $P_{i,rs}$, etc. The Einstein summation convention is used, so that for instance, we shall write $Q = Q_{ij}\, x^i\, y^j$ instead of $\sum_{i,j=0}^{\infty} Q_{ij}\, x^i\, y^j$. The terms of a series Q will always be taken to be arranged in the usual numerical order; viz.

$$(1) \qquad Q_{00} + Q_{10}\, x + Q_{01}\, y + Q_{20}\, x^2 + Q_{11}\, xy + Q_{02}\, y^2 + \cdots.$$

When the series is arranged in this manner, the term Q_{ij} occurs in the $(\frac{1}{2}(i+j)(i+j+1)+(j+1))$ place.¶ We shall call the number $\frac{1}{2}(i+j)(i+j+1)+(j+1)$ the *rank* of the coefficient Q_{ij} and denote it by q_{ij}.

3. **Evaluation of coefficients.** Suppose then that

$$P_\tau = P_\tau(x, y) = P_{\tau,rs}\, x^r\, y^s \qquad (\tau = 1, 2, \cdots, t)$$

$$(2) \qquad W(x, y) = W_{ij}\, x^i\, y^j = P_1^{m_1} \cdot P_2^{m_2} \cdot \cdots P_t^{m_t}.$$

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* Received October 4, 1927; in revised form, February 4, 1929.

† A Generalization of Recurrents. Bull. Amer. Math. Soc., vol. 33 (1927), pp. 477–492. I shall refer to this paper as R. A correction is given p. 580 in the second footnote (†).

‡ Annales de l'Ec. Norm., vol. 14 (1897), pp. 247–250.

§ Journ. de Math., vol. 8, ser. 3 (1882), pp. 61–72.

‖ Messenger of Math., vol. 21, ser. 2 (1892), pp. 177–188. See vol. 4, chapter 8 of Muir's *History* for other references.

¶ R, section 2.

Our object is *to show how to express W_{ij} in terms of the coefficients of the P series.* We may assume first of all that

$$P_{\tau,00} \neq 0 \qquad (\tau = 1, 2, \cdots, t).$$

Let

$$[P_\tau(x, y)]^{m_\tau} = \exp Q_\tau(x, y) \qquad (\tau = 1, 2, \cdots, t) \tag{3}$$

so that

$$Q_\tau(x, y) = Q_{\tau,rs}\, x^r y^s = m_\tau \log P_\tau(x, y). \tag{4}$$

On substituting from (3) into (2) we obtain

$$W(x, y) = \exp Q(x, y) \tag{5}$$

where

$$\begin{aligned} Q(x, y) &= Q_1(x, y) + Q_2(x, y) + \cdots + Q_t(x, y), \\ Q_{rs} &= Q_{1,rs} + Q_{2,rs} + \cdots + Q_{t,rs}. \end{aligned} \tag{6}$$

Our next step is to express W_{ij} as a determinant* in the Q's; but we shall explain the formation of this determinant more fully than in R.

4. **Fundamental determinants.** Consider first of all the determinants

$$\Delta_1 = 1, \quad \Delta_2 = \begin{vmatrix} Q_{10} & -1 \\ Q_{01} & \cdot \end{vmatrix}_2, \quad \Delta_3 = \begin{vmatrix} Q_{10} & -1 & \cdot & \cdot & \cdot \\ Q_{01} & \cdot & -1 & \cdot & \cdot \\ 2Q_{20} & Q_{10} & \cdot & -2 & \cdot \\ 2Q_{11} & Q_{01} & Q_{10} & \cdot & -2 \\ 2Q_{02} & \cdot & Q_{01} & \cdot & \cdot \end{vmatrix}_5,$$

$$\Delta_4 = \begin{vmatrix} Q_{10} & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Q_{01} & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2Q_{20} & Q_{10} & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2Q_{11} & Q_{01} & Q_{10} & \cdot & -2 & \cdot & \cdot & \cdot & \cdot \\ 2Q_{02} & \cdot & Q_{01} & \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\ 3Q_{30} & 2Q_{20} & \cdot & Q_{10} & \cdot & \cdot & -3 & \cdot & \cdot \\ 3Q_{21} & 2Q_{11} & 2Q_{20} & Q_{01} & Q_{10} & \cdot & \cdot & -3 & \cdot \\ 3Q_{12} & 2Q_{02} & 2Q_{11} & \cdot & Q_{01} & Q_{10} & \cdot & \cdot & -3 \\ 3Q_{03} & \cdot & 2Q_{02} & \cdot & \cdot & Q_{01} & \cdot & \cdot & \cdot \end{vmatrix}_9$$

and so on.

The dots indicate zeros and the subscripts $2, 5, 9, \cdots$ the orders of $\Delta_2, \Delta_3, \Delta_4, \cdots$. In general, Δ_n is a determinant of order $n(n+1)/2-1$ whose mode of formation is fairly apparent from the examples just given.†

We next introduce a set of N numerical functions‡ somewhat analogous to Kronecker's δ_{ij} symbol:

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* R, section 10, p. 489.

† R, pp. 490–91; 481–82.

‡ R, p. 480; p. 489.

$$(q_{t-ss};\ i+s-t, j-s) \quad \begin{pmatrix} t &= 0, 1, \cdots, i+j-1 \\ s &= 0, 1, \cdots, t \\ N &= \tfrac{1}{2}(i+j)(i+j+1) = q_{i+j0}-1 \end{pmatrix}.$$

The relations defining these functions are as follows:

$$(q_{t-ss};\ i+s-t, j-s) = 0$$

if either $i+s-t<0$ or $j-s<0$ and

$$(q_{t-ss};\ i+s-t, j-s) = (i+j-t)\, Q_{i-t+s, j-s}$$

if neither $i+s-t<0$ nor $j-s<0$.

The general coefficient W_{ij} in (5) is then given by*

$$(7) \qquad (i+j)\, e^{-q_{00}} 1^1 . 2^2 . 3^4 \cdots \cdots (i+j-1)^{i+j} W_{ij} = \begin{vmatrix} & & 0 \\ & & 0 \\ & & \cdot \\ & \Delta_{i+j} & \cdot \\ & & \cdot \\ \cdots\cdots & & -(i+j-1) \\ \cdots\cdots & (q_{t-ss};\ i+s-t, j-s) & \cdots\cdots \end{vmatrix}.$$

The determinant on the right side of (7) consists of the determinant Δ_{i+j} bordered by

$$\begin{array}{llll} (q_{00},\ i, j), & (q_{10};\ i-1, j), & (q_{01};\ i, j-1), & (q_{20};\ i-2, j), \\ (q_{11};\ i-1, j-1), & (q_{02};\ i, j-2), & \cdots, & \cdots, \\ \cdots, & \cdots, & \cdots, & \cdots, \\ \cdots, & \cdots, & (q_{t-s,s};\ i+s-t, j-s), & \cdots, \\ \cdots, & \cdots, & \cdots, & \cdots, \\ \cdots, & \cdots, & \cdots, & (q_{0\,i+j-1};\ i, 1-i) \end{array}$$

in the last row, and by

$$0, \quad 0, \quad 0, \quad \cdots, \quad -(i+j-1), \quad (q_{0\,i+j-1};\ i, 1-i)$$

in the last column.†

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* R, p. 491.

† In R, p. 491, the factor $(i+j)\, e^{-q_{00}}$ was omitted from the left-hand side of (7), and the last row of the determinant on the right-hand side of (7) was given incorrectly as $(q_{00};\ i, j)$, $(q_{10};\ i-1, j+1)$, $(q_{01};\ i-2, j+2)$, $\cdots$ etc. instead of the correct expressions above. Similar corrections should be made on p. 483 (interpreting the numerical functions appearing there as in (9) p. 480), and in the procedure sketched in section 9 R for $\log Q(x, y)$.

5. Illustrative example. To show the ease with which the general formula (7) may be applied, we shall evaluate the coefficient W_{21}. Here $i=2, j=1, i+j=3, N=\frac{3\cdot 4}{2}=6$, so that we must border the determinant Δ_3 with the row

$(q_{00}; 2, 1)$, $(q_{10}; 1, 1)$, $(q_{01}; 2, 0)$, $(q_{20}; 0, 1)$, $(q_{11}; 1, 0)$, $(q_{02}; 2, -1)$

and with the column

$$0,\ 0,\ 0,\ 0,\ -2,\ (q_{02}; 2, -1).$$

That is, with the row

$$3Q_{21},\ 2Q_{11},\ 2Q_{20},\ Q_{01},\ Q_{10},\ 0$$

and the column

$$0,\ 0,\ 0,\ 0,\ -2,\ 0.$$

Thus*

$$3e^{-Q_{00}}\,1^2\cdot 2^3\,W_{21}=\begin{vmatrix} Q_{10} & -1 & 0 & 0 & 0 & 0 \\ Q_{01} & 0 & -1 & 0 & 0 & 0 \\ 2Q_{20} & Q_{10} & 0 & -2 & 0 & 0 \\ 2Q_{11} & Q_{01} & Q_{10} & 0 & -2 & 0 \\ 2Q_{02} & 0 & Q_{01} & 0 & 0 & -2 \\ 3Q_{21} & 2Q_{11} & 2Q_{20} & Q_{01} & Q_{10} & 0 \end{vmatrix}_6.$$

6. Final evaluation of coefficients. We have thus expressed the W_{ij} as determinants in the Q_{rs}. But we can express the Q_{rs} as sums of determinants in the $P_{\tau,rs}$; for since by (4) $Q_\tau(x, y)=m_\tau \log P_\tau(x, y)$ we can apply the method developed in R section 9 to expand a logarithm. We need only to substitute in R (17) p. 483 $m_\tau \log P_{\tau,00}=Q_{\tau,00}$ for Z_{00}; $\frac{i+j}{m_\tau}Q_{\tau,ij}$ for $Z_{ij}\,(i+j>0)$; $i+j\,P_{\tau,ij}$ for P_{ij}; and $P_{\tau,ij}$ for Q_{ij}. Since P_{00} vanishes, we obtain thus†

$$(8)\quad P_{\tau,00}^{N-1}\frac{i+j}{m_\tau}Q_{\tau,ij}=\begin{vmatrix} P_{\tau,10} & P_{\tau,00} & 0 & \cdot & 0 & P_{\tau,10} \\ P_{\tau,01} & 0 & P_{\tau,00} & \cdot & 0 & P_{\tau,01} \\ P_{\tau,20} & P_{\tau,10} & 0 & \cdot & 0 & 2P_{\tau,20} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{\tau,0\,i+j-1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{\tau ij}, & (2; i-1, j), & \cdots\cdots\cdots, & & & (i+j)P_{\tau,ij} \end{vmatrix},$$

* This result may be readily checked from the equations at the foot of p. 490 in R.

† We have applied the correction mentioned in the footnote to section 4 to the last row of the determinant.

where $N = \frac{1}{2}(i+j)(i+j+1)+1$ and $1 \leqq \tau \leqq t$. By combining (8), (6) and (7) we can finally express the coefficients of $W(x, y)$ in terms of the coefficients of $P_1(x, y)$, $P_2(x, y)$, $\cdots$, $P_t(x, y)$.

7. **A special case.** In case $t = 1$ we may proceed more simply. We have

$$W(x, y) = [P(x, y)]^m, \qquad P_{00} \neq 0. \tag{9}$$

Take the logarithms of both sides of (9) and then operate with

$$W(x, y)\, P(x, y) \left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right\}.$$

We obtain

$$P_{rs}\, x^r\, y^s\, (u+v)\, W_{uv}\, x^u\, y^v = m(r+s)\, P_{rs}\, x^r\, y^s\, W_{uv}\, x^u\, y^v$$

or transposing and equating the coefficient of $x^r\, y^q$ to zero,

$$\sum_{\tau=0}^{p} \sum_{\sigma=0}^{q} [m(r+q) - (m+1)(\sigma+\tau)]\, P_{r-\tau\, q-\sigma}\, W_{\sigma\tau} = 0 \quad (r, q, = 0, 1, 2, \cdots).$$

These equations may be identified with the equations (8) on p. 479 of R on taking in R

$$\begin{aligned} P_{uv} &= 0 \ (u \neq 0,\ v \neq 0), \qquad P_{00} = P_{00}^n, \\ Q_{u-\sigma, v-\tau} &= [m(r+q) - (m+1)(\sigma+\tau)]\, P_{p-\tau, q-\sigma} \ (u \neq 0, v \neq 0), \\ Q_{00} &= 1, \\ W_{\sigma\tau} &= Z_{\sigma\tau}. \end{aligned}$$

It follows that they may be solved in the same manner by introducing a properly defined numerical function.

8. **Conclusion.** The method I have applied here for obtaining simple determinants for the expansions of various elementary functions of doubly infinite series by the introduction of suitably defined numerical functions can theoretically be extended to m-tuply infinite series.* The numerical functions which must be introduced are unfortunately of such complexity* as to reduce the results to a mere jumble of symbols. A successful application of the method to the problem of reverting two doubly infinite series would be of more interest and of some practical importance. It would seem, however, to be rather difficult.†

* R, section 8, p. 487.

† I have treated the problem for singly infinite series in a note which is to appear shortly in the Rendiconti del Circolo di Palermo.

California Institute,
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NORMAL DIVISION ALGEBRAS IN $4p^2$ UNITS, p AN ODD PRIME.*

By A. Adrian Albert.†

1. **Introduction.** We shall consider normal associative division algebras in $4p^2$ units, over any non-modular field F, satisfying special assumptions. We shall utilize the definitions and theorems of a previous paper,‡ referring to the theorems by numbers 1 to 29 and shall number the theorems of this paper beginning with 30.

In the paper above referred to we have defined "known algebras" and "the type of an irreducible equation, of an element of a normal division algebra, and of a normal division algebra".

L. E. Dickson§ has found the structure of all normal division algebras in n^2 units over F, of type R_n, that is, normal division algebras containing an element x whose minimum equation is of degree n and has the property that all of its scalar roots are rational functions, with coefficients in F, of one of them. He has also determined and constructed the algebras when the minimum equation of this x has a solvable group. These algebras are easily seen to be known algebras under our definition and the resulting criterion shall be utilized here.

2. **Algebras in $4p^2$ units of type R_p.** These are algebras containing an element x whose minimum equation $\varphi(\omega) = 0$ has degree $2p$ and the property that there exist $p - 1$ polynomials in x which are distinct and distinct from x satisfying $\varphi(\omega) = 0$. We shall use the following known Lemma.

Lemma 1. *Let x be of grade n and type R_n. Then the minimum equation of x has a regular group.*

For $n = 2p$, p an odd prime, the order of the group of $\varphi(\omega) = 0$, the minimum equation of x of grade $2p$ and type R_{2p}, is $2p$. But any group of order $2p$ is solvable for p a prime greater than 2.||

We consequently have:

Theorem 30. *Every normal division algebra in $4p^2$ units of type R_{2p} is a known algebra.*

By the use of this theorem we may prove the next important result.

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† National Research Fellow.

‡ Annals of Math., vol. 30 (1929), pp. 322-338.

§ Trans. Amer. Math. Soc., vol. 28, no. 2, pp. 207-234.

|| Netto, Theory of Substitutions, p. 289.

THEOREM 31. *Every normal division algebra in $4p^2$ units, p an odd prime, of type R_3 is known.*

Proof. Let the Algebra A contain x of type R_3 and grade $2p$. Then x is of type S_k where k is greater than or equal to 3 and is a divisor of $2p$, so that $k = p$ or $k = 2p$. If $k = 2p$ then x is of type R_{2p} and A is known by Theorem 30. Let $k = p$. Then by Theorem 24, A has a subalgebra Σ which is a normal division algebra in p^2 units over a quadratic field. Σ is evidently a cyclic algebra in this theorem for the case p an odd prime, and hence is a known algebra. But all quadratic fields are cyclic and the postulates of Theorem 27 are satisfied and A is known.

Corollary. Every normal division algebra in 36 units of type R_3 is a known algebra.

3. Elements whose minimum equations have even powers only.

THEOREM 32. *Let A be a normal division algebra. Let x in A have the property that $-x$ satisfies the minimum equation of x. Then the minimum equation of x has even powers only. Let x^2 be of grade p with $\varphi(\omega) = 0$ as its minimum equation. Then x has grade $2p$ and its minimum equation is $\varphi(\omega) = 0$.*

Proof. Let the minimum equation of x be $\Gamma(\omega) = 0$ of degree n. Since $-x$ satisfies $\Gamma(\omega) = 0$, $\Gamma(-\omega) = 0$ is satisfied by x. Hence x is a root of $\psi(\omega) \equiv \Gamma(\omega) - \Gamma(-\omega) = 0$. This equation has degree less than or equal to n and has constant term zero. Therefore $\psi(\omega) \equiv \omega\, Q(\omega)$ where $Q(\omega)$ has leading coefficient unity and degree less than n. But $\psi(x) = x \cdot Q(x) = 0$. Since A is a division algebra and $x \neq 0$, $Q(x) = 0$ and x is a root of $Q(\omega)$ of degree less than n. But the minimum equation of x is of degree n. Thus $Q(\omega) \equiv 0$, and $\Gamma(\omega) \equiv \Gamma(-\omega)$ and has even powers only. Since $\Gamma(\omega) \equiv \omega^{2m} + \alpha_1 \omega^{2(m-1)} + \cdots + \alpha_m$, x^2 is of grade $r \leqq m$. But, since x^2 is a root of $\varphi(\omega) = 0$, x is a root of $\varphi(\omega^2) = 0$ and x is of grade $\leqq 2r$, $2m \leqq 2r$ and $m \leqq r$. Hence $n = m$ and x is of grade $2r$ and satisfies its minimum equation, $\varphi(\omega^2) = 0$.

THEOREM 33. *Let ξ be an ordinary scalar root of an irreducible equation $\varphi(\omega) = 0$ of degree $2p$, p an odd prime, and type S_2. Then $F(\xi)$ contains an element η satisfying*

$$\psi(\omega) \equiv \omega^{2p} + \alpha_1 \omega^{2(p-1)} + \cdots + \alpha_p = 0$$

with coefficients in F, irreducible in F, and with even powers only.

Proof. Since $\varphi(\omega) = 0$ is of type S_2 there exists a polynomial $\theta(\xi) \neq \xi$ satisfying $\varphi(\omega) = 0$ and no other element of $F(\xi)$ distinct from $\theta(\xi)$ and ξ is a root of this equation. But $\theta^2(\xi)$ is a root of $\varphi(\omega) = 0$ since $\varphi[\theta(\omega)] = 0$ has a root in common with the irreducible $\varphi(\omega) = 0$ and hence all of its roots. Hence $\theta^2(\xi) = \xi$ or $\theta^2(\xi) = \theta(\xi)$. If $\theta^2(\xi) = \theta(\xi)$,

since the group of $\varphi(\omega) = 0$ is transitive and contains a substitution replacing $\theta(\xi)$ by ξ, we have $\theta(\xi) = \xi$ contrary to hypothesis. Hence $\theta^2(\xi) = \xi$.

If $\theta(\xi)+\xi = 0$ then $\theta(\xi) = -\xi$ and $\varphi(\omega) = 0$ is satisfied by the negative of every root and has even powers only. In this case take $\eta = \xi$.

If $\theta(\xi)+\xi \neq 0$ then it is the root of an irreducible equation of degree a divisor of $2p$ since $F(\theta(\xi)+\xi)$ is contained in $F(\xi)$ and hence has its order a divisor of $2p$. Hence the degree of this equation is $2p$, 1,2 or p. If the degree of the equation is $2p$ then the order of $F(\xi+\theta(\xi))$ is equal to the order of $F(\xi)$, whence the fields are equal and ξ is a polynomial in $\xi+\theta(\xi)$. A substitution replacing ξ by $\theta(\xi)$ replaces $\theta(\xi)$ by $\theta^2(\xi) = \xi$. Let

$$\xi = \lambda(\theta(\xi)+\xi). \tag{1}$$

Any substitution of the Galois group of $\varphi(\omega) = 0$ which replaces ξ by $\theta(\xi)$ leaves the whole equation (1) unaltered. It leaves the right side unaltered in values since it merely interchanges ξ and $\theta(\xi)$. Hence $\theta(\xi) = \xi$ a contradiction.

Hence $\xi+\theta(\xi)$ is the root of an irreducible equation of degree 1,2 or p. If $\xi+\theta(\xi)$ is in F then $\xi+\theta(\xi) = 2\lambda$. Let $\xi+\lambda = \eta$ obviously of rank $2p$. Let η satisfy $\psi(\omega) = 0$. Then $\psi(\eta) = 0$. Hence $\psi(\mu) = 0$ where $\mu = \theta(\xi)+\lambda = -\eta$, and we have the desired element. Next let $\xi+\theta(\xi)$ be not in F.

Let $\sigma = \xi - \theta(\xi)$. The element $\theta(\xi) - \xi = -\sigma$ satisfies the irreducible equation satisfied by σ. Since if this equation is $\psi(\sigma) = 0$ it is unaltered by a substitution replacing ξ by $\theta(\xi)$ and consequently $\theta(\xi)$ by $\theta^2(\xi)$ and hence σ by $-\sigma$. If σ is of grade $2p$ then it is the desired element. If not then, since $F(\sigma)$ is a proper sub-field of $F(\xi)$, its order is 1,2 or p. Since the irreducible equation satisfied by σ is satisfied by $-\sigma$ it has even powers only and is of degree 2. Hence $\sigma^2 = \alpha$ in F. If $\xi+\theta(\xi)$ is of grade 2 then $F(\sigma, \xi+\theta(\xi))$ is of order $\leqq 4$ since every element in it is expressible as a linear combination with coefficients in F of 1, $\xi+\theta(\xi)$, σ, $\sigma\xi+\theta(\xi)\sigma$. Its order is not 4 since its order is a divisor of $2p$ and $p>2$. Hence its order is <4. But it is $\geqq 2$ and a divisor of $2p$ and hence is 2. Hence it is equal to $F(\xi+\theta(\xi))$. Hence σ is a polynomial in $\xi+\theta(\xi)$ and is unaltered by replacing ξ by $\theta(\xi)$ contrary to its form. Hence $\xi+\theta(\xi)$ is of grade p. Let $\eta = \xi\sigma+\theta(\xi)\sigma$ satisfying the irreducible $\psi(\omega) = 0$. Hence $\psi(-\eta) = 0$ and $\psi(\omega)$ has even powers only. But η^2 is not in F since σ^2 is in F and $[\xi+\theta(\xi)]^2$ is not in F. Hence the degree of $\psi(\omega)$ is >2. It is not 4 since $F(\xi)$ has no quartic sub-field. Hence it is $2p$.

Corollary 1. Let A be a normal division algebra in $4p^2$ units, p a prime, containing an element q of grade $2p$ and type S_2. Then A contains an element i satisfying

$$\varphi(\omega) \equiv \omega^{2p} + \alpha_1 \omega^{2(p-1)} + \cdots + \alpha_p = 0 \qquad (\alpha_1, \cdots, \alpha_p \text{ in } F),$$

irreducible in F and with even powers only.

Corollary 2. Let A contain i as in Corollary 1. Then A contains elements $i = i_1, i_2, \cdots, i_p$ such that

$$\varphi(\omega) \equiv (\omega - i_p^2)(\omega - i_{p-1}^2) \cdots (\omega - i_1^2).$$

Proof. Let $\varrho = \omega^2$. Then the minimum equation of i^2 is

$$\varphi(\omega) \equiv \psi(\varrho) \equiv \varrho^p + \alpha_1 \varrho^{p-1} + \cdots + \alpha_p = 0.$$

Hence, by Theorem 9, there exist elements $t_1 = 1, t_2, \cdots, t_p$ such that

$$\psi(\varrho) \equiv (\varrho - t_p i^2 t_p^{-1})(\varrho - t_{p-1} i^2 t_{p-1}^{-1}) \cdots (\varrho - t_1 i^2 t_1^{-1}).$$

But $t_\alpha i^2 t_\alpha^{-1} = (t_\alpha i t_\alpha^{-1})^2$. Hence let $i_\alpha = t_\alpha i t_\alpha^{-1} (\alpha = 1, \cdots, \varrho)$ and replace ϱ by its value ω^2 and we have the desired result.

Let $\psi(\varrho) = 0$ be of degree p, p a prime. Let $\psi(\varrho) = 0$ be of type R_2. Then it is of type R_p and is cyclic. In this case we prove, in Theorem 34 that A is of type R_{2p}.

THEOREM 34. *Let A be a normal division algebra in $4p^2$ units, p an odd prime. Let A contain x satisfying the irreducible*

$$\varphi(\omega) \equiv \omega^{2p} + \alpha_1 \omega^{2(p-1)} + \cdots + \alpha_p = 0 \qquad (\alpha_1, \cdots, \alpha_p \text{ in } F),$$

such that $\varphi(\omega) \equiv \psi(\omega^2)$ and $\psi(\varrho) = 0$ is cyclic. Then A is of type R_{2p} and hence is known.

Proof. Since x is evidently of type R_2 it is of type S_2, S_p or S_{2p}. Since the negative of each root is also a root it has an even number of roots which are polynomials in x and hence is of type S_2 or S_{2p}. If x is of type S_{2p} then A is of type R_{2p}. Hence we need only consider x of type S_2. Then A contains a sub-algebra Σ which is a generalized quaternion algebra of units $(1, x, y, xy)$, $yx = -xy$, over the p-ic cyclic field $F(x^2)$. Since $F(x^2)$ is cyclic there exists an element z in A such that $z x^2 = \theta(x^2) z$. Let $x^2 = q$. Then $zq = \theta(q) z$, $z^2 q = \theta^2(q) z^2, \cdots, z^{p-1} q = \theta^{p-1}(q) z^{p-1}$, $z^p q = q z^p$. Since A is normal the only elements of A commutative with q are elements of Σ. Hence z^p is in Σ and, since Σ is of rank 2 with respect to $F(q)$, z is a root of

$$\omega^2 + a\omega + b = 0 \qquad (a, b \text{ in } F(q)).$$

Let $a = 2c$, $b = c^2 - b$. Hence $(z^p + c)^2 = b$. Let $s = z^p + c$. Let s be not in $F(q)$. Then there exists an element y of Σ such that $ys = -sy$. If b is not in F then b is of grade p and, by Theorem 33, s is of grade $2p$.

But then s is of type R_{2p} since the $2p$ distinct elements $s, zsz^{-1}, \cdots, z^{p-1}sz^{1-p}$, and their negatives, are in $F(s) = F(z^p+c)$ which contains $s^2 = b$ generating $F(q)$ and satisfy the minimum equation of s. If $b = \beta$ in F then let $t = sq$, $t^2 = q^2\beta$ generating $F(q)$ hence t is of grade $2p$ and type R_{2p}.

It remains to consider s in $F(q)$. In this case z^p is in $F(q)$. Let $z^p = \alpha(q)$. Then $z z^p z^{-1} = z^p$. Hence $\alpha(q) = \alpha(\theta(q)) = \cdots = \alpha(\theta^{p-1}(q)) = \alpha$ in F. Hence the algebra S with units $q^i z^j$ $(i, j = 0, 1, \cdots, p)$ is a cyclic normal division algebra in p^2 units over F and hence A is the direct product of S by T, a generalized quaternion normal algebra and is of type R_{2p} by Theorem 22.

The algebras of this theorem were considered by L. E. Dickson in his Bulletin note of September 1928. We have here shown them to be of type R_{2p} and, by Theorem 30, of the kind associated with an equation of solvable group. This Bulletin paper was at the printers before the author showed that this corresponding algebras in 16 units were of type R_4. We have now shown that these algebras are not new algebras.

4. **A necessary and sufficient condition that algebras in $4p^2$ units be of type R_2.** In the work on the determination of algebras in nine and sixteen units it was necessary to show the algebras of type R_2 before proceeding further. It seems evident that this will have to be accomplished in all cases if the present known methods are utilized. We shall here give a criterion that algebras in $4p^2$ units be of type R_2 as an aid in any work on the determination of such algebras.

THEOREM 35. *Let A be a normal division algebra in $4p^2$ units, p an odd prime. Let A contain x of grade 2. Then A is of type R_2.*

Proof. Without loss of generality we may take the minimum equation of x to be reduced so that $x^2 = \xi$, ξ in F. Since $-x$ satisfies the minimum equation of x there exists an element y of A such that $yxy^{-1} = -x$. Hence $y^2x = xy^2$.

Theorem 6′ states that the grade of any element of a normal division algebra of order n^2 is a divisor of n. Hence the grade of y^2 is $1, 2, p$ or $2p$. If y^2 is of grade $2p$ then x, being commutative with y^2, is a polynomial in y^2 and is commutative with y, contrary to hypothesis. If y^2 is of grade p, then since $xyx^{-1} = -y$ satisfies the minimum equation of y, y is of grade $2p$ and type R_2, by Theorem 33, and A is of type R_2. If y^2 is of grade 2 then y is of grade 4 by Theorem 33, which is impossible. Hence $y^2 = \eta$, η in F. y is not in F since $yx = -xy$. Therefore A contains the generalized quaternion algebra B with basal units $1, x, y, xy$ and multiplication table $yx = -xy$, $x^2 = \xi$, $y^2 = \eta$ over F and, by Corollary 4 of our Annals paper (loc. cit.), A is the direct product of B and a normal division algebra C of order p^2. Since C has order p^2 it

contains an element z of grade p which is commutative with every element of B. Let $u = xz$. Then $yuy^{-1} = -u$ and u has even grade. But the only sub-fields of $F(z)$, of prime order p, are F and $F(z)$ itself. Since p is odd and z^2 is not in F, $F(z^2) = F(z)$. Hence $u^2 = \xi z^2$ and has grade p. By Theorem 33, u has grade $2p$ and type R_2 and A has type R_2.

THEOREM 36. *Let x be an element of grade $2p$, p an odd prime. Let $F(x)$ contain y of grade p. Then $F(x)$ contains an element of type R_2.*

Proof. The linear set $a+bx$, where a and b range over all elements of $F(y)$ is of order $2p$ since $F(y)$ has order p and x is not in $F(y)$, an algebra of order p. Hence every element of $F(x)$ is expressible in the form $a+bx$. In particular $x^2 = a+bx$. Let $b = -2d$, $a = c-d^2$. Hence $x^2 = a-d^2-2dx$ and $(x+d)^2 = c$. Let $x+d = u$. If c is in F, then, if $v = y(x+d)$, $v^2 = y^2c$ is not in F since $F(y)$ has prime order $p>2$ and hence $F(y^2) = F(y)$ and cy^2, where c is in F, generates $F(y)$. If c is not in F then $F(c) = F(y)$ for the same reason. Hence either v^2 has grade p or u^2 has grade p. But x is not in $F(y)$ and as a result neither $y(x+d)$ nor $x+d$ is in $F(y)$. But $F(u) \geqq F(u^2)$, $F(v) \geqq F(v^2)$. Let $w = u$ or $w = v$ according as $F(u^2) = F(y)$ or $F(v^2) = F(y)$. Then $F(w) \geqq F(w^2) = F(y)$. But w is not in $F(y)$ so that $F(w) > F(y)$ and w has grade $2p$. If the minimum equation of w^2 is $\varphi(\omega) = 0$ then w satisfies $\varphi(\omega^2) = 0$ of degree $2p$ with leading coefficient unity. But w has grade $2p$ and consequently $\varphi(\omega^2) = 0$ is its minimum equation and w has type R_2.

We shall prove a general theorem in the theory of normal division algebras for use in the application of Theorem 36.

THEOREM 37. *Let A be a normal division algebra in n^2 units over F. Let y be an element of A of grade q less than n. Then A contains an element x of grade n such that y is a polynomial in x with coefficients in F.*

Proof. Let the minimum equation of y be $\psi(\omega) = 0$ and $\alpha_1, \alpha_2, \cdots, \alpha_q$ be its q distinct scalar roots. Adjoin the quantities $\alpha_1, \cdots, \alpha_q$ and sufficient further quantities to F, so that the algebra A' over the extended field F' with the same basal elements as A is equivalent to the algebra of all n-rowed square matrices with elements in F' by a transformation of basal elements. This is possible with only a finite number of adjunctions to F and we may choose $\xi_1, \xi_2, \cdots, \xi_r$ as a basis of F' relative to F. Then every element of A' is expressible in the form

$$x = x_1\xi_1 + x_2\xi_2 + \cdots + x_r\xi_r \qquad (x_1, \cdots, x_r \text{ in } A), \tag{1}$$

since we get all linear combinations with coefficients in F' of the basal elements of A in this way, and thus all elements of A'. The elements $\xi_1, \xi_2, \cdots, \xi_r$ are linearly independent with respect to A by the definition of A' and the linear independence of $\xi_1, \cdots, \xi_r$ with respect to F.

Consider the algebra $F(y) \equiv D$. Let D' be the algebra over F' with the same basal elements as D. The elements

$$u_i \equiv \frac{(y-\alpha_1)(y-\alpha_2)\cdots(y-\alpha_{i-1})(y-\alpha_{i+1})\cdots(y-\alpha_q)}{(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots(\alpha_i-\alpha_{i-1})(\alpha-\alpha_{i+1})\cdots(\alpha_i-\alpha_q)} \qquad (i=1,2,\cdots,q)$$

are idempotent elements of D' as is shown in the following.

If we replace y by a variable ω of F' the sum of the resulting fractions $u_i(\omega)$ is equal to 1 for $\omega = \alpha_i$, since then $u_i = 1$, $u_k = 0$ $(k \neq i)$. Since the sum is a polynomial of degree $q-1$ in ω, which is equal to 1 for the q values α_i $(i = 1, \cdots, q)$ of ω, it is identically equal to 1. Hence

$$u_1 + u_2 + \cdots + u_q = 1. \tag{2}$$

Since $u_i u_j$ for $i \neq j$ has the factor $\psi(y)$, it is zero. Multiplying (2) on the right by u_i we have $u_i^2 = u_i$ and have

$$u_i u_j = 0, \quad u_i^2 = u_i \qquad (i \neq j;\ i,j = 1, \cdots, q). \tag{3}$$

The elements u_i are linearly independent with respect to F' and may be taken as a basis of D' since, if not then $\sum_{i=1}^{q} \gamma_i u_i = 0$ for γ_i in F' and not all zero and thus $\sum_{i=1}^{q} \gamma_i u_i u_j = \gamma_j u_j = 0$, $\gamma_j = 0$ $(j = 1, \cdots, q)$, a contradiction. Hence y is expressible as a linear combination of u_i with coefficients in F'. But the elements u_i are expressible as the sum of primitive idempotent elements whose products in pairs are zero. We may adjoin to the s primitive idempotent elements so obtained $n-s$ further primitive idempotent elements and obtain n primitive idempotent elements, e_{ii}, which may be taken to correspond to the matrices with elements 1 in the ith row and ith column and zero elsewhere. Hence y in this representation of A' as a matric algebra, is a diagonal matrix.

Every two diagonal matrices are commutative. The matrix $z = e_{11} + 2e_{22} + 3e_{33} + \cdots + ne_{nn}$ is a diagonal matrix and is commutative with y. Since every element of A' is expressible in the form (1) we have

$$z = z_1\xi_1 + z_2\xi_2 + \cdots + z_r\xi_r \qquad (z_1, z_2, \cdots, z_r \text{ in } A). \tag{4}$$

Since $zy = yz$,

$$zy - yz = (z_1 y - y z_1)\xi_1 + \cdots + (z_r y - y z_r)\xi_r = 0,$$

so that $z_i y = y z_i$ $(i = 1, 2, \cdots, r)$ by the linear independence of $\xi_1, \cdots, \xi_r$ with respect to A.

This shows that y is commutative with

$$x_0 = z_1\eta_1 + z_2\eta_2 + \cdots + z_r\eta_r \tag{5}$$

where $\eta_1, \cdots, \eta_r$ are independent variables in F'. The rank equation of A for the element x_0 becomes

$$T(\omega; \eta_1, \cdots, \eta_r) \equiv \omega^n + f_1 \omega^{n-1} + \cdots + f_n = 0 \tag{6}$$

where $f_1, \cdots, f_n$ are polynomials in $\eta_1, \cdots, \eta_r$ with coefficients in F. The discriminant of (6) is not identically zero since for the values $\eta_i = \xi_i$ $(i = 1, \cdots, r)$ it is different from zero. Hence there exist rational values of $\eta_1, \cdots, \eta_r$ for which the discriminant of (6) does not vanish.* Let these values be $\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_r$ so that

$$x = z_1 \bar{\eta}_1 + z_2 \bar{\eta}_2 + \cdots + z_r \bar{\eta}_r$$

is in A and is commutative with y. The rank equation of A for any element x of A is a power of the minimum equation of x. But the rank equation of A for x is $T(\omega; \bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_r) = 0$ an equation with non-vanishing discriminant, so that $T(\omega; \bar{\eta}_1, \cdots, \bar{\eta}_r) = 0$ is irreducible and is the minimum equation of x and x is of grade n. But y is commutative with x, and the only elements of A commutative with any element a of grade n are polynomials in a, so that y is a polynomial in x and x is the desired element.

THEOREM 38. *Let A be a normal division algebra in $4p^2$ units, p an odd prime. Then A has type R_2 if and only if A contains an element of grade less than $2p$.*

Proof. If A contains an element x of grade less than $2p$ the grade of x is 2 or p. If the grade is 2 then, by Theorem 35, A has type R_2. If the grade of x is p then there exists an element y of grade $2p$ in A such that x is a polynomial in y and, by Theorem 36, A has type R_2.

Conversely let A have type R_2. Then A contains x of grade 2 and type R_2. If x has type S_2 then, by Corollary 1, A contains an element i of grade $2p$ whose minimum equation has even powers only so that i^2 has grade p. If x has type S_p or S_{2p} then A contains a subalgebra Σ which is a cyclic algebra in p^2 units over a quadratic field and hence A contains an element of grade 2. But if x has type R_2 then it has only types S_2, S_p or S_{2p} and the theorem is proved.

* Fricke, "Lehrbuch der Algebra", p. 96, for proof.

PRINCETON, N. J.
December 26, 1928.

DYNAMICAL TRAJECTORIES AND GEODESICS.*

By Luther Pfahler Eisenhart.

According to the general theory of relativity the motion of a material particle under the action only of inertia and gravitation is described by a geodesic of the four-dimensional space-time continuum of the gravitational field. Thus there is a correspondence between the spacial paths of particles and the geodesics of the space-time continuum. In this paper we show that the trajectories of a general holonomic conservative system in classical dynamics can be put into correspondence with the geodesics of a suitable Riemannian manifold. In § 2 we consider the general case when the time enters in the constraints and in the potential function, and in §§ 3, 4 the relation between the theory of geodesics and the Hamilton-Jacobi theory of dynamical trajectories. The relation between contact transformations of the Riemannian manifold and transformations of the Hamiltonian equations of the dynamical system is set forth in § 5; the result brings out the significance of the form of the latter transformations. In § 6 the restricted case when t enters neither in the constraints nor potential function is treated not as a special case of the more general theory, but on an independent basis.

1. **Equations of dynamics.** Consider a conservative holonomic dynamical system of n degrees of freedom such that the restraints as well as the potential function V involve the time. If we denote by $q^i (i = 1, \cdots, n)$ the independent variables determining the position of the system and put $t = q^{n+1}$, the kinetic energy is of the form† $\frac{1}{2} g_{\alpha\beta}\, \dot{q}^\alpha\, \dot{q}^\beta$, where α and β run from 1 to $n+1$ and the dot indicates differentiation with respect to the time as usual. Then the Lagrangian function L is given by

$$L = \frac{1}{2} g_{\alpha\beta}\, \dot{q}^\alpha\, \dot{q}^\beta - V. \tag{1.1}$$

When this expression is substituted in the equations of Lagrange, namely

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \tag{1.2}$$

we have as the differential equations of the trajectories of the motion

* Received April, 1929.

† Throughout this paper we use the convention that a repeated index indicates summation, unless stated otherwise; also latin indices will be supposed to take the values 1 to n.

$$g_{ij}\ddot{q}^j + [jk,i]\dot{q}^j\dot{q}^k + \left(\frac{\partial g_{ij}}{\partial t} + \frac{\partial g_{in+1}}{\partial q^j} - \frac{\partial g_{jn+1}}{\partial q^i}\right)\dot{q}^j + \frac{\partial g_{in+1}}{\partial t} - \frac{1}{2}\frac{\partial}{\partial q^i}(g_{n+1\,n+1} - 2V) = 0, \tag{1.3}$$

where $[jk,i]$ is the Christoffel symbol of the first kind, namely

$$[jk,i] = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial q^k} + \frac{\partial g_{ki}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right). \tag{1.4}$$

If we put

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{1.5}$$

we have

$$p_i = g_{ij}\dot{q}^j + g_{in+1}. \tag{1.6}$$

For a system of the kind under discussion the determinant of the quantities g_{ij} is different from zero. If then we define quantities g^{ij} by the equations

$$g^{ij}g_{jk} = \delta^i_k, \tag{1.7}$$

where the δ's are Kronecker deltas defined by

$$\delta^j_k = 1 \text{ or } 0, \text{ as } j = k \text{ or } j \neq k, \tag{1.8}$$

it follows from (1.6) that

$$\dot{q}^j = p_i g^{ij} - g^{ij}g_{in+1}. \tag{1.9}$$

The corresponding Hamiltonian function H is the function of q^i, p_i and t which results from the substitution of the values (1.9) of $\dot{q}^j$ in

$$H = p_i\dot{q}^i - L. \tag{1.10}$$

On substitution we obtain

$$H = \frac{1}{2}g^{ij}(p_ip_j - 2p_ig_{jn+1} + g_{in+1}g_{jn+1}) + V - \frac{1}{2}g_{n+1\,n+1}. \tag{1.11}$$

If we differentiate H in the form (1.10) for q^j, we have

$$\frac{\partial H}{\partial q^j} = -\frac{\partial L}{\partial q^j}, \tag{1.12}$$

in consequence of (1.5). Hence (1.9) and (1.2) can be written in the Hamiltonian form

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{1.13}$$

2. **Dynamical trajectories as geodesics.** Consider now any dynamical system of the type discussed in § 1 and in conjunction with it a Riemannian space of $n+2$ dimensions, say S_{n+2}, for which the metric is defined by

$$ds^2 = g_{ij}\,dq^i\,dq^j + 2g_{in+1}\,dq^i\,dt + A\,dt^2 + 2B\,dt\,du, \tag{2.1}$$

where g_{ij} and g_{in+1} are the functions of the q's and t for the given dynamical system, A is a function of q^i and t, to be defined fully later, and B is a constant. For convenience we write (2.1) in the condensed form

$$ds^2 = g_{\alpha\beta}\,dq^\alpha\,dq^\beta, \tag{2.2}$$

understanding that $q^{n+1} = t$, $q^{n+2} = u$.* If $\bar{g}$ and g denote the determinants of $g_{\alpha\beta}$ and g_{ij} respectively, we have $\bar{g} = -B^2 g$.

By definition the integral curves of the system of differential equations

$$g_{\alpha\beta}\frac{d^2q^\beta}{d\tau^2} + [\beta\gamma, \alpha]\frac{dq^\beta}{d\tau}\frac{dq^\gamma}{d\tau} = 0, \tag{2.3}$$

where $[\beta\gamma, \alpha]$ are defined as in (1.4) but with reference to (2.2), are the geodesics† of S_{n+2}. These equations admit the quadratic first integral

$$g_{\alpha\beta}\frac{dq^\alpha}{d\tau}\frac{dq^\beta}{d\tau} = \text{const.}$$

When the constant is different from zero, the geodesic is said to be *non-minimal;* when it is equal to zero, the geodesic is *minimal.* In the former case there is no loss in generality in taking the arc s for the parameter τ, in which case the constant in the above equation is unity, as follows from (2.2). We consider first the case of non-minimal geodesics.

When α takes the values 1 to n, equations (2.3) are

$$\begin{aligned} g_{ij}\frac{d^2q^j}{ds^2} + [jk, i]\frac{dq^j}{ds}\frac{dq^k}{ds} + \left(\frac{\partial g_{ij}}{\partial t} + \frac{\partial g_{in+1}}{\partial q^j} - \frac{\partial g_{jn+1}}{\partial q^i}\right)\frac{dq^j}{ds}\frac{dt}{ds} \\ + \left(\frac{\partial g_{in+1}}{\partial t} - \frac{1}{2}\frac{\partial A}{\partial q^i}\right)\left(\frac{dt}{ds}\right)^2 + g_{in+1}\frac{d^2t}{ds^2} = 0, \end{aligned} \tag{2.4}$$

and when α takes the values $n+1$ and $n+2$, the respective equations are

$$B\frac{d^2t}{ds^2} = 0, \tag{2.5}$$

$$\begin{aligned} g_{jn+1}\frac{d^2q^j}{ds^2} + A\frac{d^2t}{ds^2} + B\frac{d^2u}{ds^2} \\ + \frac{1}{2}\left(\frac{\partial g_{kn+1}}{\partial q^j} + \frac{\partial g_{jn+1}}{\partial q^k} - \frac{\partial g_{jk}}{\partial t}\right)\frac{dq^j}{ds}\frac{dq^k}{ds} \\ + \frac{\partial A}{\partial q^i}\frac{dq^i}{ds}\frac{dt}{ds} + \frac{1}{2}\frac{\partial A}{\partial t}\left(\frac{dt}{ds}\right)^2 = 0. \end{aligned} \tag{2.6}$$

* Throughout this section, and the next two, greek indices are understood to take the values 1 to $n+2$ and latin 1 to n.

† Cf. Eisenhart, Riemannian Geometry, p. 50. Hereafter a reference to this book will be given as E., p. 50.

We consider in this section the case when $B \neq 0$ and in all generality take it equal to unity. Then from (2.5) we have

$$\frac{dt}{ds} = a, \tag{2.7}$$

where a is a constant. Thus along any non-minimal geodesic for which $a \neq 0$ the parameter s may be chosen so that

$$t = as, \tag{2.8}$$

and consequently (2.4) become

$$g_{ij}\ddot{q}^j + [jk, i]\,\dot{q}^j\dot{q}^k + \left(\frac{\partial g_{ij}}{\partial t} + \frac{\partial g_{in+1}}{\partial q^j} - \frac{\partial g_{jn+1}}{\partial q^i}\right)\dot{q}^j + \frac{\partial g_{in+1}}{\partial t} - \frac{1}{2}\frac{\partial A}{\partial q^i} = 0. \tag{2.9}$$

Comparing these equations with (1.3), we see that they are equivalent to the latter, if we take

$$A = g_{n+1\,n+1} - 2V. \tag{2.10}$$

From (2.1) and (2.7) it follows that along any geodesic

$$\frac{du}{dt} = \frac{1}{2}\left(\frac{1}{a^2} - g_{ij}\,\dot{q}^i\,\dot{q}^j - 2g_{in+1}\,\dot{q}^i - g_{n+1\,n+1} + 2V\right) = \frac{1}{2a^2} - L. \tag{2.11}$$

On differentiating this equation with respect to t, we find that (2.6) is satisfied by any solution of (1.3). If such a solution is substituted in the right-hand member of (2.11), we have an integral of the form

$$u = \frac{1}{2}\frac{t}{a^2} - \int L\,dt + b, \tag{2.12}$$

where b is an arbitrary constant. Tbus for each set of finite values of $a(\neq 0)$ and b, we have a system of non-mininal geodesics of S_{n+2}, which are in one-to-one correspondence with the trajectories of the given dynamical system, such that the expressions for q^i as functions of t are the same as for the trajectories. Moreover, equation (2.12) reveals the relation of the variable u to Hamilton's principle.*

When the geodesics are minimal, we have $d^2t/d\tau^2 = 0$, and consequently there is no loss in generality in taking $\tau = t$. In this case the right-hand member of (2.1) is zero along any of these geodesics, and

$$u = -\int L\,dt + b. \tag{2.13}$$

* Cf. Whittaker, Analytical Dynamics, 3rd edition, Cambridge, 1927, p. 246.

3. Finite equations of geodesics and the theorem of Hamilton and Jacobi. If we define g^{ij} by (1.7) and $\overline{g}^{\alpha\beta}$ by

$$\overline{g}^{\alpha\beta} g_{\alpha\gamma} = \delta^{\beta}_{\gamma}, \tag{3.1}$$

we obtain for the form (2.1) with $B = 1$

$$\begin{aligned} &\overline{g}^{ij} = g^{ij}, \quad \overline{g}^{in+1} = \overline{g}^{n+1n+1} = 0, \quad \overline{g}^{in+2} = -g_{jn+1} g^{ij}, \\ &\overline{g}^{n+1n+2} = 1, \quad \overline{g}^{n+2n+2} = g_{in+1} g_{jn+1} g^{ij} - A. \end{aligned} \tag{3.2}$$

From the theory of geodesics in a general Riemannian space with the fundamental form (2.2) we have* that, if φ is a solution of the differential equation

$$\overline{g}^{\alpha\beta} \frac{\partial \varphi}{\partial q^{\alpha}} \frac{\partial \varphi}{\partial q^{\beta}} = 1, \tag{3.3}$$

the orthogonal trajectories of the hypersurfaces $\varphi = \text{const.}$ are non-minimal geodesics. Since $\partial\varphi/\partial q^{\alpha}$ are the covariant components of the normals to these hypersurfaces, it follows that along a normal geodesic dq^{α}/ds and $\overline{g}^{\alpha\beta}\,\partial\varphi/\partial q^{\beta}$ are proportional. From (2.2), (3.1) and (3.3) it follows that the square of this factor of proportionality is unity. Hence in all generality we may take

$$\frac{dq^{\alpha}}{ds} = \overline{g}^{\alpha\beta} \frac{\partial \varphi}{\partial q^{\beta}}. \tag{3.4}$$

When we apply this general result to the case when $\overline{g}^{\alpha\beta}$ have the values (3.2), we have for $\alpha = n+1$

$$\frac{dt}{ds} = \frac{\partial \varphi}{\partial u}.$$

From this result and (2.7) it follows that the geodesics for a given value of a are the orthogonal trajectories of the hypersurfaces $\varphi = \text{const.}$, when φ is of the form

$$\varphi = a(u + \psi),$$

where ψ does not involve u. Substituting in (3.3) and making use of (3.2) in which A has the value (2.10), we obtain

$$\frac{\partial \psi}{\partial t} + \frac{1}{2} g^{ij} \left(\frac{\partial \psi}{\partial q^{i}} \frac{\partial \psi}{\partial q^{j}} - 2 g_{jn+1} \frac{\partial \psi}{\partial q^{i}} + g_{in+1} g_{jn+1} \right) + V - \frac{1}{2} g_{n+1n+1} = \frac{1}{2a^2}.$$

Thus ψ involves a, but if we put

$$\varphi = a(u + W) + \frac{t}{2a} \tag{3.5}$$

* E., p. 58.

then W, being a solution of

$$\frac{\partial W}{\partial t}+\frac{1}{2}g^{ij}\left(\frac{\partial W}{\partial q^i}\frac{\partial W}{\partial q^j}-2g_{jn+1}\frac{\partial W}{\partial q^i}+g_{in+1}g_{jn+1}\right)+V-\frac{1}{2}g_{n+1\,n+1}=0 \tag{3.6}$$

does not involve a. The terms of this equation after the first are of the same form as (1.11) with p_i replaced by $\partial W/\partial q^i$. Hence we write (3.6) in the form

$$\frac{\partial W}{\partial t}+H\left(q,\frac{\partial W}{\partial q},t\right)=0, \tag{3.7}$$

which is the Hamiltonian partial differential equation of the given dynamical system.

If we have a complete solution of (3.7), that is a solution involving t, q^i and n arbitrary constants a^i, none of which is additive, and this solution is substituted in (3.5), the resulting expression is a complete solution of (3.3). If we substitute it in (3.3) and differentiate with respect to a and each a^i, we have

$$\bar{g}^{\alpha\beta}\frac{\partial\varphi}{\partial q^\alpha}\frac{\partial^2\varphi}{\partial q^\beta\,\partial a}=0, \qquad \bar{g}^{\alpha\beta}\frac{\partial\varphi}{\partial q^\alpha}\frac{\partial^2\varphi}{\partial q^\beta\,\partial a^i}=0.$$

Hence the hypersurfaces of each of the $n+1$ families

$$\frac{\partial\varphi}{\partial a}=\text{const.}, \qquad \frac{\partial\varphi}{\partial a^i}=\text{const.}$$

are orthogonal to the hypersurfaces $\varphi=\text{const.}$ From (3.5) it follows that the equations of the former hypersurfaces are

$$u+W+\frac{t}{2a^2}=b \tag{3.8}$$

and

$$\frac{\partial W}{\partial a^i}=b_i, \tag{3.9}$$

where the b's are arbitrary constants. Consequently these are the equations of non-minimal geodesics of S_{n+2} in finite form; moreover, from the general theory of geodesics we know that all non-minimal geodesics are given by these equations.*

Since equations (3.9) involve q^i and t, they are the equations of the dynamical trajectories in finite form. Hence if in (3.8) we assign particular values to a and b, we have a set of geodesics in S_{n+2} corresponding to

* E., p. 58.

the dynamical trajectories and these geodesics for each set of values of a^i are orthogonal to the hypersurfaces

$$a(u+W)+\frac{t}{2a}=\text{const.}$$

in which a and the a_i have these values.

Incidentally we have proved the theorem of Hamilton and Jacobi:

If W is a complete solution of equation (3.6) *involving n arbitrary constants a^i, none of which is additive, equations* (3.9) *when solved for q^i in terms of t, a^i and b_i define all the dynamical trajectories of the given system.*

Suppose that we have such a solution, that it is substituted in (3.6) and (3.9), and that the resulting identities are differentiated with respect to a^i and t respectively. This gives

$$\frac{\partial^2 W}{\partial a^i\,\partial t}+g^{jk}\left(\frac{\partial W}{\partial q^k}-g_{k\,n+1}\right)\frac{\partial^2 W}{\partial a^i\,\partial q^j}=0$$

and

$$\frac{\partial^2 W}{\partial a^i\,\partial t}+\dot{q}^j\frac{\partial^2 W}{\partial a^i\,\partial q^j}=0.$$

From the last two sets of equations we have

$$\frac{\partial^2 W}{\partial a^i\,\partial q^j}\left[\dot{q}^j-g^{jk}\left(\frac{\partial W}{\partial q^k}-g_{k\,n+1}\right)\right]=0.$$

The determinant of the quantities $\partial^2 W/\partial a^i\,\partial q^j$ is not zero, otherwise there would be a functional relation between the quantities $\partial W/\partial a^i$ not involving the q's which is impossible when the a's do not enter additively. Consequently

$$\dot{q}^j=g^{jk}\left(\frac{\partial W}{\partial q^k}-g_{k\,n+1}\right) \tag{3.10}$$

are the expressions for the q's arising from (3.9). If we put

$$p_i=\frac{\partial W}{\partial q^i}, \tag{3.11}$$

equations (3.10) become (1.9) and consequently they are the first set of (1.13). If we differentiate (3.11) with respect to t and make use of (3.10) and of the equation obtained on differentiating (3.7) with respect to q^i, we get the second set of (1.13).

4. **Dynamical trajectories as minimal geodesics.** Since the determinant of $g_{\alpha\beta}$ for (2.1), when $B=1$, is $-g$, the form (2.1) is indefinite and consequently the differential equation

$$\bar{g}^{\alpha\beta}\frac{\partial\varphi}{\partial q^\alpha}\frac{\partial\varphi}{\partial q^\beta}=0 \tag{4.1}$$

admits real solutions. Suppose that φ is such a solution and consider the system of equations

$$\frac{dq^\alpha}{d\tau} = \bar{g}^{\alpha\beta}\frac{\partial \varphi}{\partial q^\beta}, \tag{4.2}$$

where τ is a parameter. Differentiating these equations with respect to τ, we have

$$\frac{d^2 q^\alpha}{d\tau^2} = \left[\bar{g}^{\alpha\beta}\varphi_{,\beta\gamma} - \bar{g}^{\delta\beta}\frac{\partial \varphi}{\partial q^\beta}\begin{Bmatrix}\alpha\\ \gamma\delta\end{Bmatrix}\right]\frac{dq^\gamma}{d\tau},$$

where $\varphi_{,\beta\gamma}$ is the second covariant derivative* of φ with respect to the form (2.2) and $\begin{Bmatrix}\alpha\\ \gamma\delta\end{Bmatrix}$ are the Christoffel symbols of the second kind formed with respect to (2.2); they are defined by

$$g_{\alpha\beta}\begin{Bmatrix}\alpha\\ \gamma\delta\end{Bmatrix} = [\gamma\delta, \beta].$$

From (4.1) we have by covariant differentiation

$$\bar{g}^{\alpha\beta}\frac{\partial \varphi}{\partial q^\alpha}\varphi_{,\beta\gamma} = 0.$$

Since $\varphi_{,\beta\gamma} = \varphi_{,\gamma\beta}$, the above equations reduce to

$$\frac{d^2 q^\alpha}{d\tau^2} + \begin{Bmatrix}\alpha\\ \gamma\delta\end{Bmatrix}\frac{dq^\gamma}{d\tau}\frac{dq^\delta}{d\tau} = 0, \tag{4.3}$$

or in other form (2.3). From (4.1) and (4.2) we have

$$g_{\alpha\beta}\frac{dq^\alpha}{d\tau}\frac{dq^\beta}{d\tau} = 0. \tag{4.4}$$

Consequently the integral curves of the equations (4.2) are minimal geodesics of S_{n+2}.

When these results are applied to the space with the fundamental form (2.1), there is no loss of generality in taking $\tau = t$, as remarked in § 2. Then from (4.2) for $\alpha = n+1$ and from (3.2) we have

$$\varphi = u + W, \tag{4.5}$$

where W does not involve u. Substituting this expression in (4.1), we find that W is a solution of (3.6). Then equations (4.2) become

$$\frac{dq^i}{dt} = g^{ij}\left(\frac{\partial W}{\partial q^j} - g_{jn+1}\right) \tag{4.6}$$

* E., p. 29.

and

$$\frac{du}{dt} = \frac{\partial W}{\partial t} - g^{ij} g_{in+1}\left(\frac{\partial W}{\partial q^j} - g_{jn+1}\right) + 2V - g_{n+1n+1}.$$

If we eliminate V from the last equation and (3.6), we obtain

$$\frac{du}{dt} = -\frac{\partial W}{\partial t} - g^{ij}\left(\frac{\partial W}{\partial q^j} - g_{jn+1}\right)\frac{\partial W}{\partial q^i} = -\frac{dW}{dt}. \tag{4.7}$$

Suppose now that we have a complete solution of (3.6) involving n arbitrary constants a^i, none of which is additive. If we take the equations

$$\frac{\partial W}{\partial a^i} = b_i, \qquad u + W = b, \tag{4.8}$$

where the b's are constants, we show, as in § 3, that when the first n are solved for the q's as functions of t, a^i and b_i, equations (4.6) hold. From this result and (4.7), it follows that equations (4.8) define the minimal geodesics, and the first n of these equations are those of the dynamical trajectories.

5. **Contact transformations.** In this section we establish the relation between the transformations of the Hamiltonian equations (1.13) and the contact transformations of the space S_{n+2} with the fundamental form (2.1).

When for any non-minimal curve in an S_{n+2} with the general fundamental form (2.2) we put

$$P_\alpha = g_{\alpha\beta}\frac{dq^\beta}{ds}, \tag{5.1}$$

then in terms of the notation of (3.1) we have

$$\frac{dq^\beta}{ds} = \bar{g}^{\beta\alpha} P_\alpha, \tag{5.2}$$

and consequently

$$g_{\alpha\beta}\frac{dq^\alpha}{ds}\frac{dq^\beta}{ds} = \bar{g}^{\alpha\beta} P_\alpha P_\beta. \tag{5.3}$$

Moreover from (2.2) we have

$$ds = P_\alpha\, dq^\alpha. \tag{5.4}$$

If we put

$$C = \sqrt{\bar{g}^{\alpha\beta} P_\alpha P_\beta}, \tag{5.5}$$

it follows from (5.3) and (2.2) that $C = 1$. From (5.1) we have along a non-minimal geodesic, in consequence of (2.3),

$$\frac{dP_\alpha}{ds} = \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial q^\alpha}\frac{dq^\beta}{ds}\frac{dq^\gamma}{ds} = -\frac{1}{2}\frac{\partial \bar{g}^{\beta\gamma}}{\partial q^\alpha} P_\beta P_\gamma,$$

the last expression being a consequence of (5.2) and (3.1). Hence the non-minimal geodesics of S_{n+2} are defined by the equations

$$\frac{dq^\alpha}{ds} = \frac{\partial C}{\partial P_\alpha}, \qquad \frac{dP_\alpha}{ds} = -\frac{\partial C}{\partial q^\alpha}. \tag{5.6}$$

If φ^α and ψ_α are functions of $2n+4$ quantities q^α and P_α such that

$$\psi_\alpha \frac{\partial \varphi^\alpha}{\partial q^\beta} = P_\beta, \qquad \psi_\alpha \frac{\partial \varphi^\alpha}{\partial P_\beta} = 0, \tag{5.7}$$

then the equations

$$q'^\alpha = \varphi^\alpha, \qquad P'_\alpha = \psi_\alpha \tag{5.8}$$

define a homogeneous contact transformation, that is a transformation for which the condition

$$P'_\alpha \, dq'^\alpha = P_\beta \, dq^\beta \tag{5.9}$$

is satisfied for arbitrary values of the differentials dq^α and dP_α.* The conditions (5.7) are necessary as well as sufficient, and from them it follows that the functions φ^α and ψ_α are homogeneous in the P's of degrees zero and one respectively. Furthermore, when the P's are eliminated from the first set of equations (5.8), there result $p(\leqq 1)$ independent equations of the form

$$F_\sigma(q^\alpha, q'^\beta) = 0 \qquad (\sigma = 1, \cdots, p). \tag{5.10}$$

Moreover, we have the relations

$$P_\alpha = \lambda^\sigma \frac{\partial F_\sigma}{\partial q^\alpha}, \qquad P'_\alpha = -\lambda^\sigma \frac{\partial F_\sigma}{\partial q'^\alpha}, \tag{5.11}$$

where the λ's are parameters. Conversely, if F_σ are any functions such that on the elimination of the λ's from (5.11) the resulting equations and (5.10) admit unique solutions for q'^α and P'_α as functions of q^α and P_α, the latter define a homogeneous contact transformation.†

From the form of equations (5.6) in which C is homogeneous of degree one in the P's it follows that the non-minimal geodesics are the trajectories of a continuous group G_1 of contact transformations. Moreover, when a transformation (5.8) is applied to (5.6), these equations are transformed into

$$\frac{dq'^\alpha}{ds} = \frac{\partial C'}{\partial P'_\alpha}, \qquad \frac{dP'_\alpha}{ds} = -\frac{\partial C'}{\partial q'^\alpha}, \tag{5.12}$$

where C' is the transform of C.

* The proofs of the statements of this paragraph and the next are given by the author in the Annals of Mathematics, (2), vol. 30 (1929), pp. 211–249.

† For the analytical form of this condition see the paper referred to.

For the form (2.1), where $B = 1$ and A has the value (2.10), equations (5.1) are

$$P_i = (g_{ij}\dot{q}^j + g_{in+1})\frac{dt}{ds},$$

(5.13)

$$P_{n+1} = (g_{in+1}\dot{q}^i + g_{n+1n+1} - 2V)\,P_{n+2} + \frac{du}{ds}, \qquad P_{n+2} = \frac{dt}{ds}.$$

In consequence of (3.2) and (5.5) we have

$$C^2 = g^{ij}(P_i P_j - 2g_{in+1}P_j P_{n+2} + g_{in+1}g_{jn+1}) + 2P_{n+1}P_{n+2} + 2V - g_{n+1n+1}. \tag{5.14}$$

Also from (5.13) and (1.6) we have

$$P_i = p_i P_{n+2}, \tag{5.15}$$

and consequently

$$C^2 = 2P_{n+2}^2 H + 2P_{n+1}P_{n+2}. \tag{5.16}$$

In this case, since $C = 1$, equations (5.6) are reducible in consequence of (5.15) to

$$\begin{gathered}\frac{dq^i}{ds} = P_{n+2}\frac{\partial H}{\partial p_i}, \qquad \frac{dt}{ds} = P_{n+2}, \qquad \frac{du}{ds} = 2P_{n+2}H + P_{n+1}, \\ \frac{dp_i}{ds} = -P_{n+2}\frac{\partial H}{\partial q^i}, \qquad \frac{dP_{n+1}}{ds} = -P_{n+2}^2\frac{\partial H}{\partial t}, \qquad \frac{dP_{n+2}}{ds} = 0.\end{gathered} \tag{5.17}$$

Because of the last of equations (5.13) the first equations of each row of (5.17) are equivalent to (1.13).

We consider now homogeneous contact transformations such that the equation of the first set of (5.8) for $\alpha = n+2$ is of the form

$$u' = u + f, \tag{5.18}$$

where f and the functions of the other equations of both sets do not involve u. From the first of (5.7) for $\beta = n+2$ it follows that $P'_{n+2} = P_{n+2}$. We have seen in § 2 that the dynamical trajectories are in one-to-one correspondence with the geodesics for each value of a in (2.7). If we take $a = 1$, then from (5.13) we have $P_{n+2} = 1$ and equation (5.9) becomes, in consequence of (5.15),

$$p_i\,dq^i + P_{n+1}\,dt = P'_i\,dq'^i + P'_{n+1}\,dt' + df. \tag{5.19}$$

Moreover from (5.16) we have

$$P_{n+1} = \frac{1}{2} - H. \tag{5.20}$$

In this case the corresponding equations (5.10) are

$$u'-u-F_1 = 0, \qquad F_2 = \cdots = F_r = 0, \tag{5.21}$$

for $r \gtreqless 1$, the functions F being independent of u and u'. The first of (5.11) for $\alpha = n+2$ becomes $1 = -\lambda^1$, and consequently equations (5.11) become

$$\begin{aligned} p_i &= \frac{\partial F_1}{\partial q^i} + \lambda^\sigma \frac{\partial F_\sigma}{\partial q^i}, \quad P_{n+1} = \frac{\partial F_1}{\partial t} + \lambda^\sigma \frac{\partial F_\sigma}{\partial t}, \\ P_i' &= -\frac{\partial F_1}{\partial q'^i} - \lambda^\sigma \frac{\partial F_\sigma}{\partial q'^i}, \quad P_{n+1}' = -\frac{\partial F_1}{\partial t'} - \lambda^\sigma \frac{\partial F_\sigma}{\partial t'} \quad (\sigma = 2, \cdots, r). \end{aligned} \tag{5.22}$$

A contact transformation is defined by (5.21), when the functions F are such that on the elimination of the λ's from (5.22) the resulting equations and $F_2 = \cdots = F_r = 0$ can be solved for q^i, t, p_i and P_{n+1} as functions of q'^i, t', P_i' and P_{n+1}'. Suppose that these expressions are substituted in (5.20) and this equation is solved for P_{n+1}', the result being written in the form

$$P_{n+1}' + K(q'^i, t', P_i') = 0. \tag{5.23}$$

If, as in (5.12), we denote by C' the transform of C, given by (5.5), we have $C' = 1$. When the expression for P_{n+1}' from (5.23) is substituted in this equation, we have an identity in q'^i, t' and P_i'. Consequently we have

$$\frac{\partial C'}{\partial q'^i} - \frac{\partial C'}{\partial P_{n+1}'}\frac{\partial K}{\partial q'^i} = 0, \qquad \frac{\partial C'}{\partial P_i'} - \frac{\partial C'}{\partial P_{n+1}'}\frac{\partial K}{\partial P_i'} = 0.$$

By means of these identities the corresponding equations (5.12) are reducible to

$$\frac{dt'}{ds} = \frac{\partial C'}{\partial P_{n+1}'}, \qquad \frac{dP_{n+1}'}{ds} = -\frac{\partial C'}{\partial t'},$$

and

$$\frac{dq'^i}{dt'} = \frac{\partial K}{\partial P_i'}, \qquad \frac{dP_i'}{dt'} = -\frac{\partial K}{\partial q'^i}. \tag{5.24}$$

Equations (5.21), (5.22), (5.20) and (5.23) define the transformations of the Hamiltonian equations (1.13) into (5.24), and we have just seen the relation of these transformations to the general contact transformations of the space S_{n+2}.

A sub-class of these transformations is obtained when we take for the equation for $\alpha = n+1$ in the first set of (5.8) $t' = t$. Then in place of (5.21) we have

$$u'-u-F_1 = 0, \qquad t'-t = 0, \qquad F_2 = \cdots = F_r = 0,$$

for $r \geqq 1$, the functions F not involving t', u and u'. In this case the expressions (5.22) for p_i and P_i' are the same as for the general case, but in place of the other two we have

$$P'_{n+1} = -\left(\frac{\partial F_1}{\partial t} - P_{n+1} + \lambda^\sigma \frac{\partial F_\sigma}{\partial t}\right) = -\left(\frac{\partial F_1}{\partial t} + H - \frac{1}{2} + \lambda^\sigma \frac{\partial F_\sigma}{\partial t}\right).$$

Hence the corresponding function K is obtained by substituting in the quantity in parenthesis the expressions for q^i, p_i and λ^σ obtained from the equations $F_2 = \cdots = F_r = 0$ and the first equations of (5.22). Then the transforms of equations (1.13) are*

$$\frac{dq'^i}{dt} = \frac{\partial K}{\partial P_i'}, \qquad \frac{dP_i'}{dt} = -\frac{\partial K}{\partial q'^i}.$$

6. **Natural dynamical systems.** When the constraints of the dynamical system and the potential energy V do not involve t, we have that $g_{in+1} = g_{n+1\,n+1} = 0$ and that g_{ij} are independent of t. Then equations (1.3) are

$$g_{ij}\,\ddot{q}^j + [jk, i]\,\dot{q}^j\,\dot{q}^k + \frac{\partial V}{\partial q^i} = 0. \tag{6.1}$$

These equations admit the first integral

$$\frac{1}{2} g_{ij}\,\dot{q}^i\,\dot{q}^j + V = E, \tag{6.2}$$

where E is the energy constant.

The results of §§ 2 and 3 apply to this case directly, but in place of using these we consider a Riemannian space of $n+1$ dimensions, S_{n+1}, for which the fundamental form is

$$ds^2 = g_{ij}\,dq^i\,dq^j + A\,du^2, \tag{6.3}$$

where it is assumed that A does not involve u. The equations of the non-minimal geodesics of this space are readily obtained by replacing t by u in (2.4), (2.5) and (2.6), and putting $u = B = 0$ in the latter equations; this gives

$$g_{ij}\frac{d^2 q^j}{ds^2} + [jk, i]\frac{dq^j}{ds}\frac{dq^k}{ds} - \frac{1}{2}\frac{\partial A}{\partial q^i}\left(\frac{du}{ds}\right)^2 = 0, \tag{6.4}$$

$$A\frac{du}{ds} = a, \tag{6.5}$$

* Cf. Whittaker, l. c., p. 309, 310.

where a is a constant. For every finite value of a different from zero a parameter t for each non-minimal geodesic is defined by

$$t = as. \tag{6.6}$$

In terms of t as independent variable equations (6.4) and (6.5) become

$$g_{ij}\ddot{q}^j + [jk, i]\,\dot{q}^j\dot{q}^k - \frac{1}{2}\frac{1}{A^2}\frac{\partial A}{\partial q^i} = 0, \tag{6.7}$$

$$\frac{du}{dt} = \frac{1}{A}. \tag{6.8}$$

If the parameter t is identified with the time and A is defined by

$$\frac{1}{2A} = V + b, \tag{6.9}$$

where b is a constant, equations (6.7) are the same as (6.1). From (6.3), (6.6) and (6.8) we have

$$\frac{1}{a^2} = g_{ij}\dot{q}^i\dot{q}^j + \frac{1}{A} = 2(E+b). \tag{6.10}$$

When $|E|$ has an upper-bound, b can be chosen so that equation (6.10) is consistent; in this case it gives the relation between the constant a along a geodesic and the energy constant E for the corresponding trajectory. Then the non-minimal geodesics of S_{n+1} are defined by

$$q^i = f^i(t), \qquad u = 2\int V\,dt + 2bt, \tag{6.11}$$

the first n of these equations being the solutions of (6.1). In consequence of (6.2) the last of equations (6.11) may be written in the form

$$u = -2\int T\,dt + 2(E+b)\,t, \tag{6.12}$$

which reveals the relation between the coördinate u and the action.

The determinant of the form (6.3) is Ag, where g is the determinant of g_{ij}. Making use of the notation (3.1) as applied to (6.3), where now α and β take the values 1 to $n+1$, and $q^{n+1} = u$, we have in place of (3.2)

$$\bar{g}^{ij} = g^{ij}, \qquad \bar{g}^{i\,n+1} = 0, \qquad \bar{g}^{n+1\,n+1} = \frac{1}{A} = 2(V+b). \tag{6.13}$$

In this case equation (3.3) becomes

$$g^{ij}\frac{\partial\varphi}{\partial q^i}\frac{\partial\varphi}{\partial q^j} + 2(V+b)\left(\frac{\partial\varphi}{\partial u}\right)^2 = 1. \tag{6.14}$$

For equations of the form (3.4) for $\alpha = n+1$ we have

$$\frac{du}{ds} = 2(V+b)\frac{\partial \varphi}{\partial u}.$$

Comparing this result with (6.5), we have that the geodesics for a given value of a are the orthogonal trajectories of the hypersurfaces $\varphi = \text{const.}$, when

$$\varphi = a(u+\psi), \tag{6.15}$$

where ψ is independent of u. On substituting this expression in (6.14), we find that ψ must be a solution of

$$g_{ij}\frac{\partial \psi}{\partial q^i}\frac{\partial \psi}{\partial q^j} + 2(V+b) = \frac{1}{a^2}. \tag{6.16}$$

Hence if we have a solution of this equation involving $n-1$ constants a^σ, other than a, none of which is additive, then (6.15) defines a complete integral of (6.14) and the geodesics of S_{n+1} are given by

$$u+\psi+a\frac{\partial \psi}{\partial a} = c, \qquad \frac{\partial \psi}{\partial a^\sigma} = c_\sigma, \tag{6.17}$$

where the c's are arbitrary constants.

In consequence of (6.10) equation (6.16) may be written

$$g^{ij}\frac{\partial \psi}{\partial q^i}\frac{\partial \psi}{\partial q^j} = 2(E-V). \tag{6.18}$$

Hence if we have a solution of this equation involving $n-1$ arbitrary constants a^σ, none of which is additive, the last set of equations (6.17) give in finite form the trajectories for the given value of the energy E.

For an S_{n+1} with the fundamental form (6.3) we have in place of (5.13), in consequence of (6.6) and (6.5),

$$P_i = g_{ij}\frac{dq^j}{ds} = ap_i, \qquad P_{n+1} = A\frac{du}{ds} = a,$$

and in place of (5.16), because of (6.9) and (1.11)

$$C^2 = g^{ij}P_iP_j + \frac{1}{A}P_{n+1}^2 = 2a^2(H+b). \tag{6.19}$$

In view of these results equations (5.6) for $\alpha = 1, \cdots, n$ reduce to (1.13) and for $\alpha = n+1$ to (6.5).

If we take any homogeneous contact transformation such that the equation of the first set of (5.8) for $\alpha = n+1$ is of the form (5.18), where f and

the other functions of both sets do not involve u, it follows from the first of equations (5.7) for $\beta = n+1$ that $P'_{n+1} = P_{n+1} = a$. If then we replace P'_i by ap'_i, equation (5.9) becomes

$$p_i\, dq_i = p'_i\, dq'^i + df. \tag{6.20}$$

In this case the corresponding equations (5.10) are of the form (5.21), in which the F's do not involve u, u', t and t', and in place of (5.22) we have

$$p_i = \frac{\partial F_1}{\partial q^i} + \lambda^\sigma \frac{\partial F_\sigma}{\partial q^i}, \qquad p'_i = -\frac{\partial F_1}{\partial q'^i} - \lambda^\sigma \frac{\partial F_\sigma}{\partial q'^i}. \tag{6.21}$$

Equations (5.12) reduce to

$$\frac{dq'^i}{dt} = \frac{\partial H'}{\partial p'^i}, \qquad \frac{dp'^i}{dt} = -\frac{\partial H'}{\partial q'^i}, \qquad \frac{du'}{dt} = \frac{1}{A'}, \tag{6.22}$$

where H' and A' are the transforms of H and A. Conversely, if we have a set of equations (5.21) of the type under consideration defining a contact transformation, then equations (1.13) are transformed into the first two sets of (6.22).*

* Cf. Whittaker, l. c., p. 305.

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ON THE BRIOSCHI NORMAL QUINTIC.*

By Raymond Garver.

A number of transformations have been devised† which serve to reduce a sufficiently general quintic equation to the Brioschi normal form

$$y^5+10\,Cy^3+45\,C^2y+k=0, \tag{1}$$

with the aid of no irrationalities other than two square roots. The most direct of these is the transformation given first by Gordan and improved subsequently by Weber and Dickson. For this to be effective, however, a number of restrictions have to be made in the course of the work, and the significance of these restrictions seems not to be considered in the literature. The purpose of the present paper is to discuss these restrictions and to investigate the nature of the quintics to which the Gordan-Weber-Dickson transformation cannot be applied. The following theorems will be proved:

Theorem 1. *Any quintic without a double root can be reduced to* (1) *by a Tschirnhaus transformation involving at most two square roots and no further irrationality.*

Theorem 2. *Any quintic with a triple root can be reduced to* (1) *by a transformation involving a single square root and no further irrationality.*

Thus the only exceptional cases are quintics having a double root which is not a triple root. Theorem 1 does not, of course, state that such quintics cannot be reduced to the Brioschi form, but it seems quite improbable that they can by any transformation which involves only square roots, since if (1) has a double root, that root is also a triple root.

To prove the theorems of this paper it will first be necessary to outline the transformation mentioned in the first paragraph. I shall follow Dickson's presentation, omitting details which are not essential in the present work.

Consider the general quintic in the form

$$f(x)=x^5+c_3\,x^3+c_4\,x+c_5=0, \tag{2}$$

to which it can be reduced by a Tschirnhaus transformation involving at most one square root. Assume s_3,‡ and hence c_3, different from zero. Then it is easily verified that the functions

* Received October 8, 1928.

† For references, see these Annals (2), vol. 29 (1928), p. 319–333. In particular, see Gordan, Math. Annalen, vol. 28 (1887), p. 152–166; Weber, Algebra, 2nd ed., vol. I (1898), chap. VI; Dickson, Modern Algebraic Theories (1926), chap. XII.

‡ s_k means the sum of the kth powers of the roots of (2).

(3) $\quad g(x) = x^3 - s_4 x^2/s_3 - s_3/5, \qquad h(x) = x^4 - s_5 x^2/s_3 - s_4/5$

have the properties $\sum g = \sum h = \sum xg = \sum xh = 0$, where $\sum g$ means $\sum g(x_i)$, the x_i being the 5 roots of (2). The function

$$\psi(x) = zg + wh \tag{4}$$

will have the property $\sum \psi^2 = 0$ if

$$z^2 \sum g^2 + 2zw \sum gh + w^2 \sum h^2 = 0. \tag{5}$$

If $\sum g^2 = 0$, (5) may be satisfied by taking $z = 1$, $w = 0$; otherwise, w may be taken equal to 1, and z chosen as a root of the resulting quadratic. Then x and ψ have the properties

$$\sum x = \sum x^2 = \sum \psi = \sum \psi^2 = \sum x\psi = 0. \tag{6}$$

It is now possible to prove the existence of constants p, q, r, a, b (p and q not both zero) such that*

$$p\psi^2 + 2qx\psi + rx^2 - a\psi - bx \equiv f(x)\,P(x), \tag{8}$$

where $P(x)$ is a polynomial in x. The details of the proof are not needed here.

If F is used to represent $p\psi^2 + 2qx\psi + rx^2$, the identity†

$$mF \equiv (d\psi + ex)^2 - c(a\psi + bx)^2 \tag{11}$$

holds, where

$$m = pb^2 - 2qab + ra^2, \quad c = q^2 - pr, \quad d = bp - aq, \quad e = bq - ar. \tag{12}$$

The transformation which reduces the principal quintic (2) to form (1) can now be set up; it is

$$y = \frac{d\psi + ex}{a\psi + bx}. \tag{13}$$

The transformed equation is (1), with $C = -c/3$. The details of setting up the transformed equation are given by Dickson, and will not be repeated here. It is necesssary, however, to mention that the following equations are used:

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* The next few equations are numbered as in Dickson.

† Dickson's introduction of linear functions of ψ and x is unnecessary, since they cannot be made to serve the purpose desired. It is not difficult to show that the sum of the linear terms in his (10) is the same as in (8); hence his statement on page 216, lines 8–11, is incorrect. Certain other statements on page 216 must then be modified; what seem to be suitable modifications are incorporated in the paragraph to which this footnote refers.

$$y+c^{1/2}=\frac{d\psi+ex+c^{1/2}(a\psi+bx)}{a\psi+bx}, \tag{14}$$

$$m(a\psi+bx)=(d\psi+ex)^2-c(a\psi+bx)^2-mf(x)\,P(x), \tag{15}$$

$$\frac{m}{y+c^{1/2}}=d\psi+ex-c^{1/2}(a\psi+bx)+R(x), \tag{16}$$

where $R(x)$ vanishes for all the roots of (2). Equation (14) follows at once from (13); (15) comes from (8) and (11), while (16) is derived from (14) and (15).

But now for equations (13)–(16) to be valid it is evident that certain conditions must be imposed; it is with these conditions and the quintics that they characterize that the present article deals. The reader will easily see that if any of the following cases should arise the above work would not be valid, at least not without modification. First, if $m=0$, then $bd-ae=0$ and (13) is not a valid transformation. Second, if $a\psi+bx$ vanishes for one or more roots of $f(x)=0$, then (13) cannot be used. Equation (16) cannot be used if, third, the numerator of the right-hand side of (14) is identically zero, or, fourth, the same numerator vanishes for a root of $f(x)=0$. Fifth, if $a=b=0$, (13) can certainly not be used. Sixth, while $c=0$* does not affect the truth of (14)–(16), it does affect the later work, since the transformed equation cannot be set up unless $c^{1/2}$ has two values. The reader may consult Dickson, page 217, on this point. It may also be added here that while Weber and Dickson naturally recognize the existence of exceptional cases, they neither list them fully nor attempt to study them.

Of the above six cases, the third implies the first, by (11), while the fourth reduces to either the first or second, by (15). These two will no longer be considered separately. The fifth obviously implies both the first and the second, but it seems desirable to keep it separate from them. In fact, since the fifth and sixth cases are easily handled it will be assumed for the moment that neither of them holds, and also that $s_3\neq 0$; this last restriction was imposed just before equation (3). There then remain for consideration the first two cases.

That there is a close relation between the two appears from the fact that, if a is different from zero, (8) can be written in the form

$$\left(\frac{p}{a}\psi+\left(\frac{2q}{a}-\frac{pb}{a^2}\right)x-1\right)(a\psi+bx)+\frac{m}{a^2}x^2\equiv f(x)\,P(x). \tag{17}$$

* Finally, (13) would not be a valid transformation if $d=e=0$; this, however, implies either $a=b=0$ or $c=0$.

If $m = 0$, $a\psi + bx$ must vanish for at least one root of $f(x) = 0$, since the other factor on the left-hand side of (17) is of not higher than the fourth degree in x. On the other hand, if $a\psi + bx$ vanishes for a root of $f(x) = 0$, either $m = 0$ or $f(x) = 0$ has a double root. To prove this, note that, if $m \neq 0$, x must vanish for the same root as does $a\psi + bx$, that is, c_5 in (2) must be zero. Then ψ itself must vanish for $x = 0$. If $\psi = g$, this requires $s_3 = 0$, which contradicts one of our present restrictions. If $\psi = zg + h$, its constant term $(-zs_3/5 - s_4/5)$ must vanish, or z must equal $-4c_4/3c_3$. However, z is a root of (5), and this value satisfies (5) only if $27c_3^4 c_4^2 = 256c_4^5$, which is the condition that (2) with $c_5 = 0$ have a double root. To show this requires the computation of the coefficients of (5); these will be given here:

$$
\begin{aligned}
\sum g^2 &= 6c_3^2/5 + 40c_4 c_5/3c_3 - 64c_4^3/9c_3^2, \\
\sum gh &= 3c_3 c_4/5 - 80c_4^2 c_5/9c_3^2 + 25c_5^2/3c_3, \\
\sum h^2 &= 4c_4^2/5 - 2c_3 c_5 - 100c_4 c_5^2/9c_3^2.
\end{aligned} \tag{18}
$$

For the present purpose, c_5 is to be taken equal to zero.

If $a = 0$, the work of the preceeding paragraph can be carried through with but slight modification. It is merely necessary to note that (8) can then be written in the form

$$
\left(\frac{2q}{b}\psi + \frac{r}{b}x - 1\right)(bx) + \frac{m}{b^2}\psi^2 \equiv b(x)\,P(x). \tag{19}
$$

It has now been established that, except for certain quintics with a double root which will be laid aside, the vanishing of m and the vanishing of $a\psi + bx$ for at least one root of $f(x) = 0$ are equivalent. The exceptional cases being sought will then arise from the three following possibilities:

I. $m = 0$; $a\psi + bx$ vanishes for only one root of $f(x) = 0$.
II. $m = 0$; $a\psi + bx$ vanishes for only two roots of $f(x) = 0$.
III. $m = 0$; $a\psi + bx$ vanishes for more than two roots of $f(x) = 0$.

It is not difficult to show that Case I cannot arise. For $m = 0$, a, $b \neq 0$, the first factor on the left side of (17) can be written $p\psi/a + rx/b - 1$, and it must vanish for four roots of $f(x) = 0$. (If $b = 0$, we use instead $p\psi/a + 2qx/a - 1$; if $a = 0$, the first factor on the left side of (19).) Hence if we set $y = p\psi/a + rx/b - 1$ the transformed equation must then be of the form $y^5 + ky^4 = 0$. However, directly from the transformation, $\sum y = -5$, $\sum y^2 = 5$. This gives $k = 5$, but also gives a non-vanishing term in y^3. Thus a contradiction is arrived at.

Case II gives a type of quintic to which the Gordan-Weber-Dickson transformation obviously does not apply. Such quintics can, however, be reduced to the Brioschi normal form by means of a simple transformation which will be set up. First apply the transformation of the last paragraph, where $p\psi/a+rx/b-1$ now vanishes for three roots of $f(x)=0$. Using $\sum y=-5$, $\sum y^2=5$, we find the transformed equation to be $y^5+5y^4+10y^3=0$. If we now make a second transformation $v=(y-1)/(y+1)$, which is always permissible, the transformed equation in v is easily found to be $v^5-10v^3/3+5v+8/3=0$, which is of form (1) with $C=-1/3$. That there actually exist quintics falling under Case II is shown by the numerical example $x^5-20x^2/9-5x/3-34/27=0$. For this quintic, $\psi=g$ and (8) can be set up explicitly without a great deal of trouble. It is found that $p=1$, $2q=-1$, $r=-4/9$, $a=5/9$, $b=-20/27$, and hence $m=0$. The value of y for this case is $9(x^3-x^2+x/3-17/9)/5$, and it is easily verified that this vanishes for three roots of the quintic. The details of the transformation can also be set up directly, since the other two roots of the quintic are the roots of $x^2+x+2/3=0$, and since for these two roots the transformation reduces to $y=3x-1$.

A method similar to the above can be used to prove Theorem 2 of the present paper. Any principal quintic with a triple root is of the form $(x-k)^3(x^2+3kx+6k^2)=0$, which can be reduced to $y^5+5ky^4+10k^2y^3=0$ by setting $y=x-k$. The transformation $v=(y-k)/(y+k)$ then leads to (1) with $C=-1/3$. The single square root mentioned in Theorem 2 is that required (in general) to reduce the given quintic to form (2).

If the conditions of Case III are satisfied the quintic must have a double root. To prove this, note that, if $a\psi+bx$ vanishes for at least three roots of $f(x)=0$, it vanishes for all five roots, since $\sum(a\psi+bx)=\sum(a\psi+bx)^2=0$, by (6). However, $a\psi+bx$ is of not higher than the fourth degree in x; accordingly, $f(x)=0$ must have a double root. Transformation (13) does not apply to quintics coming under Case III, and as has been pointed out it is hardly to be expected that any transformation involving no irrationalities other than square roots will serve to reduce them to form (2). This does not mean that the reduction cannot be secured, since a quintic with a double root can be transformed into one with a triple root, and thence into the Brioschi form by Theorem 2. However, the first transformation will involve higher irrationalities than square roots. The present paper does not attempt to consider such transformations.

The three restrictions made in the paragraph preceeding (17) remain to be treated. First, if $c=0$, equation (8) says that

$$(p\psi + qx)^2 = p(a\psi + bx) \tag{20}$$

for all five roots of $f(x) = 0$. Hence, if the transformation $y = p\psi + qx$ is applied, it is seen, using (6) and (20), that $\sum y = 0$, $\sum y^2 = 0$, $\sum y^3 = 0$, $\sum y^4 = 0$. The transformed equation is then in the Brioschi normal form with $C = 0$, and $c \neq 0$ is thus not an essential restriction. Incidentally, quintics for which $c = 0$ have the interesting property, just proved, that they can be reduced rationally to the binomial form $y^5 + k = 0$. However, a further study of this property is not a part of this paper.

Next, quintics for which $a = b = 0$ must have a double root. For, if $p \neq 0$, equation (8) can be written $(\psi - k_1 x)(\psi - k_2 x) \equiv f(x) P(x)$, and either $\psi - k_1 x$ or $\psi - k_2 x$ must vanish for at least three roots of $f(x) = 0$. By the same argument that was used in connection with Case III above, it then vanishes for all five roots, and $f(x) = 0$ has a double root. If $p = 0$, a similar argument applies when the left side of (8) is factored differently. The proof that such quintics have a double root is sufficient from the standpoint of Theorem 1. Two further statements, however, may be of interest, the proof of which is too detailed to repeat. First, the conditions $\psi = g$, $a = b = 0$ can be shown to characterize certain quintics with two double roots. Second, if $\psi = zg + h$ and if (8) is set up by the method outlined by Dickson, a and b will both be zero if and only if $c_4 = c_5 = 0$ in (2).

Finally the restriction $s_3 \neq 0$ is rather obviously one of convenience, and not of necessity. If $s_3 = 0$, $s_4 = 0$, the equation is already in form (1) with $C = 0$. If $s_4 \neq 0$, we may return to (3), take $g = x^2$, $h = x^4 - s_5 x^3/s_4 - s_4/5$, and go through a discussion almost exactly like the one outlined in this paper. Again details will be omitted; it will merely be mentioned that the first case $z = 1$, $w = 0$ after equation (5) cannot arise in this discussion, and that, finally, the only cases which have to be excluded are quintics with a double root. This completes the proof of Theorem 1.

ON FIVE MUTUALLY ORTHOGONAL SPHERES.*

By Nathan Altshiller-Court.

Introduction. Five mutually orthogonal proper spheres are considered chiefly in their relation to the spheres and tetrahedrons determined by their centers. To abbreviate the statements of some of the results obtained the tetrahedron and the sphere determined by the centers of any four of the given spheres will be referred to, respectively, as the "central tetrahedron" and the "central sphere" associated with the fifth given sphere.

1. Theorem. *If three spheres are mutually orthogonal, the centers of any two of them are a pair of conjugate points with respect to the third sphere.*

The two given spheres (B), (C) being orthogonal to each other the sphere (BC) described on their line of centers BC as diameter is coaxal with them. Now the third given sphere (A) is orthogonal to (B), (C), by assumption, it is therefore also orthogonal to (BC), hence the ends B, C, of diameter of the latter sphere are conjugate with respect to (A).

2. (a). *If four spheres are mutually orthogonal the centers of any three of them determine a triangle conjugate with respect to the fourth sphere.*

(b). *If five spheres are mutually orthogonal, each of them is the conjugate sphere of the tetrahedron determined by the centers of the remaining four spheres.*

These propositions follow immediately from the preceding theorem (1).

3. (a). If a tetrahedron is conjugate with respect to a sphere, the tetrahedron is orthocentric, and the center of the sphere coincides with the orthocenter of the tetrahedron, hence (2a): *The centers of five mutually orthogonal spheres form an orthocentric group of points.*†

(b) Consequence. A given group of orthocentric points are the centers of one and only one group of five mutually orthogonal spheres (2b, 3a).

4. The converse of the last two propositions (2b and 3a) is true, namely: *The five conjugate spheres of the five tetrahedrons determined by an orthocentric group of five points are mutually orthogonal.*

Let A_1, A_2, A_3, A_4, A_5 be the points of the orthocentric group, and (A_5), (A_4) the conjugate spheres of the tetrahedrons $A_1 A_2 A_3 A_4$, $A_1 A_2 A_3 A_5$, respectively. If a_5, a_4 are the squares of the radii of

* Received February 4, 1929.

† J. L. Coolidge, A treatise on the circle and the sphere, p. 257, theorem 5. Oxford, 1916.

these two spheres and $H_{4,5}$ the trace of the line A_4A_5 in the plane $A_1A_2A_3$ we have, both in magnitude and in sign,

$$a_5 = A_5A_4 \cdot A_5H_{45}, \qquad a_4 = A_4A_5 \cdot A_4H_{4,5}$$

hence

$$a_4 + a_5 = A_4A_5(A_4H_{45} - A_5H_{4,5}) = A_4A_5^2.$$

Thus the square of the line of centers of (A_4), (A_5) is equal to the sum of the squares of the radii of these two spheres, hence the two spheres are orthogonal. Now the above proof is valid whether a_4 and a_5 are both positive, or one these quantities is positive and the other negative. Hence the above considerations are applicable to any two of the five spheres conjugate to the five tetrahedrons determined by the five given points, and the proposition is proved.

5. LEMMA. *The inverse of the square of the radius of the circle common to two orthogonal spheres is equal to the sum of the inverses of the squares of the radii of the two spheres.*

Let P, Q be the centers, p, q the squares of the radii of two orthogonal spheres (P), (Q), and H the trace of the radical plane of the two spheres on their line of centers PQ. The points H, Q are invere with respect to (P), and the points H, P are inverse with respect to (Q), hence we have, both in magnitude and in sign,

$$p = PH \cdot PQ, \qquad q = QH \cdot QP$$

and

$$\text{(a)} \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{PQ}\left(\frac{1}{PH} + \frac{1}{HQ}\right) = \frac{1}{PH \cdot HQ}.$$

The power r of the point H with respect to each of the two spheres (P), (Q) is the same, hence we have, both in magnitude and in sign,

$$\text{(b)} \qquad h = HP^2 - p = HQ^2 - q, \qquad p + q = PQ^2$$

and

$$\text{(c)} \qquad h = \frac{1}{2}[HP^2 + HQ^2 - (PH + HQ)^2] = -PH \cdot HQ.$$

Now h being the power of H with respect to (P) and (Q), it is also the power of H with respect to the circle (H) common to (P) and (Q), and since H is the center of (H), we have $r = -h$, (d), where r is the square of the radius of (H). The lemma follows from (a), (b), (c), (d).

From the method of proof it follows that the lemma is valid whether p and q are both positive, or one of these quantities is negative.

6. Consider five mutually orthogonal spheres (A_i) $(i = 1, 2, 3, 4, 5)$. The spheres (A_1), (A_2), (A_3) being orthogonal to the spheres (A_4), (A_5), the

plane $A_1 A_2 A_3$ determined by the centers of the first three spheres is the radical plane of the last two, hence the spheres (A_4), (A_5) are cut by the plane $A_1 A_2 A_3$ along the same circle $(H_{4,5})$.

The plane $A_1 A_2 A_3$ cuts the three spheres (A_1), (A_2), (A_3), along three great circles which are mutually orthogonal and which are also orthogonal to the circle of intersection (H_{45}) of the same plane with the sphere (A_4). Now, according to the lemma (5), the square of the radius h_{45} of (H_{45}) is expressible in terms of the squares of the radii of the spheres (A_4), (A_5), hence from every relation among the squares of the radii of these four mutually orthogonal circles may be derived a relation among the squares of the radii of the five given spheres. Thus the relation*

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{h_{45}}=0$$

gives (5)

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5}=0.$$

Thus: The sum of the inverses of the squares of the radii of five mutually orthogonal spheres is zero.†

7. Let h_{ij} denote the square of the radius of the circle common to the two spheres (A_i), (A_j) of the group of five mutually orthogonal spheres (6). We have (5 and 6):

$$\frac{1}{h_{12}}+\frac{1}{h_{13}}+\frac{1}{h_{14}}+\frac{1}{h_{15}}=3\cdot\frac{1}{a_1}+\sum_{i=1}^{5}\frac{1}{a_i}=3\cdot\frac{1}{a_1}.$$

And again

$$\frac{1}{2}\sum_{i=1}^{5}\sum_{j=1}^{5}\frac{1}{h_{ij}}=\frac{1}{2}\sum_{i=1}^{5}\sum_{j=1}^{5}\left(\frac{1}{a_i}+\frac{1}{a_j}\right)=2\sum_{i=1}^{5}\frac{1}{a_i}=0.$$

Thus: (a) *The sum of the inverses of the squares of the radii of the four circles along which one of five mutually orthogonal spheres is cut by the other four spheres is equal to three times the inverse of the square of the radius of the first sphere.* (b) *The sum of the inverses of the squares of the radii of the ten circles along which five mutually orthogonal spheres cut each other is zero.*

8. The tetrahedron $A_1 A_2 A_3 A_4$ is conjugate with respect to the sphere (A_5) (2b), and a_1, a_2, a_3, a_4 are the powers of the vertices A_1, A_2,

*) Nathan Altshiller-Court, On four mutually orthogonal circles. Annals of Math., vol. 29 (1928), p. 371.

† Coolidge, ibid., p. 257, theorem 8.

A_3, A_4 with respect to (A_5). If V_5 denotes the volume of this tetrahedron, we have*

(d) $$a_1\, a_2\, a_3\, a_4 = -36\, V_5^2\, a_5.$$

This formula may be interpreted as follows: *The square of the radius of any one of five mutually orthogonal spheres is equal to the product, with the sign changed, of the squares of the radii of the remaining four spheres divided by thirty six times the square of the volume of the tetrahedron determined by the centers of these four spheres.*

9. The formula (d) of the preceding paragraph (8) may be put into the form

(e) $$-a_1\, a_2\, a_3\, a_4\, a_5 = 36\, V_5^2\, a_5^2,$$

hence: *Given five mutually orthogonal spheres, the product of the fourth power of the radius of any one of them by the square of the volume of its associated central tetrahedron is constant.*

10. (a) Consider, for a while, four mutually orthogonal circles (A_i) $(i = 1, 2, 3, 4)$; let a_i be the square of the radius of (A_i) and S_i the area of the triangle determined by the centers of the remaining three circles. Then we have†

$$a_1\, a_2\, a_3 = -4\, S_4^2\, a_4, \quad \text{or} \quad a_1\, a_2\, a_3\, a_4 = -4\, S_4^2\, a_4^2$$

and three analogous formulas which when multiplied together give

$$(a_1\, a_2\, a_3\, a_4)^4 = (-1)^4\, 4^4\, (S_1\, S_2\, S_3\, S_4)^2 \cdot (a_1\, a_2\, a_3\, a_4)^2,$$

and this relation may be written

$$\prod_{i=1}^{4} \frac{a_i^2}{S_i^2} = (-1)^4\, (2!)^{4\times 2}.$$

Thus: *Given four mutually orthogonal circles, consider the ratio of the square of the radius of each circle to the area of the triangle determined by the centers of the remaining three circles. The product of the four ratios thus formed is a numerical constant independent of the group of circles considered. The constant is equal to sixteen.*

(b). Now returning to the consideration of the spheres, let V_i denote the volume of the tetrahedron associated with the sphere (A_i). The formula (e) of the preceding paragraph (9) gives

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* Nathan Altshiller-Court, Sur la sphère conjuguée à un tétraèdre orthocentrique. Mathesis (1928), p. 340.

† Nathan Altshiller-Court, Sur le cercle conjugué par rapport à un triangle. Mathesis (1928), p. 307, § 12. Also Annals, ibid., p. 369.

$$(a_1\, a_2\, a_3\, a_4\, a_5)^3 = -36^5\,(V_1\, V_2\, V_3\, V_4\, V_5)^2$$

and this relation may be written on the form

$$\prod_{i=1}^{5} \frac{a_i^3}{V_i^2} = (-1)^5\,(3!)^{5\times 2}.$$

Thus: *Given five mutually orthogonal spheres, consider the ratio of the sixth power of the radius of each sphere to the square of the volume of the tetrahedron determined by the centers of the remaining four spheres. The product of the five ratios thus formed is a numerical constant independent of the group of spheres considered.* This constant is equal to -6^{10}.

Remark. The form of the formula $\prod$ established in this paragraph for the two and the three dimensional spaces suggests an extension of it to spaces of higher dimensions.

11. Let R_i be the circumradius of the central tetrahedron associated with the sphere (A_i). We have*

$$R_5^2 = a_5 + \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$$

and four analogous formulas. Hence

$$\sum_{i=1}^{5} R_i^2 = 2 \sum_{i=1}^{5} a_i.$$

Thus: *The sum of the squares of the radii of five mutually orthogonal spheres is equal to half the sum of the squares of the radii of their five associated central spheres.*

12. Since (A_5) is the conjugate sphere of the tetrahedron $A_1\, A_2\, A_3\, A_4$ (2b), we have $a_5 = A_5\, A_4 \cdot A_5\, H_{45}$. Now the line $A_4\, A_5\, H_{45}$ is the altitude of the tetrahedon $A_1\, A_2\, A_3\, A_4$, hence the segment $A_5\, H_{45}$ is equal to one third of the segment determined by A_5 and the second point of intersection (besides A_4) of $A_4\, H_{45}$ with the circumsphere (O_5) of $A_1\, A_2\, A_3\, A_4$. Thus if p_5 denotes the power of A_5 with respect to (O_5), we have

$$p_5 = A_5\, A_4 \cdot 3\, A_5\, H_{45} = 3\, A_5\, A_4 \cdot A_5\, H_{45} = 3\, a_1.$$

Hence: *The square of the radius of any one of five mutually orthogonal spheres is equal to one third of the power of the center of this sphere with respect to the sphere determined by the centers of the remaining four spheres.*

13. We have (12)

$$p_1 = 3\, a_1, \qquad p_2 = 3\, a_2, \qquad \text{etc.},$$

* Mathesis, ibid., p. 342, formula L.

hence (6)

$$\sum_{i=1}^{5} \frac{1}{p_i} = \frac{1}{3} \sum_{i=1}^{5} \frac{1}{a_1} = 0.$$

Thus: *Given five mutually orthogonal spheres, the sum of the inverses of the squares of the radii of the five spheres concentric with the given spheres and orthogonal to their respective associated central spheres, is zero.*

14. The power of the centroid G_5 of the tetrahedron $A_1 A_2 A_3 A_4$ with respect to the circumsphere (O_5) of this tetrahedron is equal to* $\frac{1}{16}(a_1+a_2+a_3+a_4)$, hence we have

$$G_5 A_5^2 = a_5 + \frac{1}{16}(a_1+a_2+a_3+a_4)$$

and therefore

$$\sum_{i=1}^{5} G_i A_i^2 = \frac{5}{4} \sum_{i=1}^{5} a_i.$$

Thus: *Given five mutually orthogonal spheres, the sum of the squares of the distances of their centers from the centers of gravity of their related central tetrahedrons is equal to five fourths of the sum of the squares of the radii of these spheres.*

15. If O_5 denotes the circumcenter of the tetrahedron $A_1 A_2 A_3 A_4$, we have† $O_5 A_5 = 2 G_5 A_5$, hence (14)

$$\sum_{i=1}^{5} O_i A_i^2 = 5 \sum_{i=1}^{5} a_i$$

Thus: *Given five mutually orthogonal spheres the sum of the squares of the distances of their centers to the centers of their associated central spheres is equal to 5 times the sum of the squares of the radii of the given spheres.*

16. If q_5 denotes the power of the center O_5 with respect to the sphere (A_5), we have $q_5 = O_5 A_5^2 - a_5$, hence (15)

$$\sum_{i=1}^{5} q_5 = 4 \sum_{i=1}^{5} a_i.$$

Thus: *Given five mutually orthogonal spheres, the sum of the powers, with respect to these spheres, of the centers of their respective associated central spheres is equal to four times the sum of the squares of the radii of the five given spheres.*

17. The center of gravity L of the five points A_1, A_2, A_3, A_4, A_5, divides the segment $G_5 A_5$ into two parts such that

* Mathesis, ibid., p. 341, formula F.

† Coolidge, ibid., p. 237.

$$G_5 L : L A_5 = 1 : 4, \quad \text{hence} \quad L A_5 : G_5 A_5 = 4 : 5,$$

hence (14)

$$\sum_{i=1}^{5} L A_i^2 = \frac{16}{25} \sum_{i=1}^{5} G_i A_i^2 = \frac{4}{5} \sum_{i=1}^{5} a_5.$$

Thus: *The sum of the squares of the distances of the centers of five mutually orthogonal spheres from the common center of gravity of these five points is equal to four fifths of the sum of the squares of the radii of these spheres.*

18. If l_5 denotes the power of L with respect to the sphere (A_5), we have $l_5 = L A_5^2 - a_5$, hence (17)

$$\sum_{i=1}^{5} l_i = -\frac{1}{5} \sum a_i.$$

Thus: *Given five mutually orthogonal spheres, the sum of the powers, with respect to these spheres, of the center of gravity of their centers is equal to the arithmetic means of the squares of the radii of these spheres, with the sign changed.*

19. The four spheres (A_1), (A_2), (A_3), (A_4) taken in pairs have twelve centers of similitude which lie by sixes in twelve planes.* Four of those planes are those determined by the points A_1, A_2, A_3, A_4. The remaining eight planes may be referred to, by analogy with the case in the plane, as the "planes of similitude" of the four spheres considered.

Let σ_i be one of these eight planes of similitude and M_i its pole with respect to the tetrahedron A_1, A_2, A_3, A_4. The trace M_{i4} of the line $A_4 M_i$ in the plane $A_1 A_2 A_3$ is the trilinear pole of the line $s_{i4} = (\sigma_i, A_1 A_2 A_3)$ with respect to the triangle $A_1 A_2 A_3$. But the line s_{i4} contains three centers of similitude of the spheres (A_1), (A_2), (A_3), hence this line is an axis of similitude of the three mutually orthogonal circles along which the plane $A_1 A_2 A_3$ cuts the spheres (A_1), (A_2), (A_3), therefore the point M_{i4} is the pole of the line s_{i4} with respect to the circle orthogonal to these three circles,† which orthogonal circle is the section of the sphere (A_5) by the plane $A_1 A_2 A_3$. Consequently the polar plane of M_{i4} with respect to (A_5) passes through s_{i4}, and since s_{i4} lies in the polar plane $A_1 A_2 A_3$ of A_4 with respect to (A_5), the line s_{i4} is the conjugate line of $A_4 M_{i4}$ with respect to (A_5).

Thus the pole of the plane σ_i with respect to the sphere (A_5) must lie on the line $A_4 M_{i4}$. Similarly for the analogous lines $A_1 M_{i1}$, $A_2 M_{i2}$, $A_3 M_{i3}$. But the common point of these four lines is the point M_i, hence: *Given five mutually orthogonal spheres, the pole of a plane of similitude of four*

* Coolidge, ibid., p. 230, theorem 23.

† Annals, ibid., p. 372.

of these spheres with respect to tetrahedron determined by their centers coincides with the pole of the same plane with respect to the fifth given sphere.

20. Conversely: *Given five mutually orthogonal spheres, if the pole of a plane with respect to one of these spheres coincides with the pole of the same plane with respect to the tetrahedron determined by the centers of the remaining four spheres, the plane is a plane of similitude of the last four spheres.*

For then the point M_{i4} is the pole of the line $s_{i,4}$ with respect to the circle of intersection of (A_5) with the plane $A_1 A_2 A_3$, and also the pole of the same line with respect to the triangle $A_1 A_2 A_3$, hence s_{i4} is an axis of similitude of the three great circles along which the spheres (A_1), (A_2), (A_3) are cut by the plane $A_1 A_2 A_3$,* and therefore of the spheres themselves. Similarly for the lines s_{i1}, s_{i2}, s_{i3}. Hence the proposition.

21. (a) The preceding propositions stated with reference to five mutually orthogonal spheres may, for the most part, be readily paraphrased so as to refer to (1) a group of orthocentric points; (2) an orthocentric group of tetrahedrons; (3) the conjugate spheres of an orthocentric group of tetrahedrons. For lack of space we shall limit ourselves just to one example (17, 11). *The sum of the squares of the distances of the five points of an orthocentric group from their common center of gravity is equal to two fifths of the sum of the squares of the radii of the five spheres determined by these points.*

(b) Furthermore some propositions which are of little interest when stated with reference to five mutually orthogonal spheres may deserve more attention in connection, say, with an orthocentric group of tetrahedrons. For instance, we have (12, 11)

$$\sum_{i=1}^{5} p_i = 3 \sum_{i=1}^{5} a_i = \frac{3}{2} \sum_{i=1}^{5} R_i^2 .$$

Hence: *The triple of the sum of the squares of the radii of the circumspheres of an orthocentric group of tetrahedrons is equal to the double of the sum of the squares of the radii of the five spheres respectively orthogonal to them and having for their centers the five vertices of the tetrahedrons.*

* Annals, ibid., p. 372.

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THE STRUCTURE OF ANY ALGEBRA WHICH IS A DIRECT PRODUCT OF RATIONAL GENERALIZED QUATERNION DIVISION ALGEBRAS.*

By A. Adrian Albert.

1. **Introduction.** Before Cecioni's discovery of normal division algebras in sixteen units which are non-cyclic in form attempts were made to construct such algebras by the construction of division algebras which were the direct products of two rational generalized quaternion algebras. Such attempts are here shown to have been futile as no such direct product is ever a division algebra. We also, in the nature of a corollary, give the structure of any algebra which is the direct product of n generalized quaternion division algebras.

2. **The direct products of two generalized quaternion division algebras over R.** We shall consider rational linear associative algebras, that is algebras over R, the field of all rational numbers. Let $A = B \times C$ be the direct product of the two division algebras

$$(1) \quad B = (1, i_1, j_1, i_1 j_1), \quad i_1 j_1 = -j_1 i_1, \quad i_1^2 = \alpha, \quad j_1^2 = \beta, \quad \alpha \text{ and } \beta \text{ in } R,$$

$$(2) \quad C = (1, i_2, j_2, i_2 j_2), \quad i_2 j_2 = -j_2 i_2, \quad i_2^2 = \gamma, \quad j_2^2 = \delta, \quad \gamma \text{ and } \delta \text{ in } R.$$

The algebra A is an associative algebra of order sixteen. It has a subalgebra Σ composed of all elements

$$(3) \qquad X = x_1 + x_2 i_2, \qquad x_1 \text{ and } x_2 \text{ in } B,$$

and the elementa $1, i_2$ are linearly independent with respect to B. Every element of A is expressible in form

$$(4) \qquad E_1 + E_2 j_2, \qquad E_1 \text{ and } E_2 \text{ in } \Sigma,$$

and the elements $1, j_2$ are left linearly independent with respect to Σ. If $i^2 = \mu/\nu$ where μ and ν are rational integers then, if $i' = \nu i$, $i'^2 = \mu\nu$ a rational integer. Hence we may assume without loss of generality that $\alpha, \beta, \gamma, \delta$ are all rational integers. We shall prove the following general theorem:

Theorem 1. *Let $A = B \times C$ where B and C are given by* (1) *and* (2) *and are rational division algebras. Then A is not a division algebra.*

* Received March 10, 1929.

We shall state without proof the known theorem of the theory of numbers* as Lemma 1.

Lemma 1. Consider the form

$$q = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + a_5 x_5^2 \tag{5}$$

where $a_1, \cdots, a_5$ are integers different from zero and not all having the same sign. Then there exist integers $x_1, \cdots, x_5$ not all zero for which $q = 0$.

Lemma 2. Let $\alpha < 0$, $\beta < 0$, $\gamma > 0$, $\delta > 0$. Then there exist rational numbers ξ_1, η_2, η_3, η_4 such that

$$\delta = \xi_1^2 - \alpha\gamma\eta_2^2 - \beta\gamma\eta_3^2 + \alpha\beta\gamma\eta_4^2. \tag{6}$$

Proof. Let $\alpha = -a$, $\beta = -b$, a and b greater than zero, $c = \alpha\beta$. The form

$$-\delta x_1^2 + x_2^2 + a\gamma x_3^2 + b\gamma x_4^2 + c\gamma x_5^2$$

is a form of the type 5 with $a_1 = -\delta < 0$, $a_2 = 1 > 0$ and hence, by lemma 1, there exist integers x_1, x_2, x_3, x_4, x_5 not all zero so that

$$-\delta x_1^2 + x_2^2 + a\gamma x_3^2 + b\gamma x_4^2 + c\gamma x_5^2 = 0.$$

If $x_1 = 0$ then

$$x_2^2 + a\gamma x_3^2 + b\gamma x_4^2 + c\gamma x_5^2 = 0$$

for $x_2, \cdots, x_5$ not all zero. But this is impossible since the sum of four positive or zero numbers not all zero is greater than zero. Hence $x_1 \neq 0$. Let $\xi_1 = x_2/x_1$, $\eta_2 = x_3/x_1$, $\eta_3 = x_4/x_1$, $\eta_4 = x_5/x_1$ and we have the expression

$$\delta = \xi_1^2 - \alpha\gamma\eta_2^2 - \beta\gamma\eta_3^2 + \alpha\beta\gamma\eta_4^2$$

and the lemma is proved.

Lemma 3. Let $\alpha < 0$, $\beta < 0$, $\gamma > 0$, $\delta > 0$. Select rational numbers ξ_1, η_2, η_3, η_4 as in lemma 2. Then the product of the two non-zero elements

$$E + j_2, \qquad F + j_2 \tag{7}$$

where

$$E = -\xi_1 + f i_2, \quad F = \xi_1 + f i_2, \quad f = \eta_2 i_1 + \eta_3 j_1 + \eta_4 i_1 j_1 \tag{8}$$

is zero and A is not a division algebra.

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* A theorem of A. Meyer, Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich, vol. 29 (1884), pp. 209–222; also cf. Bachmann, Zahlentheorie, vol. IV$_1$, p. 266.

Proof. We have

$$(E+j_2)(F+j_2) = EF+j_2^2+Ej_2+j_2F$$
$$= (-\xi_1+fi_2)(\xi_1+fi_2)+\delta+Ej_2+j_2F.$$

But

$$Ej_2+j_2F = (-\xi_1+fi_2)j_2+j_2(\xi_1+fi_2) = fi_2j_2+fj_2i_2 = 0.$$

Also

$$(-\xi_1+fi_2)(\xi_1+fi_2)$$
$$= -\xi_1^2+fi_2\xi_1-\xi_1fi_2+f^2i_2^2 = -\xi_1^2+\gamma(\eta_2i_1+\eta_3i_1+\eta_4i_1j_1)^2$$
$$= -\xi_1^2+\gamma[\eta_2^2i_1^2+\eta_2i_1(\eta_3j_1+\eta_4i_1j_1)+(\eta_3+\eta_4i_1)j_1\eta_2i_1+(\eta_3+\eta_4i_1)j_1(\eta_3+\eta_4i_1)j_1].$$

But $\eta_2^2 i_1^2 = \eta_2^2\alpha$ and

$$\eta_2i_1(\eta_3j_1+\eta_4i_1j_1)+(\eta_3+\eta_4i_1)j_1\eta_2i_1$$
$$= \eta_2i_1(\eta_3j_1+\eta_4i_1j_1)-\eta_2i_1(\eta_3+\eta_4i_1)j_1 = 0.$$

Finally

$$(\eta_3+\eta_4i_1)j_1(\eta_3+\eta_4i_1)j_1 = (\eta_3+\eta_4i_1)(\eta_3-\eta_4i_1)j_1^2 = (\eta_3^2-\eta_4^2i_1^2)j_1^2$$
$$= (\eta_3^2-\eta_4^2\alpha)\beta.$$

Hence

$$(-\xi_1+fi_2)(+\xi_1+fi_2) = -(\xi_1^2-\alpha\gamma\eta_2^2-\beta\gamma\eta_3^2+\alpha\beta\gamma\eta_4^2) = -\delta,$$

so that

$$(E+j_2)(F+j_2) = \delta-\delta = 0.$$

Consider next the case where not all of the hypotheses of Lemma 2 hold. We have

Lemma 4. Let not all of the hypotheses on $\alpha, \beta, \gamma, \delta$ of Lemma 2 hold. Then there exist integers x_1, x_2, x_3, x_4, x_5 not all zero such that

$$-\delta x_1^2-\gamma x_2^2+\alpha x_3^2+\beta x_4^2-\alpha\beta x_5^2 = 0. \tag{9}$$

Proof. If $\alpha>0$ and $\beta>0$ then $-\alpha\beta<0$ and the hypotheses of Lemma 1 are fulfilled and the lemma is proved. If α and β have different signs then the hypotheses of Lemma 1 are fulfilled since α and β occur as coefficients in (9) and the present lemma is true. It remains to consider $\alpha<0$, $\beta<0$. If $\gamma<0$ then the signs of $-\gamma$ and α are different and the lemma is again true. If $\gamma>0$ then $\delta<0$ since otherwise all of the hypotheses of Lemma 2 hold and the signs of $-\gamma$ and α are different. This proves the lemma.

Let first $x_1 = 0$ in the set of integers determined in Lemma 4. Let also $x_2 = 0$. Then not all of x_3, x_4 and x_5 are zero. The element $x = x_3i_1 + (x_4+x_5-i_1)j_1$ is thus different from zero. But its square is

$$x_3^2 i_1^2 + x_3 i_1 (x_4 + x_5 i_1) j_1 + (x_4 + x_5 i_1) j_1 x_3 i_1 + (x_4 + x_5 i_1) j_1 (x_4 + x_5 i_1) j_1$$
$$= \alpha x_3^2 + x_3 i_1 (x_4 + x_5 i_1) j_1 - x_3 i_1 (x_4 + x_5 i_1) j_1 + (x_4 + x_5 i_1)(x_4 - x_5 i_1) j_1^2$$
$$= \alpha x_3^2 + \beta x_4^2 - \alpha\beta x_5^2 = 0.$$

Thus B contains an element $x \neq 0$ whose square is zero, so that A contains an element which is a divisor of zero and A is not a division algebra.

Let next $x_1 = 0$, $x_2 \neq 0$. Let $\xi_2 = x_3/x_2$, $\xi_3 = x_4/x_2$, $\xi_4 = x_5/x_2$. Then $\gamma = \alpha\xi_2 + \beta\xi_4^2 - \alpha\beta\xi_5^2$. The element $e = \xi_2 i_1 + (\xi_3 + \xi_4 i_1) j_1$ is in B. Hence the element $e + i_2$ is not zero and the element $-e + i_2$ is not zero. But

$$(-e + i_2)(e + i_2) = -e_2 + (-e_2 + e) i_2 + i_2^2 = -e^2 + \gamma.$$

But

$$e^2 = [\xi_2 i_2 + (\xi_3 + \xi_4 i_1) j_1][\xi_2 i_1 + (\xi_3 + \xi_4 i_1) j_1] = \gamma.$$

Hence $-e^2 + \gamma = 0$ and the product of two non-zero elements of A is zero and A is not a division algebra.

Next let $x_1 \neq 0$. Let $\eta_1 = x_2/x_1$, $\xi_2 = x_3/x_1$, $\xi_3 = x_4/x_1$, $\xi_4 = x_5/x_1$. The elements $E = -f + \eta_1 i_2$ and $F = f + \eta_1 i_2$ where $f = \xi_2 i_1 + (\xi_3 + \xi_4 i_1) j_1$ are in Σ. Hence $E + j_2$ and $F + j_2$ are both not zero. But $\delta = -\gamma\eta_1^2 + \alpha\xi_2^2 + \beta\xi_3^2 - \alpha\beta\xi_4^2$ by our assumption that $x_1 \neq 0$ and by Lemma 4 so that

$$(E + j_2)(F + j_2) = EF + \delta + Ej_2 + j_2 F,$$
$$Ej_2 + j_2 F = (-f + \eta_1 i_2) j_2 + j_2 (f + \eta_1 i_2) = Ef + f + [(\eta_1 - \eta_1) i_2] j_2 = 0.$$

Moreover

$$EF = (-f + \eta_1 i_2)(f + \eta_1 i_2) = -f^2 + \eta_1^2 i_2 + (\eta_1 f - f\eta_1) i_2$$
$$= -f^2 + \gamma\eta_1^2 = \gamma\eta_1^2 - [\xi_2^2\alpha + \xi_3^2\beta - \xi_4^2\alpha\beta] = -\delta,$$

so that

$$(E + j_2)(F + j_2) = -\gamma + \gamma = 0.$$

In all cases we have shown that A is not a division algebra.

Corollary. Let A be any normal division algebras in sixteen units over R. Then A contains no rational generalized quaternion sub-algebra.

For if A were to contain a generalized quaternion sub-algebra, B over R, then A would be the direct product of B and another generalized quaternion algebra C, contrary to the hypothesis that A is a division algebra.

3. **The structure of $A = B_1 \times B_2 \times \cdots \times B_n$.** We shall now consider briefly the possible structure of the algebras A of the preceding section. Let A^1 be the algebra with the same basis and multiplication table as A but over $R(\sqrt{\alpha}, \sqrt{\gamma})$. Then A^1 is a total four rowed matric algebra

and is a simple algebra. Consequently A is semi-simple.* If A were not simple then A would be reducible and $A = A_1 \oplus A_2$. Then $A' = A_1' \oplus A_2'$ over $R(\sqrt{\alpha}, \sqrt{\gamma})$ contrary to the hypothesis that A^1 is simple. Thus we may write immediately

THEOREM 2. *Let A be the direct product of n rational generalized quaternion algebras. Then, if n is odd, A is the direct product of a generalized quaternion algebra and a total matric algebra of order $(2^{n-1})^2$. However, if n is even, then A has either the above structure as for n odd or is a total matric algebra of order $(2^n)^2$.*

For A is evidently a normal algebra and the direct product of two generalized quaternion rational algebras has been seen to be either a total matric algebra of order sixteen or the direct product of a generalized quaternion algebra and a two rowed matric algebra, the only possible simple normal algebras in sixteen units.

* Cf. L. E. Dickson "Algebren und ihre Zahlentheorie", p. 110.

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NOTE ON THE EXISTENCE OF ANALYTIC SOLUTIONS OF NON-HOMOGENEOUS LINEAR q-DIFFERENCE EQUATIONS, ORDINARY AND PARTIAL.

By C. R. Adams.*

We wish to take this opportunity to correct a small error in a paper published in these Annals, vol. 27 (1925), pp. 73–83, and at the same time slightly extend its results. In the treatment of the case $\chi = 0$ on p. 78 the statements are made as though $\arg[\gamma - \varrho_k]$ were zero. If this is not the case, the procedure depends on the value of

$$D = (\log|q|) R[\varrho_k - \gamma] - \arg q \arg[\varrho_k - \gamma].$$

It is always possible to define $\arg q$ so that D is positive; then the evaluation of $\sum$ given by (12) can be used. If $\arg q$ is defined as indicated by the footnote on p. 77 and $D < 0$, it is necessary to divide (17) into parts and proceed as before. If D is a negative integer, three parts should be used, the middle one of the form

$$\theta_w x^{iW},$$

W being real; for this part the evaluation

$$\sum \theta_w x^{iW} = \frac{\theta_w x^{iW}}{q^{iW} - 1}$$

should be used. Similar remarks apply to the proof of Theorem III, especially with reference to the first footnote on p. 82. Clearly the R may now be removed from the statements of Theorems I and III.

* Received March 26, 1929.